HYDROMAGNETIC STABILITY OF A FLOW BETWEEN TWO CO-AXIAL CYLINDERS

by

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ABSTRACT

Hydromagnetic stabilities of an inhomogeneous, incompressible, viscous fluid are investigated. The fluid is rotating between two infinite perfectly conducting co-axial rigid cylinders and is subjected to an axial and a toroidal magnetic field. Both the axisymmetric and non-axisymmetric perturbations are examined by employing the inner product technique. We have also discussed the instability with respect to axisymmetric disturbance of an inviscid fluid rotating differentially between the cylinders in the presence of an axial and a toroidal magnetic field. By introducing an appropriate transformation, the characteristic equation is simplified and sufficient conditions for oscillatory, stable, and unstable motion are derived. A sufficient condition for stability when the perturbation is axisymmetric and the magnetic field is purely toroidal are also deduced.
DEDICATION

TO MY PARENTS
ACKNOWLEDGMENT

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CHAPTER 1

1. INTRODUCTION

Magnetohydrodynamics deals with the dynamics of a conducting fluid which interacts with a magnetic field. It has been used to describe a variety of configurations, which include incompressible or compressible flows, liquid or gaseous state, dynamic or static case, and continuum fluid analyses. Considerable interest in the dynamics of conducting fluid has arisen during the past decade in connection with efforts to harness fusion energy leading to problems in missile and spacecraft dynamics, propulsion, and communications. This discipline has been applied extensively in astrophysical and geophysical phenomena. Observed effects as well as experiments show a comfortable correspondence with the mathematical predictions of MHD continuum fluid relations at macroscopic level combined with Maxwell equations. Even with continuum structure, further idealizations need to be made in order to simplify a problem or to approximate a physical situation. For instance, when discussing the flow of a conducting fluid through a parallel channel in the presence of a transverse magnetic field, a closed form solution can be obtained if the fluid is viscous and has finite conductivity. On the other hand, however, to discuss the Couette flow of a rotating fluid, it is necessary to consider only inviscid and perfectly conducting fluids to derive an analytic solution. The flow in the magnetic field gives rise to mechanical forces which in turn modify the motion. It is then this interaction between the field and the motion which invites the interest in the subject, particularly in studies related to Earth's interior, the Sun, the stars or interstellar space.
It is generally believed that the existence of geomagnetic field is a consequence of finite amplitude instability of the Earth's liquid core, and that the rotation of the Earth has a pronounced effect on hydromagnetic oscillations of the liquid core. This has inspired considerable activity in magnetohydrodynamic stability of rotating flows. On account of significant effects of rotation, stratification, and the imposed magnetic fields on the stability, several researchers including Helmholtz [19], Kelvin [23], Rayleigh [33], Taylor [44], Hide [20,21], Michael [31], Chandrasekhar [8], Acheson [1,2], Acheson and Hide [3] have investigated hydromagnetic stability of fluid flows in various configurations.

This dissertation is entirely devoted to MHD stability. In hydrodynamics, instability is of two kinds. The first occurs when laminar motion breaks down to give rise to turbulence. The criterion for this instability is that the Reynolds number should be large. The second type is thermal instability leading to convection in a fluid heated from below, and the criterion for instability is the size of Rayleigh number. The presence of a magnetic field induces something like a viscous drag on a conducting fluid and also imparts some degree of rigidity. The imposition of a magnetic field thus inhibits instability. The intuitive way to approach stability is by determining whether or not it departs far from equilibrium when it is subjected to an arbitrary small disturbance and is then left to itself.

The magnetohydrodynamic equations are nonlinear, but they can be linearized for a small amplitude motion. The linearization procedure rests on writing the velocity and the magnetic field, respectively, as:

\[
\begin{align*}
U &= U_0 + u, & |u| &< |U_0| \\
B &= B_0 + b, & |b| &< |B_0|
\end{align*}
\]
where \( u \) and \( b \) are perturbations on an otherwise equilibrium state.

The disturbances imposed are assumed to be so small that the terms quadratic or higher in perturbed quantities and their derivatives can be ignored so that the equation of motion and associated equations become linear. The linear system of equations so obtained does not contain time \( t \) explicitly, but is time dependent only through derivatives with respect to \( t \). Instability is then investigated by the so-called normal mode method. The small perturbation imposed on the original state of equilibrium or steady motion is assumed to involve a time factor \( e^{\sigma t} \). If all the characteristic values of \( \sigma \) have negative real parts only, then the motion is stable with respect to infinitesimal disturbances. However, even if only one characteristic value has positive real part then the motion becomes unstable.

Chandrasekhar [8] has investigated the effect of a magnetic field on the stability of a liquid flow between coaxial cylinders rotating at different rates. In hydrodynamics, this problem was studied by Taylor [44]. In this presentation, we consider the role of magnetic fields when they are the direct cause of instabilities. It is well recognized that weak magnetic fields destabilize stable hydrodynamical flows by inducing axisymmetric instabilities. We shall confine our attention to the effects of only axisymmetric perturbations in an axially infinite, incompressible, rotating Couette flow, whose initial equilibrium state includes a magnetic field of various kinds: a purely toroidal field, a purely axial field, or a combination of both. This model is ideally appropriate for linear stability analysis because it allows us to look into oscillatory (normal) modes in rotating magnetized flows, and to understand the role of the magnetic field components in the appearance of various unstable axisymmetric modes.
When the velocity components $U_i(x,y,z)$, pressure $P(x,y,z)$, and magnetic field $H_i(x,y,z)$ representing a solution to the system of hydromagnetic equation are time independent, then this state is known as the basic state. When an infinitesimal disturbance is superposed on such a basic state and if the ensuing solution approaches to the steady basic state as $t \to \infty$, we say the system is stable. Otherwise, the motion is characterized as unstable.

The hydrodynamic instabilities of an inviscid flow between two concentric cylinders which has only swirl velocity component $V_\theta(r)$ in the direction of increasing azimuthal angle $\theta$ for axisymmetric disturbances has been well studied by Rayleigh [33]. He proved that the necessary and sufficient condition for stability is that the square of the circulation velocity should not decrease anywhere as $r$ increases from the inner to the outer cylinder. Chandrasekhar [8] has considered the same problem with both axial and swirl components of the velocity, and concluded that the stability can be determined by the swirl component alone. However, if the fluid is a perfect electrical conductor and subjected to a transverse magnetic field, then in the case of zero axial flow, the magnetic field has a similar effect as that of swirl velocity and also yields Rayleigh's criterion.

Howard and Gupta [22] investigate the hydromagnetic instability with respect to axisymmetric disturbances of a non-dissipative helical flow, with velocity components $(0, u_\theta(r), u_z(r))$ in cylindrical polar coordinate system. They take a conducting fluid permeated by an axial or an azimuthal magnetic field. However, hydromagnetic instability of a rotating fluid when the disturbance is non-axisymmetric and the fluid is viscous has not been examined in detail so far. In these problems, the mathematical
complexities are rather difficult to treat.

Using a technique due to Barston [5], Ganguly and Gupta [16] investigate the instabilities of a non-dissipative helical flow of an incompressible conducting fluid subjected to a helical field \((0, H_y(r), H_z(r))\) with respect to non-axisymmetric disturbances. Employing the same technique, Lucus [26] derived a sufficient condition for stability in the case of a rotating conducting fluid permeated by a helical magnetic field but when disturbances are non-axisymmetric. More recently, Bhattacharyya, Gupta, and Ganguly [7] examined the stability of a rotating liquid column in the presence of a magnetic field with non-zero radial component, disturbances again being non-axisymmetric.

Acheson [2] studied a class of hydromagnetic instabilities for an inviscid fluid rotating uniformly in the annular region between two infinitely long rigid cylinders. These instabilities arise due to the variations of the azimuthal magnetic field intensity with respect to distance from the axis of rotation. In the subsequent paper, Acheson [1] investigates the instabilities of a radially stratified fluid rotating between two co-axial cylinders with particular emphasis on the case when the angular velocity greatly exceeds both buoyancy and Alfvén frequencies.

The present thesis consists of two parts. In the first part, we have investigated the hydromagnetic instability making a normal mode assumption for an inhomogeneous, incompressible, conducting fluid, rotating differentially between two rigid infinite co-axial cylinders in the presence of an axial and a toroidal magnetic field. All the transport processes such as viscosity, magnetic resistivity, and thermal diffusivity are not taken into account. We have derived the criteria for the motion to be oscillatory, stable, or unstable. We have then extended the study of this problem to non-axisymmetric
perturbation. We find that regardless of the details of the magnetic field profile, any stable disturbance must propagate along the basic rotation. Furthermore, for non-axisymmetric perturbation, stable disturbances may propagate if either the magnetic field is purely axial or purely toroidal. And, in the case when both fields are present, we have derived a condition which must be satisfied for stable propagation.

The second part of this dissertation is devoted to investigate MHD stability with respect to axisymmetric perturbation of an inhomogeneous, incompressible, viscous fluid rotating between two perfectly conducting, infinite co-axial cylinders. The magnetic field taken here is axial as well as toroidal. Earlier, Sung [43] and Acheson [1] investigated this problem for an inviscid fluid. Using inner product method for axisymmetric perturbation, Sung [43] deduced sufficient stability conditions. We have derived sufficient conditions for stability, by extending the inner product method to the case of perturbations of a viscous fluid.
CHAPTER 2

HYDROMAGNETIC STABILITY OF A ROTATING NONHOMOGENEOUS
INCOMPRESSIBLE FLUID WITH RESPECT TO AXISYMMETRIC
AND NON-AXISYMMETRIC PERTURBATIONS

In investigating hydromagnetic stability in a uniformly rotating homogeneous
incompressible fluid with variations of the azimuthal magnetic field, Acheson [1] has
shown that the fluid system is stable with respect to axisymmetric disturbances if the
rotation is sufficiently rapid. On the other hand, the system is shown to be unstable
with respect to non-axisymmetric disturbances no matter how rapid the rotation. Several
authors including Chandrasekhar [8], Acheson [2], Acheson and Hide [3], Rudraiah [36]
have based their analysis on the so called Boussinesq approximation which amounts to
the neglect of density variation. In this section, we take the fluid to be inhomogeneous
with density varying in the radial direction. We have considered stability with a wide
range of magnetic field profiles such as purely axial, purely azimuthal, or when both
axial and azimuthal components are present. We have shown that any unstable
disturbance must propagate against the basic rotation. We have also deduced sufficient
stability conditions when the magnetic field is azimuthal.
2.1 **MATHEMATICAL FORMULATION**:

An incompressible, inhomogeneous, inviscid fluid is rotating with angular velocity $\Omega$ between two coaxial rigid cylinders of radii $r_1$ and $r_2$. The magnetic field is axial and toroidal. Quantities such as viscosity, magnetic resistivity and thermal diffusivity are not taken into account.

The governing equations of motion in the inertial frame in cylindrical polar coordinates $(r, \theta, z)$ are

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P + \frac{\mu}{\rho} (\mathbf{H} \cdot \nabla) \mathbf{H} + \mathbf{g} ,
\]

\[
\frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{H}) ,
\]

\[
\nabla \cdot \mathbf{u} = 0 ,
\]

\[
\nabla \cdot \mathbf{H} = 0 ,
\]

and

\[
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0 .
\]

Here

$\mathbf{u}$ is the velocity of the fluid,

$\rho$ denotes the local fluid density,

$t$ time,
\( \mu \) magnetic permeability, 
\( \mathbf{g} = (g(r), 0, 0) \), net radial body force per unit mass acting on the fluid, 
\( \mathbf{H}(=\frac{\mathbf{B}}{\mu}) \) magnetic fluid intensity, 
\( P = P_F + \frac{\mu}{2} \mathbf{H} \cdot \mathbf{H} \) is the total pressure including the actual fluid pressure \( P_F \) and the magnetic pressure \( \frac{\mu}{2} \mathbf{H} \cdot \mathbf{H} \).

The basic equilibrium state is described by

\[
\mathbf{U}_0 = (0, r[\Omega(r) + \Omega_0], 0),
\]

\[
\mathbf{H}_0 = [0, H_\theta(r), H_z(r)].
\]  
(2-6)

\( \rho_0 = \rho_0(r) \).

This state represents a flow in the presence of a radial gradient of both the magnetic field and the density. The rotation consists of a uniform part \( \Omega_0 \) and a differential part \( \Omega(r) \).

We perturb the equilibrium state given by (2-6) through infinitesimal amounts such that the resulting perturbed state of flow is

\[
\mathbf{U} = \{u_r, u_r + r[\Omega(r) + \Omega_0], u_z\},
\]

\( \mathbf{H} = \{h_r, H_\theta(r) + h_\theta, H_z(r) + h_z\}.\)  
(2-7)
To investigate stability, we look for the solution of the perturbed state in the form such that all perturbed quantities $\psi$ may be written as

$$\psi = \phi(r)e^{i(\sigma t + m\theta + k\zeta)}, \quad (2-8)$$

where $\sigma$ is in general a complex number ($\sigma = \sigma_r + i\sigma_i$), $m$ is an integer, and $k$ any real number. This will imply that,

(i) the solution is stable if $\sigma_i$ is non-zero and $< 0$,

(ii) and unstable if $\sigma_i$ is non-zero and $> 0$.

On the other hand if $\sigma_i = 0$, and $\sigma_r \neq 0$, then the motion becomes oscillatory.

With (2-8), equation (2-5) yields

$$\frac{\partial \rho}{\partial t} + u_r \frac{\partial \rho}{\partial r} + [\Omega(r) + \Omega_o] \frac{\partial \rho}{\partial \theta} = 0,$$

or

$$i[\sigma + m(\Omega(r) + \Omega_o)]\rho + u_r \rho' = 0,$$

or

$$\frac{\rho}{\rho'} = \frac{iu_r}{\omega} = \psi(r), \quad (2-9)$$

where

$$\omega = \sigma + m[\Omega(r) + \Omega_o]. \quad (2-10)$$

Here, prime denotes differentiation with respect to $r$ and $\omega$ is the modified frequency of oscillation.
Equation (2-1), in component form is

$$
\frac{\partial U_r}{\partial t} + (U_r \nabla) U_r - \frac{U_r^2}{r} = - \frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{\mu}{\rho} [(H_r \nabla) H_r] + g, \quad (2-11)
$$

$$
\frac{\partial U_\theta}{\partial t} + (U_\theta \nabla) U_\theta + \frac{U_r U_\theta}{r} = - \frac{1}{\rho r} \frac{\partial P}{\partial \theta} + \frac{\mu}{\rho} [(H_\theta \nabla) H_\theta + \frac{H_\theta H_r}{r}], \quad (2-12)
$$

and

$$
\frac{\partial U_z}{\partial t} + (U_z \nabla) U_z = - \frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\mu}{\rho} [(H_z \nabla) H_z]. \quad (2-13)
$$

With use of perturbed state (2-7), the linearized equations (2-11), (2-12), and (2-13) can be written as

$$
\frac{\partial U_r}{\partial t} + \left[ \Omega(r) + \Omega_0 \right] \frac{\partial U_r}{\partial \theta} - r[\Omega(r) + \Omega_0]^2 - 2u_\theta[\Omega(r) + \Omega_0] \\
= - \frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{\mu}{\rho} \left[ \frac{H_\theta}{r} \frac{\partial h_\theta}{\partial \theta} + H_z \frac{\partial h_z}{\partial z} - \frac{H_\theta^2 + 2H_\theta h_\theta}{r} \right] + g, \quad (2-14)
$$

$$
\frac{\partial U_\theta}{\partial t} + u_r \frac{\partial}{\partial r} \left[ r[\Omega(r) + \Omega_0] \right] + [\Omega(r) + \Omega_0] \frac{\partial U_\theta}{\partial \theta} + [\Omega(r) + \Omega_0] u_\theta \\
= - \frac{1}{\rho r} \frac{\partial P}{\partial \theta} + \frac{\mu}{\rho} \left[ \frac{h_\theta}{r} \frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r} \frac{\partial h_\theta}{\partial \theta} + H_z \frac{\partial h_z}{\partial z} + \frac{H_\theta h_\theta}{r} \right], \quad (2-15)
$$

and

$$
\frac{\partial U_z}{\partial t} + \left[ \Omega(r) + \Omega_0 \right] \frac{\partial U_z}{\partial \theta} = - \frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\mu}{\rho} \left[ \frac{h_\theta}{r} \frac{\partial H_z}{\partial r} + \frac{H_\theta}{r} \frac{\partial h_z}{\partial \theta} + H_z \frac{\partial h_z}{\partial z} \right]. \quad (2-16)
$$
Where the squares and products of the infinitesimal quantities $u_r, u_\theta, u_z, h_\theta, h_z$ can be neglected.

Using (2-8) into equations (2-14) to (2-16), we have

\[ i\omega u_r - r[\Omega(r) + \Omega_0]^2 - 2[\Omega(r) + \Omega_0]u_\theta \]
\[ = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{\mu}{\rho} [i\left(\frac{mH_\theta}{r} + kh_\theta\right)h_\theta - \frac{H_\theta^2 + 2H_\theta h_\theta}{r}] + g. \quad (2-17) \]

\[ i\omega u_\theta + u_r [r(\Omega(r) + \Omega_0)]' + [\Omega(r) + \Omega_0]u_\theta \]
\[ = -\frac{1}{\rho} imP + \frac{\mu}{\rho} [h_\theta H'_\theta + i\left(\frac{mH_\theta}{r} + kh_\theta\right)h_\theta + \frac{H_\theta h_\theta}{r}], \quad (2-18) \]

and

\[ i\omega u_z = -\frac{1}{\rho} ikP + \frac{\mu}{\rho} [h_z H'_z + i\left(\frac{mH_\theta}{r} + kh_z\right)h_z]. \quad (2-19) \]

In the basic state described by equation (2-6), radial component of equation (2-1) is

\[ \frac{\partial P_0}{\partial r} = \rho_0 [r(\Omega(r) + \Omega_0)]^2 + g] - \frac{\mu H_\theta^2}{r}, \quad (2-20) \]

where $P_0(r)$ represents the basic pressure.

With use of equations (2-7) and (2-8), from equation (2-3), we get

\[ \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0, \]

or
\[ u_z = \frac{i (ru_r)' - mu_\theta}{kr} , \]  

(2-21)

whereas equation (2-4) yields

\[ \frac{\partial h_r}{\partial r} + \frac{h_r}{r} + \frac{1}{r} \frac{\partial h_\theta}{\partial \theta} + \frac{\partial h_z}{\partial z} = 0 , \]

or

\[ h_z = \frac{i}{kr} (ru_r)' - \frac{mh_\theta}{kr} . \]  

(2-22)

Equation (2-2) can now be written as

\[ \frac{\partial h_r}{\partial t} + [\Omega(r) + \Omega_o] \frac{\partial h_r}{\partial \theta} - \frac{H_\theta}{r} \frac{\partial u_r}{\partial \theta} + H_z \frac{\partial u_z}{\partial z} , \]

\[ \frac{\partial h_\theta}{\partial t} + u_r \frac{\partial h_\theta}{\partial r} + [\Omega(r) + \Omega_o] \frac{\partial h_\theta}{\partial \theta} - [h_\theta \left( \frac{\partial}{\partial r} (r(\Omega(r) + \Omega_o)) \right)] \]

\[ + \frac{H_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + H_z \frac{\partial u_\theta}{\partial z} + [\Omega(r) + \Omega_o] h_r - \frac{H_\theta (r) u_r}{r} = 0 , \]

and

\[ \frac{\partial h_z}{\partial t} + u_r \frac{\partial h_z}{\partial r} + [\Omega(r) + \Omega_o] \frac{\partial h_z}{\partial \theta} = \frac{H_\theta}{r} \frac{\partial u_z}{\partial \theta} + H_z \frac{\partial u_z}{\partial z} . \]

With the help of equations (2-8) and (2-9), the above equations can be expressed as

\[ h_r = -i \left[ \frac{mH_\theta}{r} + kH_z \right] \psi , \]  

(2-23)
\[ h_\theta = \frac{1}{\omega} \left( \frac{mH_\theta}{r} + kH_\zeta \right) u_\theta - \left( \frac{H_\theta}{r} - H_\theta' \right) \psi - \frac{r\Omega(r)}{\omega} \left( \frac{mH_\theta}{r} + kH_\zeta \right) \psi, \]  

(2.24)

and

\[ h_\zeta = H_\zeta \psi + \frac{1}{\omega} \left[ \frac{mH_\theta}{r} + kH_\zeta \right] u_\zeta. \]  

(2.25)

Multiplying equation (2.18) by \( k \), equation (2.19) by \( \frac{m}{r} \), and then substracting, we get

\[ i\omega k u_\theta + k u_\zeta [r(\Omega(r) + \Omega_0)]' + k[\Omega(r) + \Omega_0] u_\tau - \frac{i\omega m u_\zeta}{r} \]

\[ = -\frac{H}{\rho} \left[ (kH_\theta' + \frac{kH_\theta}{r} - \frac{mH_\theta'}{r}) h_\tau + i(\frac{mH_\theta}{r} + kH_\zeta)(kh_\theta - \frac{m}{r} h_\zeta) \right]. \]  

(2.26)

Substituting the values for \( u_\tau, u_\zeta \) from equations (2.17) and (2.19) respectively, and for \( h_\tau, h_\theta, \) and \( h_\zeta \) from equations (2.23) to (2.25) respectively, equation (2.26) gives

\[ u_\theta = \left[ k^2 r^3 \right] \frac{\Omega'(r)}{M(r)} - \frac{2(\Omega(r) + \Omega_0)}{M(r)E(r)} k^2 r^2 \omega^2 + \frac{m\omega'(r)}{M(r)} \]

\[ + \frac{m\omega(r)}{M(r)} + \frac{2r \omega V_s G k^2}{M(r)E(r)} \]  

\[ \psi + \frac{m\omega r}{M(r)} \psi'. \]  

(2.27)

Here, we have defined the local Alfvén speeds as
\[ V_\theta(r) = H_\theta(r) \sqrt{\frac{\mu}{\rho}} \quad , \quad V_\zeta(r) = H_\zeta(r) \sqrt{\frac{\mu}{\rho}} \quad , \]

(2-28)

and the Buoyancy frequency as

\[ N(r) = \left( \frac{\rho'}{\rho} (g + r(\Omega(r) + \Omega_0)^2) \right)^{\frac{1}{2}} \quad . \]  
(2-29)

We also introduce the following functions:

\[ M(r) = k^2 r^2 + m^2 \quad , \]  
(2-30)

\[ G(r) = \frac{mv_\perp^2 + kV_z}{r} \quad , \]  
(2-31)

\[ E(r) = G^2(r) - \omega^2 \quad , \]  
(2-32)

and

\[ F(r) = 2[\omega(\Omega(r) + \Omega_0)] - \frac{V_\theta}{r} G(r) \quad . \]  
(2-33)

Differentiating equation (2-17) with respect to \( \theta \) and equation (2-18) with respect to \( r \), and then subtracting, we get

\[
[-m \omega - 4r \Omega'(r) - r^2 \Omega''(r) - 2(\Omega(r) + \Omega_0)] u_r
\]

\[-[r^2 \Omega'(r) + 2(\Omega(r) + \Omega_0) r] u_r' - N^2 im \psi - i[2(\Omega(r) + \Omega_0) m
\]

\[ + \omega + r \omega'(r) - i r \omega] u_\theta' = -[mG(r) + r H_\theta'' + 2 H_\theta'] h_\theta \]

(2-34)

\[-[rH_\theta' + H_\theta] h_\theta' - i(\frac{2mH_\theta}{r} + r G'(r) + G) h_\theta - i r G h_\theta' \quad . \]

Substitution of the values of \( u_r, u_r', u_\theta, u_\theta', h_r, h_r', h_\theta, h_\theta' \), and \( h_\theta'' \) in equation (2-34), furnishes
\[ E \psi'' + \left[ E' + \frac{E}{r} \left( \frac{r^2 k^2 + 3m^2}{M(r)} \right) \right] \psi' + Q \psi = 0 \quad (2-35) \]

where

\[
Q(r) = \frac{2k^2 V_z V'}{r} + k^2 r \left[ \frac{V_\theta^2}{r^2} \right]' - k^2 r \left[ (\Omega(r) + \Omega_0)^2 \right]' - N^2 \frac{M(r)}{r^2} + \frac{F^2(r)k^2}{E(r)} - \frac{2mF(r)k^2}{M(r)}
- \frac{Ek^2}{M(r)} \left[ k^2 r^2 + 1 + 2m^2 + \frac{m^2}{k^2 r^2 (m^2 - 1)} \right] . \quad (2-36)
\]

The boundary condition of course are

\[ \psi(r_1) = \psi(r_2) = 0 . \quad (2-37) \]

### 2.2. STABILITY WHEN DISTURBANCES ARE AXISYMMETRIC:

We shall confine our attention here to the case in which the magnetic field is purely azimuthal, i.e \( V_z = 0 \).

For axisymmetric case, \( m = 0 \). The value of \( Q(r) \) in equation (2-36) then is

\[ Q(r) = k^2 r \left( \frac{V_\theta}{r^2} \right)' - \frac{k^2}{r^3} \left[ r^4 (\Omega(r) + \Omega_0)^2 \right]' - N^2 k^2 + \omega^2 \left( k^2 + \frac{1}{r^2} \right) . \]

Hence, equation (2-35) can be rewritten as

\[ \psi'' + \frac{\psi'}{r} - \left( \frac{k^2 S(r)}{\omega^2} + k^2 + \frac{1}{r^2} \right) \psi = 0 , \quad (2-38) \]
where,

\[ S(r) = r \frac{\partial}{\partial r} \left( \frac{V_0^2}{r^2} \right)' - N^2 - \frac{1}{r^3} \left[ r^4 \Omega(r) + \Omega_0 \right]^2 \left( \frac{\partial}{\partial r} \right)' \]  

(2-39)

It should be noted that the frequency \( \omega \) becomes identical to \( \sigma \) (see equation (2-10)).

Introducing the transformation

\[ \xi = r^2 \phi \]

equation (2-38) yields

\[ \xi'' - \left( \frac{k^2 S(r)}{\omega^2} + k^2 + \frac{3}{4r^2} \right) \xi = 0 \]  

(2-40)

Multiplying equation (2-40) by the complex conjugate of \( \xi \), then integrating over the interval \( r_1 \leq r \leq r_2 \), and using the boundary condition \( \xi(r_1) = \xi(r_2) = 0 \), we obtain

\[ \omega^2 \int_{r_1}^{r_2} \left( |\xi|^2 + (k^2 + \frac{3}{4r^2}) |\xi|^2 \right) dr = -\int_{r_1}^{r_2} S(r) k^2 |\xi|^2 dr \]  

(2-41)

**CONCLUSIONS:**

From equation (2-41), it is clear that \( \omega^2 \) is real and positive if \( S(r) \) given in equation (2-39) is negative.
Hence, for \( S(r) < 0 \),

\[
\omega = \pm (-A^{1/2}) ,
\]

(2-42)

where

\[
A = \frac{\int_{r_1}^{r_2} S(r) k^2 |\xi|^2 \, dr}{\int_{r_1}^{r_2} \left(|\xi|^2 + (k^2 + \frac{3}{4r^2})|\xi|^2 \right) \, dr} .
\]

(2-43)

It is quite obvious that \( A \) is a negative quantity. We may therefore conclude that the motion is oscillatory.

However, if \( S(r) > 0 \), then \( A \) is positive. We can deduce from equation (2-41) that

\[
\omega = \pm i\sqrt{A} .
\]

We may, therefore, realize that:

(i) When \( \omega = +i\sqrt{A} \), the motion is stable.

(ii) When \( \omega = -i\sqrt{A} \), the motion is unstable.

2.3 NON-AXISYMMERIC STABILITY:

In this section, we shall examine instability condition for non-axisymmetric flow under the influence of both toroidal and axial magnetic
fields. That is, when \( \mathbf{H} = (0, H_\theta(r), H_\phi(r)) \).

From equation (2-35),

\[
(E\psi')' + \frac{E}{r} \left[ \frac{r^2 k^2 + 3m^2}{r^2 + m^2 k^2} \right] \psi' + Q\psi = 0 .
\]  

(2-44)

Multiplying equation (2-44) by \( \frac{r^3}{r^2 + m^2 k^2} \), and after considerable simplification, we obtain

\[
\left( \frac{r^3 E\psi'}{r^2 + m^2 k^2} \right)' + \frac{r^3 Q(r)}{r^2 + m^2 k^2} \psi = 0 .
\]  

(2-45)

We shall now multiply equation (2-45) by the complex conjugate \( \overline{\psi} \) and integrate between the boundaries \( r = r_1 \) and \( r = r_2 \). Realizing that at the boundaries, \( \psi \) must vanish. We deduce

\[
\int_{r_1}^{r_2} \left( \frac{r^3 E\psi'}{r^2 + m^2 k^2} \right)' \overline{\psi} dr + \int_{r_1}^{r_2} \left( \frac{r^3 Q(r)}{r^2 + m^2 k^2} \right) \psi \overline{\psi} dr = 0 .
\]

or

\[
\left[ \frac{r^3 E\psi' \overline{\psi}}{r^2 + m^2 k^2} \right]_{r_1}^{r_2} - \int_{r_1}^{r_2} \frac{r^3 E(r)}{r^2 + m^2 k^2} \psi' \overline{\psi} dr + \int_{r_1}^{r_2} \left( \frac{r^3 Q(r)}{r^2 + m^2 k^2} \right) \psi \overline{\psi} dr = 0 ,
\]

or

\[
\int_{r_1}^{r_2} \frac{r^3}{r^2 + m^2 k^2} \{ Q(r) \psi \overline{\psi}^2 - E(r) \psi' \overline{\psi} \} dr = 0 .
\]  

(2-46)
Letting \( \omega = \omega_r + i\omega_i \), and multiplying equation (2-46) by \( \omega^2 \), we equate the imaginary part of (2-46) to zero. This furnishes

\[
2\omega_r\omega_i \int_{r_1}^{r_2} \frac{r^3}{r^2 + m^2k^{-2}} \left[ |\psi'|^2 + \left( S_1(r) - \frac{2m[\Omega(r) + \Omega_0]}{\omega_r[r^2 + m^2k^{-2}]} + \frac{4k^2}{|E|^2} S_2(r)\right)|\psi|^2 \right] dr = 0, \tag{2-47}
\]

where,

\[
S_1(r) = \frac{1}{r^2 + m^2k^{-2}} \left( k^2r^2 + 2m^2 + \frac{m^2}{k^2r^2} (m^2 - 1) \right), \tag{2-48}
\]

and

\[
S_2(r) = \left( \frac{mV_\theta}{r} + kV_z \right)^2 \left[ \frac{V_\theta^2}{r^2} + \{\Omega(r) + \Omega_0\}^2 \right]
- \frac{\{\Omega(r) + \Omega_0\}V_\theta}{r\omega_r} \left[ \frac{mV_\theta}{r} + kV_z \right] \left[ \left( \frac{mV_\theta}{r} + kV_z \right)^2 + \Omega^2 + \omega_i^2 \right]. \tag{2-49}
\]

**CONCLUSIONS:**

**A. TOROIDAL PROPAGATION:**

If the differential part of the rotation \( \Omega(r) \) becomes uniform and is against the basic rotation \( \Omega_0 \), i.e. \( \Omega(r) + \Omega_0 = 0 \), then equation (2-47) reduces to

\[
2\omega_r\omega_i \int_{r_1}^{r_2} \frac{r^3}{r^2 + m^2k^{-2}} \left[ |\psi'|^2 + \left( S_1(r) + \frac{4k^2}{|E|^2} \frac{mV_\theta}{r} + kV_z \right)^2 \frac{V_\theta^2}{r^2} \right] |\psi|^2 \right] dr = 0. \tag{2-50}
\]
Since \( m \) takes only the integral values, the integrand on the left in equation (2-50) is always positive in the interval \( r_1 \leq r \leq r_2 \).

The integral therefore can only vanish if \( \omega, \omega_i = 0 \).

Leaving aside the trivial case when both \( \omega, \text{ and } \omega_i \) are zero, two possibilities arise:

(i) When \( \omega = 0 \), but \( \omega_i \neq 0 \), the motion is stable. The unstable modes do not propagate.

(ii) When \( \omega \neq 0 \), but \( \omega_i = 0 \), the motion is oscillatory.

On the other hand, when \( \Omega(r) + \Omega_0 \neq 0 \), the non-axisymmetric disturbances may propagate and grow in amplitude. To study such modes for which \( \omega, \omega_i \neq 0 \), we express equation (2-47) as

\[
\int_{r_1}^{r_2} \frac{r^3}{r^2 + m^2 k^2} |\psi|^2 + (S_1(r) + S_2(r) - S_3(r)) |\psi|^2 \, dr = 0 .
\] (2-51)

Here, \( S_1(r) \) is given by equation (2-48), and

\[
S_3(r) = \frac{4k^2}{r^2} \left( \frac{mV_\theta}{r} + kV_z \right)^2 [V_\theta^2 + (r\Omega(r) + \Omega_0)^2] ,
\] (2-52)
It is obvious that both \( S_s(r) \) and \( S_t(r) \) in this case are positive.

However, unless \( S_s(r) \) is also positive somewhere in the interval \( r_1 \leq r \leq r_2 \), the integral in equation (2-51) cannot vanish in view of our assumption that \( \omega_s \omega_r \neq 0 \). Thus, for \( \omega_s \omega_r \neq 0 \) to hold, we must have

\[
S_s(r) = \frac{r[\Omega(r) + \Omega_0]V_\theta}{\omega_r} \left( \frac{mV_\theta}{r} + kV_z \right) \left( \frac{mV_\theta}{r} + kV_z \right)^2 + \omega_s^2 + \omega_r^2 \right) \\
+ \frac{2m[\Omega(r) + \Omega_0]}{\omega_r(r^2 + m^2k^2)} \left[ \left( \frac{mV_\theta}{r} + kV_z \right)^2 - \omega_s^2 + \omega_r^2 \right]^2 + 4\omega_s^2 \omega_r^2 \right] .
\] (2-53)

somewhere in the interval \( r_1 \leq r \leq r_2 \).

We shall now consider the case when the magnetic field is purely toroidal. Then, equation (2-52) and inequality (2-54) reduce respectively to

\[
S_s(r) = \frac{4k^2}{r^4} m^2 V_\theta^2 \left[ V_\theta^2 + r^2 \left( \frac{\Omega(r) + \Omega_0}{r^2} \right)^2 \right] .
\] (2-55)

and
\[
\frac{m[\Omega(r) + \Omega_0]}{\omega_r} \left( \frac{m^2 V^2_\theta}{r^2} + \omega_r^2 + \omega_i^2 \right) \\
+ \frac{2m[\Omega(r) + \Omega_0]}{\omega_r (r^2 + m^2 k^{-2})} \left( \frac{m^2 V^2_\theta}{r^2} - \omega_r^2 + \omega_i^2 \right)^2 + 4 \omega_r^2 \omega_i^2 | > 0 .
\]

(2-56)

From inequality (2-56), it is clear that for any stable disturbance, we must have

\[
\frac{m[\Omega(r) + \Omega_0]}{\omega_r} = \frac{(\Omega(r) + \Omega_0)}{\omega_{\theta r}} > 0 ,
\]

(2-57)

where \( \omega_{\theta r} = \frac{\omega_i}{m} \) is the phase velocity in the toroidal direction and must therefore propagate along the basic rotation.

We shall now take up the case when the magnetic field is purely axial. Then equation (2-52) and inequality (2-54) can be written as

\[
S_\zeta(r) = 4k^4 V^2_\zeta \{ \Omega(r) + \Omega_0 \}^2 ,
\]

(2-58)

and

\[
\frac{2m[\Omega(r) + \Omega_0]}{\omega_r (r^2 + m^2 k^{-2})} \{ (k^2 V^2_\zeta - \omega_r^2 + \omega_i^2)^2 + 4 \omega_r^2 \omega_i^2 \} > 0 .
\]

(2-59)

For stable disturbances in this case, we must have
(i) \[ \frac{m(\Omega(r) + \Omega_0)}{\omega_r} > 0 \quad (2-60) \]

and

(ii) \[ kV_z \geq 0 \quad (2-61) \]

everywhere in the interval \( r_1 \leq r \leq r_2 \). The disturbance therefore must propagate along the basic rotation.

**B. SUFFICIENT CONDITION:**

We shall now attempt to derive sufficient condition for stability when the magnetic field is only azimutal. We shall take both \( \omega_1 \) and \( \omega_2 \) as non-zero.

Equation (2-46) when divided by \( \omega^2 \) becomes

\[
\int_{r_1}^{r_2} \frac{r^3}{r^2 + m^2 k^2} \left[ \frac{Q(r)}{\omega^2} |\psi|^2 \right. - \left. \frac{E(r)}{\omega^2} |\psi'|^2 \right] dr = 0 \quad (2-62)
\]

It can now be calculated that

\[
\text{Im} \left[ \frac{E(r)}{\omega^2} \right] = -2 \omega_1 \omega_r \frac{m^2 V_\theta^2}{r^2 |\omega|^4},
\]

and
Equating imaginary part of equation (2-62) to zero, we find

\[
\frac{2 \omega_1 \omega_i}{|\omega|^2} \int_{r_i}^{r_2} \frac{r^3}{r^2 + m^2 k^{-2}} \left[ \frac{m^2 V_\theta^2}{r^2 |\omega|^2} |\psi|^2 + S_\psi(r) |\psi|^2 \right] dr = 0 , \tag{2-63}
\]

where

\[
S_\psi(r) = \frac{4 k^2 [\Omega(r) + \Omega_0] |\omega|^2}{|E|^2 r^2} - \frac{k^2}{|\omega|^2} \left[ \frac{V_\theta^2}{r^2} \right]^{r^2 + N_2 (1 + \frac{m^2}{k^2 r^2})} \]

\[
+ \frac{2 m [\Omega(r) + \Omega_0]}{|E|^2 r^2} \left[ \frac{2 k^2 V_\theta^2}{r^2} \left( \frac{m^2 V_\theta^2}{r^2} - 3 \omega_i^2 + \omega_i^2 \right) + \frac{1}{r^2 + m^2 k^{-2}} \right] \tag{2-64}
\]

\[
+ \frac{m^2 V_\theta^2}{r^2 |\omega|^2} S_\psi(r) - \frac{4}{r^2 + m^2 k^{-2}} - \frac{4 k^2}{m^2} \left( \frac{m^2 V_\theta^2}{r^2} - 2 \omega_i^2 + 2 \omega_i^2 \right)^{-1} \tag{2-65}
\]

and

\[
E(r) = \frac{m V_\theta^2}{r} - \omega_i^2 \tag{2-65}
\]

Since we are considering axisymmetric disturbance, \( m = 0 \). Equations (2-64) and (2-65) reduce to

\[
S_\psi(r) = \frac{k^2}{|\omega|^2} \left[ 4 [\Omega(r) + \Omega_0]^2 - \frac{V_\theta^2}{r^2} \right] \tag{2-65}
\]

and
\[ E = -\omega^2. \quad (2-66) \]

From equation (2-63) it can be deduced that if

\[ T = 4[\Omega(r) + \Omega_0]^2 - r \left( \frac{V^2}{r^2} \right)' + N^2. \quad (2-67) \]

If \( T \) does not change sign in the interval \( r_1 \leq r \leq r_2 \), then either \( \omega_r \) or \( \omega_i \) must vanish. However, Michael [31] has proved that the system is stable (i.e. \( \omega_i > 0 \)) if and only if \( T > 0 \). Physically, that would imply that the system is stable if the rotation is very rapid.
CHAPTER 3

HYDROMAGNETIC STABILITY OF A VISCOS FLUID ROTATING BETWEEN TWO CYLINDERS WITH RESPECT TO AXISYMMETRIC PERTURBATIONS

We shall discuss hydromagnetic stability with respect to axisymmetric disturbances for an inhomogeneous, incompressible, viscous fluid rotating between two perfectly conducting rigid infinite coaxial cylinders. The magnetic field to which the system is subjected has an axial as well as toroidal components. The method we use is that of inner product. Sung [43] used this method to analyse this kind of stability for an inviscid fluid. He deduced the characteristic equation in the form

$$\sigma^2 \xi - 2i\sigma A \xi - B \xi = 0.$$  

He had shown the inner products $<\xi, iA\xi>$ and $<\xi, B\xi>$ were both real due to the Hermitian property of the operators iA and B. This led him to deduce the sufficient stability condition. This condition is similar to the one derived by Howard and Gupta [22] when $\rho$ and $H_z$ are constant and the acceleration due to gravity is not taken into account. However, the technique of inner product method for a viscous fluid is much more complex. We have attempted to discuss the sufficient conditions for stability by separating the real and imaginary parts of the inner products. Such an approach does yield conditions for the propagation to be unstable, oscillatory, or stable.
3.1 MATHEMATICAL FORMULATION:

We consider a viscous fluid rotating with angular velocity $\Omega$ between two rigid infinite coaxial cylinders of radii $r_1$ and $r_2$. The cylinders are perfectly conducting. An axial and a toroidal magnetic field is present. The fluid is incompressible and nonhomogeneous. The dissipative mechanisms such as magnetic resistivity and thermal diffusivity are neglected.

The governing equations of motions in cylindrical polar coordinates $(r, \theta, z)$ are

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P + \frac{\mu}{\rho} (\mathbf{H} \cdot \nabla) \mathbf{H} + \nu \nabla^2 \mathbf{u} - \mathbf{g} ,
\]

(3-1)

\[
\frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{H}) ,
\]

(3-2)

\[
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0 ,
\]

(3-3)

\[
\nabla \cdot \mathbf{u} = 0 ,
\]

(3-4)

and

\[
\nabla \cdot \mathbf{H} = 0 .
\]

(3-5)

Here,

$\mathbf{u}$ is the fluid velocity,

$\rho$ is the density,

$\mathbf{H}$ the magnetic intensity,
the kinematic viscosity coefficient,

\[ \nu = (-g(r), 0, 0) \text{, net radial body force per unit mass acting on the fluid,} \]

and \( P \) represents the total pressure

\[
P = P_f + \frac{\mu}{2} H \cdot H ,
\]

(3-6)

where \( P_f \) is the actual fluid pressure and \( \frac{\mu}{2} H \cdot H \) represents the magnetic pressure.

The boundary conditions are those of perfectly conducting rigid walls:

\[
u_r(r_1) = u_r(r_2) = 0 .
\]

(3-7)

In addition, there are no slip boundary conditions at \( r = r_1, r_2 \). However, since we are going to consider only azimuthal disturbances, no slip boundary conditions do not matter for our stability analysis.

The solution given below is the stationary solution:

\[
\mathcal{U}_0 = (0, r\Omega(r), 0) ,
\]

\[
\mathcal{H}_0 = (0, H_\theta(r), H_z(r)) ,
\]

(3-8)

\[
\rho_0 = \rho_0(r) .
\]

We plan to investigate the stability when the stationary state (3-8) is disturbed slightly. We superimpose an infinitesimal perturbation of the viscous flow so that the perturbed state is described by
$U = (u_r, r\Omega(r) + u_\theta, u_z)$,

$H = (h_r, H_\theta(r) + h_\theta, H_z(r) + h_z)$,  \hspace{1cm} (3-9)

$\rho_0 + \rho, \quad P_0 + P$.

Equation (3-1) in component form is

$$
\frac{\partial U_r}{\partial t} + (U \cdot \nabla) U_r - \frac{U_r^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{\mu}{\rho} [(H \cdot \nabla) H_r - \frac{H_r^2}{r}] \\
+ \mu U_r \frac{\partial U_r}{\partial \theta} - \frac{U_r}{r^2} + g,  \hspace{1cm} (3-10)
$$

$$
\frac{\partial U_\theta}{\partial t} + (U \cdot \nabla) U_\theta + \frac{U_r U_\theta}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial \theta} + \frac{\mu}{\rho} [(H \cdot \nabla) H_\theta + \frac{H_\theta H_r}{r}] \\
+ \mu \frac{\partial U_r}{\partial \theta} + \frac{U_\theta}{r^2} + \frac{U_r}{r^2},  \hspace{1cm} (3-11)
$$

and

$$
\frac{\partial U_z}{\partial t} + (U \cdot \nabla) U_z = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\mu}{\rho} [(H \cdot \nabla) H_z] + \nu \nabla^2 U_z.  \hspace{1cm} (3-12)
$$

We substitute now the perturbed state described by (3-9). The perturbation quantities are assumed to be very small. Neglecting square and product terms of the perturbed quantities in equations (3-10) to (3-12), we obtain the linearized equations of motion:
\[
\frac{\partial u_r}{\partial t} + \frac{\partial u_r}{\partial \theta} - r \Omega^2 - 2 \Omega u_\theta = -\frac{1}{\rho} \frac{\partial P}{\partial r} \\
\frac{\partial u_\theta}{\partial t} + \frac{\partial u_\theta}{\partial \theta} + u_r \frac{\partial \Omega}{\partial r} + \Omega u_r = -\frac{1}{\rho r} \frac{\partial P}{\partial \theta} \\
+ \frac{\mu}{\rho} \left[ \frac{H_\theta}{r} \frac{\partial H_\theta}{\partial \theta} + H_\theta \frac{\partial H_\theta}{\partial z} - \frac{2 H_\theta H_r}{r} \right] + \nu \left[ \nabla^2 u_r + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r^2} \right] - g ,
\]

(3-13)

\[
\frac{\partial u_z}{\partial t} + \frac{\partial u_z}{\partial \theta} \frac{\partial \Omega}{\partial r} + \Omega u_z = -\frac{1}{\rho \frac{\partial z}{\partial \theta}} \frac{\partial P}{\partial \theta} \\
+ \frac{\mu}{\rho} \left[ \frac{H_\theta}{r} \frac{\partial H_\theta}{\partial r} + H_\theta \frac{\partial H_\theta}{\partial z} + \frac{H_\theta H_r}{r} \right] + \nu \left[ \nabla^2 u_z + \frac{2}{r^2} \frac{\partial u_z}{\partial \theta} - \frac{u_\theta^2}{r^2} \right] ,
\]

(3-14)

and

\[
\frac{\partial u_z}{\partial t} + \frac{\partial u_z}{\partial \theta} = -\frac{1}{\rho} \frac{\partial P}{\partial \theta} + \frac{\mu}{\rho} \left[ \frac{H_\theta}{r} \frac{\partial H_\theta}{\partial r} + H_\theta \frac{\partial H_\theta}{\partial z} + \frac{H_\theta H_r}{r} \right] + \nu \nabla^2 u_z .
\]

(3-15)

Using the stationary solution (3-8), equation (3-13) yields

\[
g - r \Omega^2 = G_r - \frac{1}{2} r^2 B_r - 2 r \Omega_\Lambda^2 - \frac{\mu}{\rho} H_\theta D H_z ,
\]

(3-16)

where

\[
G_r = -\frac{1}{\rho} DP_0 ,
\]

\[
B_r = \frac{\mu}{\rho} D \left( \frac{H_\theta}{r} \right)^2 = D \Omega_\Lambda^2 + \frac{1}{\rho} D \rho \Omega_\Lambda^2 ,
\]

\[
D = \frac{d}{dr} ,
\]

\[
\Omega_\Lambda^2 = \frac{\mu H_\theta^2}{\rho} .
\]
The boundary conditions (3-7) become

\[ u_r(r_1) = u_r(r_2) = 0 \quad \text{(3-17)} \]

We shall investigate the normal mode solution for the Lagrangian displacement:

\[ \xi = \xi(r)e^{i(\omega t + m\theta + kz)} \quad \text{(3-18)} \]

where

\[ \xi = (\xi_r, \xi_\theta, \xi_z) \quad \omega \text{ is in general a complex number, } m \text{ is an integer, and } k \text{ any real number.} \]

Since the Lagrangian displacement vector \( \xi \) and velocity vector \( u \) are related by

\[ U = \frac{\partial \xi}{\partial t} + U_0 \cdot \nabla \xi - \xi \cdot \nabla U_0 + \Omega(r) \xi_r, \xi_\theta \]

we have

\[ u_r = \frac{\partial \xi_r}{\partial t} + \Omega(r) \frac{\partial \xi_r}{\partial \theta} \quad \text{(3-19)} \]

\[ u_\theta = \frac{\partial \xi_\theta}{\partial t} + \Omega(r) \frac{\partial \xi_\theta}{\partial \theta} - r \xi_r \frac{d\Omega}{dr} \]

and

\[ u_z = \frac{\partial \xi_z}{\partial t} + \Omega(r) \frac{\partial \xi_z}{\partial \theta} \]

With substitution of the normal mode solution (3-18), the quantities (3-19) reduce to
\[ u_r = i \sigma \xi_r , \]  
\[ u_\theta = i \sigma \xi_\theta - r \xi_r \frac{d\Omega}{dr} , \]  
\[ u_z = i \sigma \xi_z , \]

where
\[ \sigma = \omega + m\Omega(r) . \]

We write equation (3-2) in component form:

\[ \frac{\partial H_r}{\partial t} + (U \cdot \nabla) H_r - (H \cdot \nabla) U_r = 0 , \]

\[ \frac{\partial H_\theta}{\partial t} + (U \cdot \nabla) H_\theta - (H \cdot \nabla) U_\theta + \frac{2}{r}(U_\theta H_r - U_r H_\theta) = 0 , \]

and
\[ \frac{\partial H_z}{\partial t} + (U \cdot \nabla) H_z - (H \cdot \nabla) U_z = 0 . \]

Using the perturbed state (3-9) in the above equations, we get

\[ \frac{\partial \xi_r}{\partial t} + \Omega(r) \frac{\partial \xi_r}{\partial \theta} = \left( \frac{H_\theta}{r} \frac{\partial u_r}{\partial \theta} + H_z \frac{\partial u_r}{\partial z} \right) , \]  
\[ \frac{\partial \xi_\theta}{\partial t} + \Omega(r) \frac{\partial \xi_\theta}{\partial \theta} + u_r \frac{\partial \xi_\theta}{\partial r} = -\Omega(r) h_r + \frac{u_r}{r} H_\theta \]
\[ + h_r [\Omega(r) + r\Omega'(r)] + \frac{H_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + H_z \frac{\partial u_\theta}{\partial z} . \]
\[
\frac{\partial h_z}{\partial t} + u_r \frac{\partial H_z}{\partial r} + \Omega(r) \frac{\partial h_z}{\partial \theta} = \frac{H_\theta}{r} \frac{\partial u_z}{\partial \theta} + H_z \frac{\partial u_z}{\partial z}.
\] 

(3-25)

With substitution of the normal mode solution, equations (3-23) to (3-25) reduce to

\[
h_r = \frac{u_r}{\sigma} \left[ \frac{mH_\theta}{r} + kH_z \right] ,
\]

(3-26)

\[
h_\theta = \frac{1}{i\sigma} \left[ \frac{H_\theta}{r} u_r + r\Omega'(r)h_r + \left( \frac{mH_\theta}{r} + kH_z \right) \omega_\theta \right] ,
\]

(3-27)

and

\[
h_z = \frac{1}{i\sigma} \left[ \left( \frac{mH_\theta}{r} + kH_z \right) u_z - \frac{1}{i\sigma} H'_z u_z \right] .
\]

(3-28)

Use of quantities (3-20) to (3-22) in equations (3-26) to (3-28) gives

\[
h_r = \frac{imH_\theta}{r} (1 + \alpha) \xi_r ,
\]

(3-29)

\[
h_\theta = \frac{imH_\theta}{r} (1 + \alpha) \xi_\theta + \left( \frac{H_\theta}{r} - DH_\theta \right) \xi_r ,
\]

(3-30)

and

\[
h_z = \frac{mH_\theta}{r} (1 + \alpha) \xi_z - (DH_z) \xi_r ,
\]

(3-31)

where,

\[\alpha = \frac{rkH_z}{mH_\theta} , \quad D = \frac{d}{dr} .\]
When the perturbed state (3-9) is used in conjunction with the normal mode solution (3-20) to (3-22), the linearized form of equation (3-3) becomes

\[ i(\omega + m\Omega)\rho + u,\rho' = 0 , \]

or

\[ i\sigma\rho + i\xi_r\rho'\sigma = 0 , \]

thus giving

\[ \xi_r = -\frac{\rho'}{\rho} . \]

Substituting the normal mode solutions (3-20) to (3-22) in equations of motion (3-13) to (3-15), we obtain

\[ i\sigma u_r - 2\Omega u_\theta - r\Omega^2 + g \]

\[ = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{\mu}{\rho} \left[ \left( \frac{mH_\theta}{r} + kH_z \right) h_r - \frac{2H_\theta h_r}{r} \right] + \nu((DD^* - M^2)u_r - \frac{2imv}{r^2}u_\theta) \]

or

\[ \sigma^2 \xi_r + 2i\sigma\Omega \xi_\theta - 2\Omega r(DD^*)\xi_r - \frac{1}{\rho} \frac{\partial P}{\partial r} - m^2\Omega_\Lambda^2(1 + \alpha)^2 \xi_r - 2im\Omega_\Lambda^2(1 + \alpha)\xi_\theta \]

\[ + rB_\xi_r - N^2 \xi_r + \nu[DD^* - M^2] i\sigma \xi_r - \frac{2imv}{r^2} [i\sigma \xi_\theta - r\xi_r, D\Omega] = 0 \]

\[ i\sigma u_\theta + \left[ \frac{\partial}{\partial r}(r\Omega(r) + \Omega)u_r + \frac{imP}{pr} - \frac{\mu}{\rho} \left[ h_r \frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r} \frac{\partial h_r}{\partial \theta} + H_z \frac{\partial h_\theta}{\partial z} + h_r \frac{\partial h_z}{\partial r} \right] \right] \]

\[ - \nu((DD^* - M^2)u_\theta - \frac{2im}{r^2}u_r) = 0 \],

or
\[
\begin{align*}
\sigma^2 \xi_t - 2 i \sigma \Omega \xi_{\theta} & - \frac{imP}{\rho r} - m^2 \Omega^2 (1 + \alpha)^2 \xi_{\theta} + 2 im \Omega^2 (1 + \alpha) \xi_r, \\
+ u(DD^* - M^2)(i \sigma \xi_t - r \xi_r D \Omega) + \frac{2im}{r^2} u i \sigma \xi_r &= 0, \\
\end{align*}
\]

(3-34)

and

\[
\begin{align*}
\sigma u_z &= - \frac{ikP}{\rho} + \frac{\mu}{\rho} \left[ \left( \frac{imH_z}{r} + kH_z \right) h_z + h_r \frac{\partial H_z}{\partial r} \right] + \nu [(DD^* - M^2) u_z] + \frac{im}{r^2} u_z, \\
\end{align*}
\]

or

\[
\begin{align*}
\sigma^2 \xi_z - \frac{imP}{\rho} - m^2 \Omega^2 (1 + \alpha)^2 \xi_z + \nu (DD^* - M^2) i \sigma \xi_z + \frac{\nu}{r^2} i \sigma \xi_z &= 0, \\
\end{align*}
\]

(3-35)

where

\[
M^2 = \frac{m^2}{r^2} + k^2, \\
\Omega^2_{\alpha} = \frac{\mu H^2_{\alpha}}{\rho r^2}, \\
\Omega^2_{\theta} = \frac{\mu H^2_{\theta}}{\rho r^2}, \\
D^* = D + \frac{1}{r}, \\
\text{and} \\
N^2 = -\frac{\rho '}{\rho} (g - r \Omega^2).
\]

Multiplying equations (3-33), (3-34) and (3-35) by \( \xi_r, \xi_\theta \) and \( \xi_z \) respectively, and then adding, we obtain after considerable simplification the characteristic equation:

\[
\begin{align*}
\sigma^2 \xi - 2 i \sigma E \xi - F \xi &= 0, \\
\end{align*}
\]

(3-36)
where

\[
iE_\xi = -\left[\left(\Omega - \frac{im\nu}{r^2}\right)(\xi_t, - \xi, \xi_\theta) + \frac{\nu L}{2} \xi + \frac{\nu}{2r^2} \xi_\theta \xi_\phi\right], \quad (3-37)
\]

\[
F_\xi = \frac{1}{\rho} S + m^2\Omega^2(1 + \alpha)^2 \xi + 2i\Omega^2(1 + \alpha)(\xi_t, - \xi_\theta)
+ \xi_\theta([\phi - 4\Omega^2] - 2i\Omega)\xi_\phi + \nu L(r\Omega)(\xi_t, \xi_\phi), \quad (3-38)
\]

\[
S = \frac{dP}{dr} + \frac{imP}{r} + i\kappa P, \quad (3-39)
\]

\[
L = DD^* - M^2, \quad (3-40)
\]

and

\[
\phi = \frac{1}{r} D(r^4\Omega^2). \quad (3-41)
\]

Equations (3-20) to (3-22) can now be expressed as a single equation

\[
u = i\sigma_\xi - r(D\Omega)(\xi_t, \xi_\phi). \quad (3-42)
\]

Similarly, equations (3-29) to (3-31) can be rewritten as

\[
H = \frac{imH_\phi}{r}(1 + \alpha)\xi_t - \xi_\phi(DH_\phi - \frac{H_\theta}{r})\xi_\phi - \xi_\xi(DH_\phi)\xi_\phi. \quad (3-43)
\]

The linearized version of the boundary condition that the radial component \( u \),
of velocity vanishes at both walls \( r = r_1, r_2 \) is

\[
\xi_r(r_1) = \xi_r(r_2) = 0. \quad (3-44)
\]
These conditions are consistent with the assumption of rigid perfectly conducting walls at \( r = r_1, r_2 \). The continuity equation

\[
\nabla \cdot \xi = 0 ,
\]

becomes

\[
\left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \xi_r + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} = 0 ,
\]

or

\[
D^* \xi_r + \frac{im}{r} \xi_\theta + ik \xi_z = 0 , \tag{3-43}
\]

where \( D^* = (D + \frac{1}{r}) \).

Equation (3-36) along with equations (3-37), (3-38), (3-39), (3-42) and (3-43) constitute the basic equations for stability analysis.

To analyse stability, we substitute \( \sigma = \omega + m \Omega \) in equation (3-36) to obtain

\[
\omega^2 \xi_z - 2i \omega A_\xi - B_\xi = 0 , \tag{3-44}
\]

where

\[
iA_z = -m \Omega \xi_z - i[(\Omega - \frac{imv}{r^2})(\xi_\theta \xi_r - \xi_r \xi_\theta) + \frac{vL}{2} \xi + \frac{\nu}{2r^2} \xi_z \xi_z] , \tag{3-45}
\]

and

\[
B_z = -m^2 \Omega^2 \xi_z - 2im \Omega [(\Omega - \frac{imv}{r^2})(\xi_\theta \xi_r - \xi_r \xi_\theta) + \frac{vL}{2} \xi + \frac{\nu}{2r^2} \xi_z \xi_z] + F_\xi , \tag{3-46}
\]

with \( F_\xi \) already defined in (3-38).
3.2 STABILITY OF AXISYMMETRIC PERTURBATIONS (m=0):

In this case, we have from equation (3-45) and (3-46):

\[ iA \xi = -i[\Omega(\xi_\theta \xi_r - \xi_r \xi_\theta) + \frac{vL}{2} \xi + \frac{v}{2r^2} \xi_z \xi_z] , \]  

(3-47)

and

\[ B \xi = \frac{1}{\rho} \left[ S + \Omega^2 \left( \frac{r^2 k^2 H^2}{H^2} \xi + 2vLrD\Omega \xi_\theta \xi_\theta \right) \right. \]
\[ + 2i\Omega \left( \frac{rkH}{H} (\xi_\theta \xi_r - \xi_r \xi_\theta) + \xi_r (N^2 + (\phi - 4\Omega^2) - rB) \xi_r \right) . \]

(3-48)

3.3 INNER PRODUCT:

we shall now proceed to define inner product. Let \( E \) be a function space consisting of the set of all complex valued vectors (T denotes transpose of a matrix).

\[ \xi = [\xi_r, \xi_\theta, \xi_z]^T, \]

such that each of the components appearing in equations (3-33), (3-34), (3-35), (3-42) and (3-43) is a continuous function of \( r \) on the interval \([r_1, r_2]\). For every \( \xi \) and \( \eta \) in \( E \), we define the inner product as

\[ \langle \xi, \eta \rangle = \int_{r_1}^{r_2} \left( \xi_r \eta_r + \xi_\theta \eta_\theta + \xi_z \eta_z \right) rdr , \]

(3-49)
where $\bar{\xi}$ is the complex conjugate of $\xi$.

Taking inner product with $\xi$ in equation (3-43) yields

$$\omega^2 < \xi, \bar{\xi} > - 2\omega < \xi, iA\bar{\xi} > - < \xi, B\bar{\xi} > = 0 \quad \text{(3-50)}$$

Now, we define the following inner products that we plan to use:

$$< \xi, \bar{\xi} > = \int_{r_1}^{r_2} (\xi, \bar{\xi}) r dr = M_1 \quad \text{(3-51)}$$

which is obviously real.

$$< \xi, \overline{S} > = \int_{r_1}^{r_2} \left[ \frac{\overline{P}}{r} + \frac{\overline{P}}{r} \right] r dr$$

where prime and overbar represent differentiation with respect to $r$ and complex conjugate, respectively. The quantity $S$ is defined as in equation (3-39). By virtue of equations (3-42) and (3-43), the above inner product

$$< \xi, \overline{S} > = 0$$

and

$$< \xi, iA\bar{\xi} > = -i \int_{r_1}^{r_2} \left[ \Omega \bar{\xi} - \frac{\overline{\xi}}{2r} \frac{\overline{\xi}}{2} + \frac{\overline{\xi}}{2} \frac{\overline{\xi}}{2} \right] r dr$$

$$= M_2 + iM_3$$
where

\[ M_2 = \int_{r_1}^{r_2} (\Omega R_t) r dr , \]  

\[ M_3 = -\int_{r_1}^{r_2} \left[ \frac{vL}{2} \xi \cdot \xi + \frac{v}{2r^2} \bar{\xi} \cdot \bar{\xi} \right] r dr , \]  

and \( \bar{\xi}_{r_0} \bar{\xi}_r - \bar{\xi}_{r_0} \bar{\xi}_r = iR_t \).

Here, \( R_t \) is real.

Again,

\[ <\bar{\xi}, B \xi > = \int_{r_1}^{r_2} \left[ r^2 k^2 \Omega_x^2 \bar{\xi}_r \bar{\xi}_r - \frac{2H_0}{H_z} \Omega_y^2 r k R_t + vLr(\Omega \xi, \bar{\xi}) \right] r dr 
\]

\[ + \{N^2 + (\phi - 4\Omega^2) - rB_0, \|\xi, \|^2 \} r dr \]

\[ = M_4 + iM_5 , \]

where

\[ M_4 = \int_{r_1}^{r_2} \left[ r^2 k^2 \Omega_x^2 \|\xi\|^2 - \frac{2H_0}{H_z} \Omega_y^2 r k R_t + \text{Real}[vLr(\Omega \xi, \bar{\xi})] \right] \]

\[ + \{N^2 + (\phi - 4\Omega^2) - rB_0, \|\xi, \|^2 \} r dr , \]

and

\[ M_5 = \int_{r_1}^{r_2} \text{Im}[vLr(\Omega \xi, \bar{\xi})] r dr . \]  

Equation (3-50) is quadratic in \( \omega \) with complex coefficients. Its roots are:

\[ \omega = \frac{<\xi, iA \xi > + \{<\xi, iA \xi >\}^2 + <\xi, \xi > <\xi, B \xi > \}^{1/2}}{<\xi, \xi >} . \]  

(3-58)
With the help of equations (3-51) to (3-55), we can rewrite \( \omega \) in (3-58) as

\[
\omega = \frac{(M_2 + iM_3) \pm [M_6 + iM_7]^{1/2}}{M_1},
\]

where

\[
M_6 = M_2^2 - M_3^2 + M_4,
\]

and

\[
M_7 = 2M_2M_3 + M_4M_5.
\]

Letting

\[
M_6 = r \cos \theta, \quad M_7 = r \sin \theta,
\]

we obtain

\[
\omega = \frac{(M_2 + iM_3) \pm [(M_6^2 + M_7^2)^{1/2} \{\cos(\tan^{-1} x) + i\sin(\tan^{-1} x)\}]^{1/2}}{M_1},
\]

or

\[
\omega = \frac{(M_2 + iM_3) \pm [\lambda^{1/4} \{\cos(\frac{1}{2}\tan^{-1} x) + i\sin(\frac{1}{2}\tan^{-1} x)\}]}{M_1},
\]

where,

\[
\lambda = M_6^2 + M_7^2,
\]

and

\[
x = M_7 / M_6.
\]

Separating the real and imaginary parts of \( \omega = \omega_r + i \omega_i \), we obtain
\[ \omega_r = \frac{M_2 \pm \lambda^{1/2} \cos \left( \frac{1}{2} \tan^{-1} x \right)}{M_1}, \quad (3-61) \]

and

\[ \omega_i = \frac{M_3 \pm \lambda^{1/2} \sin \left( \frac{1}{2} \tan^{-1} x \right)}{M_1}, \quad (3-62) \]

where

\[ \sin \left( \frac{1}{2} \tan^{-1} x \right) = \frac{-1 \pm \sqrt{1 + x^2}}{\{x^2 + (1 + \sqrt{1 + x^2})^2\}^{1/2}} L_1 \quad (3-63) \]

and

\[ L_1 = \sqrt{x^2 + (1 + \sqrt{1 + x^2})^2} \]

3.4 CONCLUSIONS:

For the propagation of an unstable mode \( \omega_i \neq 0 \).

Therefore, equation (3-62) yields

\[ \omega_i = \frac{M_3 \pm \lambda^{1/2} (-1 \pm \sqrt{1 + x^2}) \sqrt{L_1}}{M_1} \]

where

\[ L_1 = \sqrt{x^2 + (1 + \sqrt{1 + x^2})^2} > 0 \]
The following are therefore the possible values of \( \omega_i \):

\[
\omega_1 = \frac{M_3 + [\lambda^4 (-1 + \sqrt{1 + x^2})]/L_1}{M_1},
\]

\[
\omega_2 = \frac{M_3 + [\lambda^4 (-1 - \sqrt{1 + x^2})]/L_1}{M_1},
\]

\[
\omega_3 = \frac{M_3 - [\lambda^4 (-1 + \sqrt{1 + x^2})]/L_1}{M_1},
\]

\[
\omega_4 = \frac{M_3 - [\lambda^4 (-1 - \sqrt{1 + x^2})]/L_1}{M_1}.
\]

From equation (3-54), it is clear that

\[
M_3 < 0.
\]

Therefore:

(i) Propagation will be unstable, oscillatory, or stable according as

\[
|M_3| \geq \frac{\lambda^4 (-1 + \sqrt{1 + x^2})}{L_1}.
\]

(ii) Since \( \omega_2 \) and \( \omega_3 \) are always \( < 0 \), the mode propagating in these cases shall be unstable.
REFERENCES


