AMENABILITY AND WEAK AMENABILITY OF BANACH ALGEBRAS

BY

YONG ZHANG

A Thesis
Submitted to the Faculty of Graduate Studies
In Partial Fulfillment of the Requirements
For the Degree of

Doctor of Philosophy

Department of Mathematics
University of Manitoba
Winnipeg, Manitoba

©April 1999
The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L’auteur conserve la propriété du droit d’auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

0-612-41635-6
AMENABILITY AND WEAK AMENABILITY OF BANACH ALGEBRAS

BY

YONG ZHANG

A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree of

Doctor of Philosophy

Yong Zhang©1999

Permission has been granted to the Library of The University of Manitoba to lend or sell copies of this thesis/practicum, to the National Library of Canada to microfilm this thesis and to lend or sell copies of the film, and to Dissertations Abstracts International to publish an abstract of this thesis/practicum.

The author reserves other publication rights, and neither this thesis/practicum nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.
ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor, Professor F. Ghahramani, who was always there for me – with his guidance, caring, support and encouragement. His extensive mathematical knowledge and research insight benefited me greatly.

I wish to thank Professors W. G. Bade, H. G. Dales, N. Gronbaek and G. Willis for their interest in my work and for their inspiration.

I would also like to thank Professors P. McClure and J. Williams for carefully reading the thesis and for their valuable advice.

My special thanks go to my dear wife Hong Wang. Her understanding confirmed in me my decision to pursue Mathematics.

My special thanks also go to my daughter Hannah Zhang, whose smiles always brought me out of my frustration and exhaustion.

Finally, I would like to extend my thanks to the University of Manitoba and the Department of Mathematics for their generous financial support.
ABSTRACT

We start with the investigation of (2m)-th conjugate algebras of Banach algebras (equipped with Arens products) and related higher order dual Banach modules. In the process we explore important properties of the higher order dual operators of module morphisms and of derivations. After this basic investigation we consider various problems in or related to the theory of amenability and weak amenability of Banach algebras. The work is described by the following three themes:

Theme 1. Weak Amenability of Banach Algebras

We first discuss the conditions under which n-weak amenability implies (n+2)-weak amenability for n>0, and the conditions for the unitization of a Banach algebra to be n-weakly amenable. After an extensive study of module extensions of Banach algebras, we give a necessary and sufficient condition for a Banach algebra of this kind to be n-weakly amenable, separately for odd numbers n and even numbers n. Then we discuss several concrete cases, especially those related to operator algebras. These finally offer us a way to construct a counter-example to the open question of whether weak amenability implies 3-weak amenability.

Theme 2. Structure of Contractible Banach Algebras and Amenable Banach Algebras with the Underlying Spaces Reflexive

By exploiting properties of minimal idempotents we show that a Banach algebra of this kind is finite dimensional if every maximal ideal of it is contained in a maximal left ideal which is complemented as a subspace. This result improves several known results concerning the subject.

Theme 3. Nilpotent Ideals in a Class of Banach Algebras

We introduce the concept of approximately complemented subspaces in a Banach space. Then we discuss the relation between this notion and some well-known ones like approximation property and weakcomplementability. The category of this notion includes many interesting cases. Tensor products of subspaces of this kind have some natural properties. We prove that in a class of Banach algebras called approximately biprojective Banach algebras, nilpotent ideals, if any, lack certain geometric properties such as being approximately complemented.
Contents

Chapter 1. Introduction 1

1.1. Preliminaries 1
1.2. Outline of the content 8

Chapter 2. Dual Modules 10

2.1. Arens products 10
2.2. Bimodule actions of \( \mathfrak{A}^{(2m)} \) on \( X^{(2m)} \) 11

Chapter 3. The \( n \)-Weak Amenability of Banach Algebras 16

3.1. \((n + 2)\)-Weak amenability and \( n \)-weak amenability 16
3.2. The \( n \)-weak amenability of \( \mathfrak{A}^t \) 21

Chapter 4. The \( n \)-Weak Amenability of Module Extensions of Banach Algebras 28

4.1. Lifting derivations 30
4.2. On \( n \)-weak amenability of \( \mathfrak{A} \oplus X \) 40
4.3. The algebra \( \mathfrak{A} \oplus \mathfrak{A} \) 52
4.4. The algebra \( \mathfrak{A} \oplus X_0 \) 61

Chapter 5. Weak Amenability does not imply 3-Weak Amenability 69

5.1. Weak amenability of the algebra \( \mathfrak{A} \oplus (X_1 \oplus X_2) \) 69
## CONTENTS

5.2. Weak amenability is not 3-weak amenability 72

Chapter 6. Contractible and Reflexive Amenable Banach Algebras 77

6.1. Approximation property and contractible Banach algebras 77

6.2. Maximal ideals and dimensions 81

Chapter 7. Approximate Complementability and Nilpotent Ideals 91

7.1. Approximately Complemented Subspaces 92

7.2. Approximately Biprojective Banach Algebras 100

7.3. Nilpotent ideals 101

Bibliography 106
CHAPTER 1

Introduction

1.1. Preliminaries

All Banach spaces and algebras we deal with in this thesis are assumed to be over $\mathbb{C}$, the complex field. Suppose that $\mathfrak{A}$ is a Banach algebra. A Banach space $X$ is said to be a Banach left $\mathfrak{A}$-module if a bilinear mapping: $(a, x) \mapsto ax$ (called the left module multiplication) of $\mathfrak{A} \times X$ into $X$ is defined which satisfies, for some number $k > 0$, the following axioms.

(i) $a(bx) = (ab)x$, $a, b \in \mathfrak{A}, x \in X$;

(ii) $\|ax\| \leq k\|a\|\|x\|$, $a \in \mathfrak{A}, x \in X$.

Banach right $\mathfrak{A}$-modules are defined similarly. $X$ is said to be a Banach $\mathfrak{A}$-bimodule if it is both a Banach left $\mathfrak{A}$-module and a Banach right $\mathfrak{A}$-module and the module multiplications are related by

(iii) $a(xb) = (ax)b$, $a, b \in \mathfrak{A}, x \in X$;

Sometimes we use $a \cdot x$ instead of $ax$ to denote the module multiplication to avoid any possible confusion with other products involved. For any Banach algebra $\mathfrak{A}$, $\mathfrak{A}$ itself is a Banach $\mathfrak{A}$-bimodule with the product of $\mathfrak{A}$ giving the module multiplications. If $X$ is a Banach left (right) $\mathfrak{A}$-module, then $X^*$, the conjugate space of $X$, is a Banach right (resp. left) $\mathfrak{A}$-module with the
natural module multiplications defined by

\[(1.1) \quad \langle x, fa \rangle = \langle ax, f \rangle \quad \text{(resp.} \quad \langle x, af \rangle = \langle xa, f \rangle \text{)},\]

for \( f \in X^*, \ a \in \mathcal{A} \) and \( x \in X \). We call this module the dual right (resp. left) module of \( X \). Here and throughout this thesis, for \( x \in X \) and \( f \in X^* \), \( \langle x, f \rangle \) denotes the value \( f(x) \). If \( X \) is a Banach \( \mathcal{A} \)-bimodule, then multiplications given by Equation (1.1) make \( X^* \) into a Banach \( \mathcal{A} \)-bimodule, called the dual module of \( X \). The dual (left, right) module of \( X^* \) is called the second dual (resp. left, right) module of \( X \). In this thesis we use \( \sigma(X^*, X) \) to denote the weak* topology on the conjugate space \( X^* \). A limit under this topology is called weak* limit, briefly, wk*-lim. Suppose that \( \phi \in X^{**} \), Then by a theorem of Goldstine [23, Theorem V.4.5], there exists a net \( (x_\alpha) \subset X \) such that wk*-lim \( x_\alpha = \phi \). Then for each \( a \in \mathcal{A} \) it is easy to see that

\[ a\phi = \text{wk*-lim} ax_\alpha \quad (\phi a = \text{wk*-lim} x_\alpha a). \]

The \( n \)th dual (left, right) module of \( X \) can be defined by induction; it will always be denoted by \( X^{(n)} \).

Suppose that \( X \) is a Banach left (right) \( \mathcal{A} \)-module. A net \( (e_i) \subset \mathcal{A} \) is called a \emph{left (resp. right) approximate identity} for \( X \) if \( \lim_i e_i x = x \) (resp. \( \lim_i xe_i = x \)) in norm for all \( x \in X \). If in addition, \( (e_i) \) is also a bounded net, then \( (e_i) \) is a \emph{left (resp. right) bounded approximate identity}, briefly denoted by left (resp. right) b.a.i.. If \( X \) is a Banach \( \mathcal{A} \)-bimodule and \( (e_i) \) is both a left and a right (bounded) approximate identity for \( X \), then \( (e_i) \)
1.1. PRELIMINARIES

is called a \textit{(bounded) approximate identity} for \(X\). When \(X = \mathcal{A}\), a (left, right, bounded) approximate identity for \(X\) will be called a (resp. left, right, bounded) approximate identity of \(\mathcal{A}\).

The following factorization theorem due to P. J. Cohen (see [4, Theorem 11.10] or [43, Theorem 32.22]) will be often used:

\textbf{THEOREM 1.1.} Suppose that \(X\) is a Banach left \(\mathcal{A}\)-module and \(\mathcal{A}\) has a left b.a.i. for \(X\). Then for any \(x \in X\) and \(\delta > 0\) there exist \(a \in \mathcal{A}\), and \(y \in X\) such that \(x = ay\) and \(\|x - y\| \leq \delta\).

A special kind of operator which plays an important role in this thesis is a derivation, defined below:

\textbf{DEFINITION 1.2.} Suppose that \(\mathcal{A}\) is a Banach algebra and \(X\) is a Banach \(\mathcal{A}\)-bimodule. A (continuous) \textit{derivation} from \(\mathcal{A}\) into \(X\) is a (continuous) linear mapping \(D: \mathcal{A} \rightarrow X\) which satisfies

\[D(ab) = a \cdot D(b) + D(a) \cdot b, \quad (a, b \in \mathcal{A}).\]

For any \(x \in X\), the mapping \(\delta_x: \mathcal{A} \rightarrow X\) given by

\[\delta_x(a) = ax - xa, \quad (a \in \mathcal{A})\]

is a continuous derivation, called an \textit{inner derivation}. Denote by \(\mathcal{Z}^1(\mathcal{A}, X)\) the space of all continuous derivations from \(\mathcal{A}\) into \(X\) and by \(\mathcal{N}^1(\mathcal{A}, X)\) the space of all inner derivations from \(\mathcal{A}\) into \(X\). Then \(\mathcal{N}^1(\mathcal{A}, X)\) is a subspace
1.1. PRELIMINARIES

of $\mathcal{Z}^1(\mathfrak{A}, X)$. The quotient space $\mathcal{H}^1(\mathfrak{A}, X) = \mathcal{Z}^1(\mathfrak{A}, X)/\mathcal{N}^1(\mathfrak{A}, X)$ is called the first cohomology group of $\mathfrak{A}$ with coefficients in $X$. All the amenability theories are concerned with the question of whether $\mathcal{H}^1(\mathfrak{A}, X) = \{0\}$ for specifically chosen classes of Banach $\mathfrak{A}$-modules.

A Banach algebra $\mathfrak{A}$ is said to be contractible if $\mathcal{H}^1(\mathfrak{A}, X) = \{0\}$ for all Banach $\mathfrak{A}$-bimodules $X$, amenable if $\mathcal{H}^1(\mathfrak{A}, X^*) = \{0\}$ for all Banach $\mathfrak{A}$-bimodules $X$, and weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$. A contractible Banach algebra has an identity and an amenable Banach algebra has a b.a.i. ([16] and [49]).

A contractible Banach algebra is finite dimensional if it has the bounded approximation property [67] or if it is commutative [16]. We do not know that a contractible Banach algebra is always finite dimensional or not. We refer to [39, VII § 1.4–1.5] as well as [67] and [65] for properties of contractibility.

Amenability theory of Banach algebras was originated by B. E. Johnson in [49], where he proved that a locally compact group $G$ is amenable in the classical sense (see [19] and [27]) if and only if the Banach algebra $L^1(G)$ with the convolution product is amenable. Other approaches to this concept have been discussed in [47] and [39, Chapter VII]. For second dual algebras of group algebras we know that $L^1(G)^{**}$ is amenable if and only if $G$ is a finite group [26]. Another remarkable result is that a C*-algebra is amenable
1.1. Preliminaries

If and only if it is nuclear, proved by U. Haagerup and A. Connes in [36] and [12].

Suppose that $M$ is a subset of a Banach space $X$. We denote by $M^\perp$ the annihilator of $M$ in $X^*$, i.e.

$$M^\perp = \{ f \in X^* ; f|_M = 0 \}$$

We will use the following result in this thesis.

**Theorem 1.3** ([16], Theorem 3.7). Suppose that $\mathcal{A}$ is an amenable Banach algebra. Let $J \neq 0$ be a closed left, right or two-sided ideal in $\mathcal{A}$. Then $J$ has a right, left, or two-sided b.a.i. if and only if $J^\perp$ is complemented in $\mathcal{A}^*$ (i.e., $J$ is weakly complemented; see page 96 for the definition of weak complementability.).

It has been conjectured that every reflexive, amenable Banach algebra is finite dimensional. In 1992 J. E. Gale, T. J. Ransford and M. C. White proved that this is true if irreducible representations of $\mathcal{A}$ are finite dimensional [25]. This result was much improved in [44] by Johnson, who showed that this is the case if each maximal left ideal of $\mathcal{A}$ is complemented. In 1996 F. Ghahramani, R. J. Loy and G. A. Willis proved that this is true if the underlying space of $\mathcal{A}$ is a Hilbert space [26]. Although this result is covered by Johnson’s preceding result, the method is totally different and is of special interest. Recently, V. Runde [61] proved a new result: if each maximal ideal
of $\mathfrak{A}$ is finite codimensional then the conjecture is true. In Chapter 6, we will prove a theorem which improves both Johnson’s and Runde’s results.

W. G. Bade, P. C. Curtis and H. G. Dales first introduced the concept of weak amenability in [2] for commutative Banach algebras. The original definition used there is that a commutative Banach algebra $\mathfrak{A}$ is weakly amenable if $\mathcal{H}^1(\mathfrak{A}, X) = \{0\}$ for all symmetric Banach $\mathfrak{A}$-bimodules $X$. They proved that this is actually equivalent to $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$. Using this equivalence, Johnson called a Banach algebra (not necessarily commutative) weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$ [46].

In the rest of this section, we will mention several Banach algebras of concrete types. Their definitions can be found in the mentioned references. Since they are not the objects of this thesis, we will not give formal definitions for them.

It is known that the list of weakly amenable Banach algebras includes $L^1(G)$ [45] [20], C*-algebras [36], the tensor algebra $E \otimes E^*$ of any Banach space $E$ [17], and some Lipschitz algebras [2]. By using a result of [9] Ghahramani, Loy and Willis proved in [26] that both $M(G)$ and $L^1(G)^{**}$ fail to be weakly amenable for all infinite, nondiscrete abelian locally compact groups $G$. Other related results can be found in [52]. Many interesting properties of weak amenability have also been investigated by N. Grønbæk in [30]–[33].
1.1. PRELIMINARIES

In [17], Dales, Ghahramani and Grønbæk introduced the concept of \( n \)-weak amenability as stated below.

**DEFINITION 1.4.** A Banach algebra \( \mathfrak{A} \) is \( n \)-weakly amenable, \( n \geq 0 \), if \( \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\} \), and is permanently weakly amenable if it is \( n \)-weakly amenable for all \( n \geq 1 \).

It is pointed out in [17] that \((n + 2)\)-weak amenability implies \( n \)-weak amenability for \( n \geq 1 \), and if \( \mathfrak{A} \) is an ideal in \( \mathfrak{A}^{**} \) then weak amenability of \( \mathfrak{A} \) implies \((2n + 1)\)-weak amenability. A variety of concrete algebras have been investigated there: \( L^1(G) \) is \((2n + 1)\)-weakly amenable for all \( n \) (later Johnson proved that \( l^1(\mathbb{F}_2) \) is permanently weakly amenable [48]); \( C^* \)-algebras are permanently weakly amenable; for each infinite compact metric space \( K \), \( \text{lip}_\alpha K \) is permanently weakly amenable if \( \alpha < 1/2 \), and is \( 2m \)-weakly amenable if \( 0 < \alpha < 1 \), but \( \text{lip}_\alpha \mathbb{T} \) is not weakly amenable for \( \alpha > 1/2 \); \( \mathcal{N}(E) \), the algebra of nuclear operators on \( E \) with \( E \) a reflexive Banach space having the approximation property, is \((2n + 1)\)-weakly amenable but not \( 2n \)-weakly amenable. An open question raised there is as follows.

**QUESTION 1.5.** Does weak amenability imply \( 3 \)-weak amenability?

After an extensive study of the Banach algebra \( \mathfrak{A} \oplus X \) in Chapter 4, we will answer this question in Chapter 5; the answer is negative.

Since there exists a permanently weakly amenable Banach algebra of the form \( \mathfrak{A} \oplus X \) (see Remark 4.36), a permanently weakly amenable Banach
algebra may have a nilpotent ideal that is complemented. This is not the case in amenable Banach algebras. In fact, from Theorem 1.3 one can see that non-zero nilpotent ideals in amenable Banach algebras, if any, can not be weakly complemented. It is known that there do exist amenable Banach algebras that contain non-zero nilpotent ideals [68]. Loy and Willis in [54] proved that for biprojective Banach algebras with a central approximate identity any non-zero nilpotent ideal can not have the approximation property. Here a Banach algebra \( \mathfrak{A} \) is biprojective if the continuous bimodule morphism \( \pi: \mathfrak{A} \hat{\otimes} \mathfrak{A} \to \mathfrak{A} \), specified by \( \pi(a \otimes b) = ab \), has a continuous bimodule morphism right inverse \( T: \mathfrak{A} \to \mathfrak{A} \hat{\otimes} \mathfrak{A} \), i.e. \( \pi \circ T = I_\mathfrak{A} \), the identity operator on \( \mathfrak{A} \), where \( \mathfrak{A} \hat{\otimes} \mathfrak{A} \) denotes the projective tensor product. One can see [64] for further properties of biprojective Banach algebras. A contractible Banach algebra is in fact a biprojective Banach algebra with an identity ([39, VII §1.4]). We will introduce the concept of approximate complementability for subspaces of Banach spaces in Chapter 7 and improve Loy’s and Willis’ above result by showing that non-zero nilpotent ideals in a biprojective Banach algebra with both left and right approximate identities can not be approximately complemented.

1.2. Outline of the content

In Chapter 2, we construct, for any Banach algebra \( \mathfrak{A} \) and \( \mathfrak{A} \)-bimodule \( X \), the \( \mathfrak{A}^{(2m)} \)-bimodule actions on \( X^{(2m)} \) for \( m \geq 1 \), assuming \( \mathfrak{A}^{(2m)} \) being the \( 2m \)th dual algebra of \( \mathfrak{A} \) equipped with the Arens first product. This
discussion sets a cornerstone for the three Chapters following it, especially Chapter 4 and 5. Chapter 3 concerns the general theory about \( n \)-weak amenability of Banach algebras. We will improve or extend some crucial results of [17]. In Chapter 4 we discuss the module extensions of Banach algebras, give a necessary and sufficient condition for them to be \( n \)-weakly amenable, and consider several important special cases. Concrete examples will also be discussed there illustrating the relations between the weak amenability of different orders. In the chapter following it we are constructing an example of a weakly amenable Banach algebra which is not 3-weakly amenable, answering the open question mentioned in the preceding section.

Chapter 6 deals with reflexive amenable Banach algebras. We will prove a theorem which improves Johnson's theorem in [44] and Runde's theorem in [61]. In the last Chapter, Chapter 7, we consider nilpotent ideals. We will give the concept of approximate complementability and discuss the relations between it and other well-known concepts such as weak complementability and the approximation property. Then we will consider approximately biprojective Banach algebras, showing that this kind of algebra can not have a non-trivial nilpotent ideal that is approximately complemented.
CHAPTER 2

Dual Modules

2.1. Arens products

Suppose that $\mathfrak{A}$ is a Banach algebra. In [1] Arens defined two Banach algebra products on $\mathfrak{A}^{**}$, the second conjugate space of $\mathfrak{A}$, each extending the product of $\mathfrak{A}$ as canonically embedded in $\mathfrak{A}^{**}$. The first (or left) Arens product $(\Phi, \Psi) \mapsto \Phi \Psi: \mathfrak{A}^{**} \times \mathfrak{A}^{**} \to \mathfrak{A}^{**}$ is given by

$$\langle f, \Phi \Psi \rangle = \langle \Psi f, \Phi \rangle, \quad (f \in \mathfrak{A}^{**})$$

where $\Psi f \in \mathfrak{A}^{*}$ is defined by

$$\langle a, \Psi f \rangle = \langle fa, \Psi \rangle, \quad (a \in \mathfrak{A}).$$

The second (or right) Arens product $\Phi \cdot \Psi$ on $\mathfrak{A}^{**}$ is given by

$$\langle f, \Phi \cdot \Psi \rangle = \langle f \cdot \Phi, \Psi \rangle, \quad (f \in \mathfrak{A}^{*})$$

where $f \cdot \Phi \in \mathfrak{A}^{*}$ is defined by

$$\langle a, f \cdot \Phi \rangle = \langle af, \Phi \rangle, \quad (a \in \mathfrak{A}).$$

These products can be different for some $\mathfrak{A}$. If $\Phi \Psi = \Phi \cdot \Psi$ for all $\Phi, \Psi \in \mathfrak{A}^{**}$, then $\mathfrak{A}$ is called Arens regular. $L^1(G)$ is Arens regular only
when $G$ is a finite group [70], while a C*-algebra is always Arens regular [11]. See the survey article [22] for further properties of Arens products.

Viewing $\mathfrak{A}^{(2m+2)}$ as the second dual of $\mathfrak{A}^{(2m)}$, $m \geq 0$, we can define Arens products in $\mathfrak{A}^{(2n)}$ for each $n \geq 1$ inductively. In this thesis we always take the first Arens product for all the $2n$th dual algebras $\mathfrak{A}^{(2n)}$. If $\mu, \nu \in \mathfrak{A}^{(2n+2)}$ such that $\mu = \text{wk}^*\text{-lim} u_\alpha$, $\nu = \text{wk}^*\text{-lim} v_\beta$ for nets $(u_\alpha), (v_\beta) \subset \mathfrak{A}^{(2n)}$, then we have the following:

$$\mu \nu = \text{wk}^*\text{-lim}_{\alpha, \beta} u_\alpha v_\beta.$$

### 2.2. Bimodule actions of $\mathfrak{A}^{(2m)}$ on $X^{(2m)}$

Suppose that $\mathfrak{A}$ is a Banach algebra, and $X$ is a Banach $\mathfrak{A}$-bimodule. Then according to [17, pp. 27-28], $X^{**}$ is a Banach $\mathfrak{A}^{**}$-bimodule. The module actions are successively defined as follows:

First, for $x \in X$, $f \in X^*$, $\phi \in X^{**}$ and $u \in \mathfrak{A}^{**}$ define $\phi f$, $fx \in \mathfrak{A}^*$ and $uf \in X^*$ by

$$\langle a, \phi f \rangle = \langle fa, \phi \rangle, \quad \langle a, fx \rangle = \langle xa, f \rangle \quad (a \in \mathfrak{A}),$$

$$\langle x, uf \rangle = \langle fx, u \rangle \quad (x \in X).$$

Then for $\phi \in X^{**}$, $u \in \mathfrak{A}^{**}$, define $u\phi$, $\phi u \in X^{**}$ by

$$\langle f, u\phi \rangle = \langle \phi f, u \rangle, \quad \langle f, \phi u \rangle = \langle uf, \phi \rangle \quad (f \in X^*).$$

They give the bimodule actions of $\mathfrak{A}^{**}$ on $X^{**}$. Also the definition for $uf$ with $u \in \mathfrak{A}^{**}$ and $f \in X^*$ gives a left Banach module action of $\mathfrak{A}^{**}$ on $X^*$. 
2.2. BIMODULE ACTIONS OF $\mathfrak{A}^{(2m)}$ ON $X^{(2m)}$

When $u = a \in \mathfrak{A}$, these module actions agree with the module actions of $\mathfrak{A}$ on the corresponding dual modules $X^*$ and $X^{**}$.

Viewing $\mathfrak{A}^{(2m)}$ as a new $\mathfrak{A}$ and $X^{(2m)}$ as a new $X$, the preceding procedure will successively define $X^{(2m+2)}$ to be a Banach $\mathfrak{A}^{(2m+2)}$-bimodule for $m \geq 0$. So $X^{(2m)}$ is naturally a Banach $\mathfrak{A}^{(2m)}$-bimodule for all non-negative integers $m$. Since some relations arising from this procedure are important for later use, we now give the definition in detail as follows.

Suppose that the bimodule actions of $\mathfrak{A}^{(2m)}$ on $X^{(2m)}$ have been defined, where $m \geq 0$. Then in a natural way, $X^{(2m+k)}$, $k \geq 1$, is a Banach $\mathfrak{A}^{(2m)}$-bimodule with the following module actions:

For $\Lambda \in X^{(2m+k)}$ and $u \in \mathfrak{A}^{(2m)}$, $u\Lambda$, $\Lambda u \in X^{(2m+k)}$ are defined by

\[
\langle \gamma, u\Lambda \rangle = \langle u\gamma, \Lambda \rangle, \quad \langle \gamma, \Lambda u \rangle = \langle u\gamma, \Lambda \rangle \quad (\gamma \in X^{(2m+k-1)}).
\]

If $u = a \in \mathfrak{A}$, these module actions agree with $\mathfrak{A}$-bimodule actions on $X^{(2m+k)}$.

Then define, for $F \in X^{(2m+1)}$ and $\Phi \in X^{(2m+2)}$, $F\Phi$, $\Phi F \in \mathfrak{A}^{(2m+1)}$ by

\[
\langle u, F\Phi \rangle = \langle F, \Phi u \rangle (= \langle uF, \Phi \rangle)
\]

and

\[
\langle u, \Phi F \rangle = \langle Fu, \Phi \rangle (= \langle F, u\Phi \rangle), \quad (u \in \mathfrak{A}^{(2m)}).
\]
Throughout this thesis, for any Banach space $Y$ and an element $y \in Y$, $\hat{y}$ always denotes the canonical image of $y$ in $Y^{**}$ (but to avoid unnecessary complicated notations, we often use the same notation $y$ to represent its canonical image in any $2m$th dual space $Y^{(2m)}$). When $F \in X^{(2m+1)}$ and $\phi \in X^{(2m)}$, we denote $F\hat{\phi}$ by $F\phi$ and $\hat{\phi}F$ by $\phi F$. It is easy to check that

$$\langle u, F\phi \rangle = \langle \phi u, F \rangle, \quad \langle u, \phi F \rangle = \langle u\phi, F \rangle \quad \text{for } u \in \mathcal{X}^{(2m)}.$$  

(2.1)

By using the canonical image of $F$ or $\Phi$ in the appropriate $2l$th dual space, we can then signify a meaning to $F\Phi$ and $\Phi F$ for any $F \in X^{(2m+1)}$ and $\Phi \in X^{(2n)}$. They are elements of $\mathcal{X}^{(2k+1)}$, where $k = \max\{m, n - 1\}$.

Now for $\mu \in \mathcal{X}^{(2m+2)}$ and $F \in X^{(2m+1)}$, define $\mu F \in X^{(2m+1)}$ by

$$\langle \phi, \mu F \rangle = \langle F\phi, \mu \rangle, \quad (\phi \in X^{(2m)}).$$

This actually defines a left Banach $\mathcal{X}^{(2m+2)}$-module action on $X^{(2m+1)}$.

Finally, for $\mu \in \mathcal{X}^{(2m+2)}$ and $\Phi \in X^{(2m+2)}$, define $\mu\Phi, \Phi\mu \in X^{(2m+2)}$ by

$$\langle F, \mu\Phi \rangle = \langle \Phi F, \mu \rangle, \quad \langle F, \Phi\mu \rangle = \langle \mu F, \Phi \rangle \quad (F \in X^{(2m+1)}).$$

They give the $\mathcal{X}^{(2m+2)}$-bimodule actions on $X^{(2m+2)}$ and complete our definition.

If $\lim u_\alpha = \mu$ in $\sigma(\mathcal{X}^{(2m+2)}, \mathcal{X}^{(2m+1)})$ and $\lim \phi_\beta = \Phi$ in $\sigma(X^{(2m+2)}, X^{(2m+1)})$, where $(u_\alpha) \subset \mathcal{X}^{(2m)}$ and $(\phi_\beta) \subset X^{(2m)}$, then

$$\mu\Phi = \lim_{\alpha, \beta} u_\alpha \phi_\beta, \quad \Phi\mu = \lim_{\beta} \lim_{\alpha} \phi_\beta u_\alpha \quad \text{in } \sigma(X^{(2m+2)}, X^{(2m+1)}).$$
For $\mu \in \mathfrak{A}^{(2m+2)}$ and $\phi \in X^{(2m)}$, since $\mu\phi = \mu\hat{\phi}$, $\phi\mu = \hat{\phi}\mu$ we have

\begin{equation}
\langle F, \mu\phi \rangle = \langle \phi F, \mu \rangle, \quad \langle F, \phi\mu \rangle = \langle F\phi, \mu \rangle \quad (F \in X^{(2m+1)}).
\end{equation}

One can also easily check the following relations:

\[
\begin{align*}
    u\hat{f} &= \hat{u}\hat{f} = (uf)^\sim, \\
    \hat{f}\hat{\phi} &= (f\phi)^\sim, & \hat{\Phi}\hat{f} &= (\Phi f)^\sim, \\
    \hat{u}\hat{\Phi} &= (u\Phi)^\sim, & \hat{\Phi}\hat{u} &= (\Phi u)^\sim,
\end{align*}
\]

where $f \in X^{(2m-1)}$, $\phi \in X^{(2m-2)}$, $\Phi \in X^{(2m)}$ and $u \in \mathfrak{A}^{(2m)}$ ($m \geq 1$).

Therefore each product agrees with those previously defined.

Concerning dual module morphisms we have the following.

**Proposition 2.1.** Suppose that $X$ and $Y$ are Banach $\mathfrak{A}$-bimodules. Then for each integer $m \geq 1$ and each continuous $\mathfrak{A}$-bimodule morphism $\tau: X \to Y$, $\tau^{(2m)}: X^{(2m)} \to Y^{(2m)}$, the $2m$th dual operator of $\tau$, is an $\mathfrak{A}^{(2m)}$-bimodule morphism.

**Proof.** It suffices to prove the proposition in the case $m = 1$. Suppose that $\Phi \in X^{**}$, $u \in \mathfrak{A}^{**}$, and

\[
a_\alpha \xrightarrow{\text{wk}^*} u, \quad x_\beta \xrightarrow{\text{wk}^*} \Phi,
\]
2.2. BIMODULE ACTIONS OF $\mathfrak{A}^{(2m)}$ ON $X^{(2m)}$

where $(a_\alpha) \subset \mathfrak{A}$, $(x_\beta) \subset X$. Since $\tau^{**}$ is weak* continuous, we have

\[
\tau^{**}(\Phi u) = \tau^{**}(\text{wk}^*\lim_\alpha \hat{x}_\beta \hat{a}_\alpha) = \text{wk}^*\lim_\alpha \tau^{**}(\hat{x}_\beta \hat{a}_\alpha)
\]

\[
= \text{wk}^*\lim_\beta \tau (x_\beta a_\alpha)^* = \text{wk}^*\lim_\beta \tau (x_\beta)^* \hat{a}_\alpha
\]

\[
= \text{wk}^*\lim_\beta \tau^{**}(\hat{x}_\beta)^* \hat{a}_\alpha = \tau^{**}(\Phi) u.
\]

Similar calculations show the identity for the left action. \hfill \square

Concerning derivations we have the following proposition which is an extension of [17, Proposition 1.7].

**Proposition 2.2.** Suppose that $\mathfrak{A}$ is a Banach algebra and $X$ is a Banach $\mathfrak{A}$-bimodule. If $D: \mathfrak{A} \to X$ is a continuous derivation, then for $n \geq 0$

$D^{(2n)}: \mathfrak{A}^{(2n)} \to X^{(2n)}$, the 2nth dual operator of $D$, is also a continuous derivation.

**Proof.** For $n = 1$, the proof is similar to the proof of the preceding proposition, and by induction on $n$ the proof can be easily completed. \hfill \square
CHAPTER 3

The \( n \)-Weak Amenability of Banach Algebras

The theory of \( n \)-weak amenability of Banach algebras was established by H. G. Dales, F. Ghahramani and N. Grønbæk in [17]. It extends the theory of weak amenability. In Chapter 1 we already gave the definition and narrated the backgroud of this concept.

3.1. \((n + 2)\)-Weak amenability and \( n \)-weak amenability

In this thesis we use the notation \( + \) for the direct sum of Banach spaces, or the direct sum of modules, while we save the notation \( \oplus \) for module extentions of Banach algebras.

Suppose that \( D: \mathcal{A} \rightarrow \mathcal{A}^{(n)} \) is a continuous derivation, \( n \geq 1 \). Since

\[
\mathcal{A}^{(n+2)} = \mathcal{A}^{(n)} \oplus \mathcal{A}^{(n-1)\perp}
\]

is an \( \mathcal{A} \)-bimodule decomposition, \( D \) can be viewed as a continuous derivation from \( \mathcal{A} \) into \( \mathcal{A}^{(n+2)} \). If \( \mathcal{A} \) is \((n + 2)\)-weakly amenable then \( D \) is inner in \( \mathcal{A}^{(n+2)} \).

By considering the \( \mathcal{A} \)-bimodule projection from \( \mathcal{A}^{(n+2)} \) onto \( \mathcal{A}^{(n)} \) it follows that \( D \) is inner in \( \mathcal{A}^{(n)} \), showing that \( \mathcal{A} \) is \( n \)-weakly amenable. This was observed by Dales, Ghahramani and Grønbæk. So we have:
Proposition 3.1 ([17], Proposition 1.2). Suppose that $\mathcal{A}$ is an $(n + 2)$-weakly amenable Banach algebra, $n \geq 1$. Then $\mathcal{A}$ is $n$-weakly amenable.

From this proposition any $(2m + 1)$-weakly amenable Banach algebra must be weakly amenable. It is of course interesting to know if the converse is also true. This question was raised and left open in [17]. We will give a negative answer to it in Chapter 5. Here we are interested in conditions that can guarantee the converse. Before this discussion we give an important property of weak amenability, which is taken from [17] (see also [32] for the commutative algebra version). We include the proof for completeness.

Proposition 3.2 ([17], Proposition 1.3). Suppose that $\mathcal{A}$ is a weakly amenable Banach algebra. Then $\mathcal{A}^2 = \text{span}\{ab; a, b \in \mathcal{A}\}$ is dense in $\mathcal{A}$.

Proof. Let $\lambda \in \mathcal{A}^*$ satisfy $\lambda|_{\mathcal{A}^2} = 0$. Let $D: \mathcal{A} \to \mathcal{A}^*$ be the bounded operator given by

$$D(a) = \langle a, \lambda \rangle \lambda, \quad a \in \mathcal{A}.$$ 

Then $D$ is a derivation and hence is inner. There is an $f \in \mathcal{A}^*$ such that

$$\langle a, \lambda \rangle \langle b, \lambda \rangle = \langle b, af - fa, \rangle, \quad a, b \in \mathcal{A}.$$ 

Taking $b = a$, we have $\langle a, \lambda \rangle^2 = 0$ for all $a \in \mathcal{A}$. So $\lambda$ must be 0, which implies that $\mathcal{A}^2$ is dense in $\mathcal{A}$ by the Hahn-Banach Theorem. \qed
LEMMA 3.3. Suppose that \( \mathfrak{A} \) is a left (right) ideal in \( \mathfrak{A}^{**} \). Then it is also a left (resp. right) ideal in \( \mathfrak{A}^{(2m)} \) for all \( m \geq 1 \).

PROOF. We show that if \( \mathfrak{A} \) is a left (right) ideal of \( \mathfrak{A}^{(2m)} \), \( m \geq 1 \), then it is a left (resp. right) ideal of \( \mathfrak{A}^{(2m+2)} \). Then by using induction we will have the conclusion.

In fact,

\[
(3.1) \quad \mathfrak{A}^{(2m+2)} = (\mathfrak{A}^*) \cap (\mathfrak{A}^{**})^{\perp}
\]

and

\[
(3.2) \quad \mathfrak{A}^{(2m+1)} = (\mathfrak{A}) \cap (\mathfrak{A}^*),
\]

as \( \mathfrak{A} \)-bimodules. For any \( F \in \mathfrak{A}^{(2m+1)} \), write \( F = f_1 + f_2 \), where \( f_1 \in \mathfrak{A}^{\perp} \) and \( f_2 \in \mathfrak{A}^* \). If \( \mathfrak{A} \) is a left (right) ideal of \( \mathfrak{A}^{(2m)} \), then \( af_1 = 0 \) (resp. \( f_1 a = 0 \)) for \( a \in \mathfrak{A} \). So

\[
aF = a\hat{f}_2 = (af_2)^\perp \quad (\text{resp. } Fa = \hat{f}_2 a = (f_2 a)^\perp).
\]

For any \( \Phi \in \mathfrak{A}^{(2m+2)} \), let \( \Phi = \phi + \hat{u} \), where \( \phi \in (\mathfrak{A}^*)^{\perp} \) and \( u \in \mathfrak{A}^{**} \). Then

\[
\langle F, \Phi a \rangle = \langle (af_2)^\perp, \hat{u} \rangle = \langle F, (ua)^\perp \rangle
\]

(\text{resp. } \langle F, a\Phi \rangle = \langle F, (au)^\perp \rangle).

So \( \Phi a = (ua)^\perp \in \widehat{\mathfrak{A}} \) (resp. \( a\Phi = (au)^\perp \in \widehat{\mathfrak{A}} \)) for \( a \in \mathfrak{A} \). Thus \( \mathfrak{A} \) is a left (resp. right) ideal of \( \mathfrak{A}^{(2m+2)} \). The proof is complete. \( \square \)
**Remark 3.4.** Suppose that $\mathcal{B}$ is a Banach algebra and $\mathfrak{A} = \mathcal{B}^{**}$. If $\mathcal{B}$ is an ideal in $\mathcal{B}^{**}$, then the natural way to make $\mathcal{B}$ an $\mathfrak{A}$-bimodule is using the Arens product to give the module actions; but, when we go to higher duals, the distinction between algebra products and module actions makes superficial sense. For instance, since $\mathcal{B}$ is an $\mathfrak{A}$-bimodule, $\mathcal{B}^{**}$ is an $\mathfrak{A}^{**}$-bimodule; for $b \in \mathcal{B} \subset \mathcal{B}^{**}$ and $u \in \mathfrak{A}^{**}$, the module actions $u \cdot b$, $b \cdot u$ make sense and give elements of $\mathcal{B}^{**}$. Since $\mathcal{B} \subset \mathcal{B}^{**} \subset \mathcal{B}^{(4)}$, $ub$, $bu$ also make sense and are elements in $\mathcal{B}^{(4)}$ thanks to the Arens product in $\mathcal{B}^{(4)}$. But from the above lemma, $ub, bu \in \mathcal{B} \subset \mathcal{B}^{**}$. It is routine to check that as elements in $\mathcal{B}^{**}$, $u \cdot b = ub$ and $b \cdot u = bu$. This fact will be used later in Chapter 5.

**Theorem 3.5 ([17], Proposition 1.13).** Suppose that $\mathfrak{A}$ is a weakly amenable Banach algebra and is an ideal in $\mathfrak{A}^{**}$. Then $\mathfrak{A}$ is $(2m+1)$-weakly amenable for all $m \geq 1$.

**Proof.** Since $\langle u, aF \rangle = \langle ua, F \rangle$ and $\langle u, Fa \rangle = \langle au, F \rangle$ for $u \in \mathfrak{A}^{(2m)}$, $a \in \mathfrak{A}$ and $F \in \mathfrak{A}^{(2m+1)}$, from Lemma 3.3, $aF = Fa = 0$ for all $a \in \mathfrak{A}$ and $F \in \mathfrak{A}^\perp \cap \mathfrak{A}^{(2m+1)}$. Suppose that $D: \mathfrak{A} \to \mathfrak{A}^{(2m+1)}$ is a continuous derivation and $P: \mathfrak{A}^{(2m+1)} \to \mathfrak{A}^\perp$ is the $\mathfrak{A}$-module projection with the kernel $\mathfrak{A}^*$ (see equation (3.2)). Then

$$P \circ D(ab) = aP \circ D(b) + P \circ D(a)b = 0 \quad (a, b \in \mathfrak{A}).$$
This shows that $P \circ D = 0$ from Proposition 3.2. Therefore $D$ is in fact a derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$. So $D$ is inner from the assumption. It follows that $\mathfrak{A}$ is $(2m + 1)$-weakly amenable.

\[ \square \]

**Lemma 3.6.** Suppose that $\mathfrak{A}$ is a Banach algebra with a left (right) b.a.i.. Suppose that $X$ is a Banach $\mathfrak{A}$-bimodule and $Y$ is a weak* closed submodule of the dual module $X^*$. If the left (resp. right) $\mathfrak{A}$-module action on $Y$ is trivial, then $\mathcal{H}^1(\mathfrak{A}, Y) = \{0\}$.

**Proof.** We prove the result only in the case $\mathfrak{A}$ has a left b.a.i.. The proof for the other case is similar. Suppose that $D: \mathfrak{A} \to Y$ is a continuous derivation. Let $(e_i)$ be a left b.a.i. of $\mathfrak{A}$, and $f \in Y$ be a weak* cluster point of $(D(e_i))$. Since $2\mathfrak{A}Y = \{0\}$, we have

\[ D(a) = \lim D(e_i a) = fa = fa - af, \quad a \in \mathfrak{A}. \]

Hence $D$ is inner, showing that $\mathcal{H}^1(\mathfrak{A}, Y) = \{0\}$. \[ \square \]

When $Y = X^*$ and the right (left) $\mathfrak{A}$-module action on $X$ is trivial, the above lemma gives the following result due to Johnson (we indicate that in Johnson's original result the condition of the existence of a b.a.i. is unnecessary and can be replaced by an appropriate one-sided b.a.i.).
3.2. The n-weak amenability of $\mathbb{A}$

Corollary 3.7 ([49], Proposition 1.5). Suppose that $\mathbb{A}$ is a Banach algebra with a left (right) b.a.i. and $X$ is a Banach $\mathbb{A}$-bimodule with the right (resp. left) module action trivial. Then $\mathcal{H}^1(\mathbb{A}, X^*) = \{0\}$.

With Lemma 3.6, we can prove another partial converse result to Proposition 3.1 as follows.

Theorem 3.8. Suppose that $\mathbb{A}$ is a weakly amenable Banach algebra. If $\mathbb{A}$ has a left (right) b.a.i. and is a left (resp. right) ideal in $\mathbb{A}^{**}$, then $\mathbb{A}$ is $(2m + 1)$-weakly amenable for $m \geq 1$.

Proof. We prove the result only in the case that $\mathbb{A}$ has a left b.a.i. and is a left ideal in $\mathbb{A}^{**}$. From the $\mathbb{A}$-bimodule decomposition (3.2) we have the cohomology group decomposition

$$\mathcal{H}^1(\mathbb{A}, \mathbb{A}^{(2m+1)}) = \mathcal{H}^1(\mathbb{A}, \mathbb{A}^*) \oplus \mathcal{H}^1(\mathbb{A}, \mathbb{A}^\perp).$$

If $\mathbb{A}$ is weakly amenable, we have $\mathcal{H}^1(\mathbb{A}, \mathbb{A}^*) = \{0\}$. $\mathbb{A}^\perp$ is clearly weak* closed submodule of $\mathbb{A}^{(2m+1)}$. Since $\mathbb{A}$ is a left ideal in $\mathbb{A}^{**}$, it is a left ideal in $\mathbb{A}^{(2m)}$ from Lemma 3.3. It follows that the left $\mathbb{A}$-module action on $\mathbb{A}^\perp$ is trivial. Then Lemma 3.6 leads to that $\mathcal{H}^1(\mathbb{A}, \mathbb{A}^\perp) = \{0\}$. As a consequence we have $\mathcal{H}^1(\mathbb{A}, \mathbb{A}^{(2m+1)}) = \{0\}$ and so $\mathbb{A}$ is $(2m + 1)$-weakly amenable. \(\square\)

3.2. The n-weak amenability of $\mathbb{A}$

Suppose that $\mathbb{A}$ is a Banach algebra. Let $\mathbb{A}^\ell$ be the unitization of $\mathbb{A}$, the Banach algebra formed by adjoining an identity $e$ to $\mathbb{A}$, so that $\mathbb{A}^\ell = \mathbb{A} \oplus Ce$
3.2. THE $n$-WEAK AMENABILITY OF $\mathfrak{A}$

with the product given by

$$(a + \alpha e)(b + \beta e) = ab + \beta a + \alpha b + \alpha \beta e \quad (a, b \in \mathfrak{A}, \alpha, \beta \in \mathbb{C}).$$

Here we use $\oplus$ instead of $\oplus$ to denote the direct sum, because the latter will have special meaning in this thesis. In this section we are interested in finding the relations between the weak amenability of $\mathfrak{A}$ and $\mathfrak{A}^f$. If $\mathfrak{A}$ is commutative, then $\mathfrak{A}^f$ is $n$-weakly amenable if and only if $\mathfrak{A}$ is $n$-weakly amenable ([17]). In the general case things seem complicated. It is known that if $\mathfrak{A}$ is $(2n + 1)$-weakly amenable then so is $\mathfrak{A}^f$, and if $\mathfrak{A}^f$ is $2n$-weakly amenable then so is $\mathfrak{A}$ ([17]). The converses remain open [28].

It is easy to see that

$$\mathfrak{A}^{(2n+1)} = \mathfrak{A}^{(2n+1)} \oplus \mathbb{C}e^*;$$

$$\mathfrak{A}^{(2n)} = \mathfrak{A}^{(2n)} \oplus \mathbb{C}e,$$

where $n \geq 0$, and $e^* \in \mathfrak{A}^{*}$ satisfies $e^*|_{\mathfrak{A}} = 0$ and $\langle e, e^* \rangle = 1$. The first Arens product on $\mathfrak{A}^{(2n)}$ is given by

$$(u + \alpha e)(v + \beta e) = uv + \beta u + \alpha v + \alpha \beta e \quad (u, v \in \mathfrak{A}^{(2n)}, \alpha, \beta \in \mathbb{C}),$$

and the module operations of $\mathfrak{A}^{(2n)}$ on $\mathfrak{A}^{(2m+1)}$ for $m \geq n \geq 0$ are given by

$$(u + \alpha e) \cdot (F + \beta e^*) = (uF + \alpha F) + ((u, F) + \alpha \beta)e^*,$$

$$(F + \beta e^*) \cdot (u + \alpha e) = (Fu + \alpha F) + ((u, F) + \alpha \beta)e^*,$$
for $u \in A^{(2n)}$, $F \in A^{(2m+1)}$, and $\alpha, \beta \in \mathbb{C}$. Especially the $A^*$-bimodule actions on $A^{(2m+1)}$ are given by the formulas:

\begin{align*}
(a + \alpha e) \cdot (F + \beta e^*) &= (aF + \alpha F) + (\langle a, F \rangle + \alpha \beta) e^*, \\
(F + \beta e^*) \cdot (a + \alpha e) &= (Fa + \alpha F) + (\langle a, F \rangle + \alpha \beta) e^*,
\end{align*}

for $a \in A$, $F \in A^{(2m+1)}$, and $\alpha, \beta \in \mathbb{C}$.

**Proposition 3.9.** For $m \geq 0$, $A^*$ is $(2m + 1)$-weakly amenable if and only if $A^2$ is dense in $A$ and every bounded derivation $D: A \to A^{(2m+1)}$, with the condition that there is a $T \in A^*$ such that

\begin{equation}
\langle ab, T \rangle = \langle a, D(b) \rangle + \langle b, D(a) \rangle \quad (a, b \in A),
\end{equation}

is inner. If this is the case, then $T = 0$.

**Proof.** For necessity, assume $A^*$ is $(2m + 1)$-weakly amenable. Let $f \in A^*$ be such that $f|_{A^2} = 0$, and let $\Delta: A^* \to A^{(2m+1)}$ be defined by

$$
\Delta(a + \alpha e) = \langle a, f \rangle e^* \quad (a \in A, \alpha \in \mathbb{C}).
$$

Then $\Delta$ is a continuous derivation. So $\Delta$ is inner. A simple calculation by using formulas (3.3) and (3.4) shows that $f = 0$. Therefore $A^2$ is dense in $A$. Suppose that $D: A \to A^{(2n+1)}$ is a bounded derivation satisfying (3.5). Then $\overline{D}: A^* \to A^{(2m+1)}$ defined by

$$
\overline{D}(a + \alpha e) = D(a) + \langle a, T \rangle e^*, \quad a \in A, \alpha \in \mathbb{C},
$$

is weakly amenable.
is a continuous derivation. In fact,

\[
\overline{D}((a + \alpha e)(b + \beta e)) = D(ab + \alpha b + \beta a) + \langle ab + \alpha b + \beta a, T \rangle e^* \\
= (D(a) + \langle a, T \rangle e^*) \cdot (b + \beta e) + (a + \alpha e) \cdot (D(b) + \langle b, T \rangle e^*) \\
= \overline{D}(a + \alpha e) \cdot (b + \beta e) + (a + \alpha e) \cdot \overline{D}(b + \beta e).
\]

So \(\overline{D}\) is inner. It then follows that \(D\) is inner and \(T = 0\).

For the sufficiency, let \(\Delta: \mathfrak{A}^d \to \mathfrak{A}^f(2m+1)\) be a continuous derivation. Since \(\Delta(e) = 0\) we can assume

\[
\Delta(a + \alpha e) = D(a) + \langle a, T \rangle e^*, \quad a \in \mathfrak{A}, \alpha \in \mathbb{C},
\]

where \(D: \mathfrak{A} \to \mathfrak{A}^{(2m+1)}\) is a bounded operator and \(T \in \mathfrak{A}^*.\) We have from formulas (3.3) and (3.4)

\[
D(ab) + \langle ab, T \rangle e^* = \Delta(ab) = a \cdot \Delta(\hat{o}) + \Delta(a) \cdot b \\
= aD(b) + D(a)b + \langle (a, D(b)) + \langle b, D(a) \rangle \rangle e^* \quad (a, b \in \mathfrak{A}).
\]

This shows that \(D\) is a derivation from \(\mathfrak{A}\) into \(\mathfrak{A}^{(2m+1)}\) and \(T\) satisfies (3.5).

Thus \(D\) is inner, which in turn implies that \(\langle ab, T \rangle = 0\) for all \(a, b \in \mathfrak{A}\). So \(T = 0\). It follows that \(\Delta\) is inner. So \(\mathfrak{A}^d\) is \((2m + 1)\)-weakly amenable. The proof is complete. \(\square\)

**Corollary 3.10.** Suppose that \(\mathfrak{A}\) has a b.a.i.. Then for \(m \geq 0\) the algebra \(\mathfrak{A}^d\) is \((2m + 1)\)-weakly amenable if and only if \(\mathfrak{A}\) is \((2m + 1)\)-weakly amenable.
3.2. THE $n$-WEAK AMENABILITY OF $\mathfrak{A}$

**Proof.** Let $(e_i)$ be a b.a.i. of $\mathfrak{A}$, and let $E \in \mathfrak{A}^{(2m+2)}$ be a weak* cluster point of $(e_i)$. For any continuous derivation $D: \mathfrak{A} \to \mathfrak{A}^{(2m+1)}$, define $T \in \mathfrak{A}^*$ by

$$\langle a, T \rangle = \langle D(a), E \rangle, \quad a \in \mathfrak{A}.$$ 

Then

$$\langle ab, T \rangle = \lim \langle e_i a, D(b) \rangle + \langle b e_i, D(a) \rangle = \langle a, D(b) \rangle + \langle b, D(a) \rangle, \quad a, b \in \mathfrak{A}.$$ 

Therefore Equation (3.5) holds for any continuous derivation $D: \mathfrak{A} \to \mathfrak{A}^{(2m+1)}$. The rest is clear. □

**Remark** 3.11. For $m = 0$ Corollary 3.10 was obtained by Grønbæk in [30].

**Corollary 3.12.** Suppose that $\mathfrak{A}$ is a weakly amenable Banach algebra. Then $\mathfrak{A}^f$ is $(2m + 1)$-weakly amenable if and only if $\mathfrak{A}$ is $(2m + 1)$-weakly amenable.

**Proof.** We only need to prove the necessity. Suppose that $D: \mathfrak{A} \to \mathfrak{A}^{(2m+1)}$ is a continuous derivation. Let $P: \mathfrak{A}^{(2m+1)} \to \mathfrak{A}^*$ be the projection with the kernel $\mathfrak{A}^\perp$. Then $P \circ D: \mathfrak{A} \to \mathfrak{A}^*$ is an inner derivation. On the other hand, the bounded derivation $(I - P) \circ D: \mathfrak{A} \to \mathfrak{A}^\perp$ satisfies Equation (3.5) with $T = 0$. From Proposition 3.9, $(I - P) \circ D$ is inner. This shows that $D$ is inner. So $\mathfrak{A}$ is $(2m + 1)$-weakly amenable. □
It was shown in [17, Proposition 1.4] that if $\mathfrak{U}^\sharp$ is $2m$-weakly amenable, $m \geq 0$, then $\mathfrak{A}$ is $2m$-weakly amenable. For the converse we have the following:

**Proposition 3.13.** Suppose that $\mathfrak{A}$ is $2m$-weakly amenable, $m \geq 0$, and $\mathfrak{U}^2$ is dense in $\mathfrak{A}$. Then $\mathfrak{U}^\sharp$ is $2m$-weakly amenable.

**Proof.** Let $\Delta: \mathfrak{U}^\sharp \rightarrow \mathfrak{U}^{2(2m)}$ be a continuous derivation. Then there is $f \in \mathfrak{A}^*$ and a bounded operator $D: \mathfrak{A} \rightarrow \mathfrak{A}^{(2m)}$ such that

$$\Delta(a + \alpha e) = D(a) + \langle a, f \rangle e, \quad a \in \mathfrak{A}, \alpha \in \mathbb{C}.$$  

Routine calculations lead to the following equalities:

$$D(ab) = aD(b) + D(a)b + \langle b, f \rangle a + \langle a, f \rangle b,$$

$$\langle ab, f \rangle = 0, \quad a, b \in \mathfrak{A}.$$

The last equality shows that $f = 0$ since $\mathfrak{U}^2$ is dense in $\mathfrak{A}$. It follows that $D$ is a derivation and hence it is inner, and so there is $\phi \in \mathfrak{A}^{(2m)}$ such that

$$\Delta(a + \alpha e) = a\phi - \phi a = (a + \alpha e) \cdot \phi - \phi \cdot (a + \alpha e), \quad a \in \mathfrak{A}, \alpha \in \mathbb{C}.$$  

Thus $\Delta$ is inner and consequently $\mathfrak{U}^\sharp$ is $2m$-weakly amenable. \hfill $\Box$

From Cohen factorization theorem (Theorem 1.1) and using the above proposition, we have immediately the following.

**Corollary 3.14.** If $\mathfrak{A}$ has a b.a.i., then $\mathfrak{U}^\sharp$ is $2m$-weakly amenable if and only if $\mathfrak{A}$ is $2m$-weakly amenable.
From Proposition 3.2, Proposition 3.13 and [17, Proposition 1.4] we obtain also the following:

**Corollary 3.15.** Suppose that $\mathfrak{A}$ is weakly amenable. Then $\mathfrak{A}^n$ is $2m$-weakly amenable if and only if $\mathfrak{A}$ is $2m$-weakly amenable.

To end this Chapter we combine corollaries 3.10, 3.12, 3.14 and 3.15 to a theorem as follows.

**Theorem 3.16.** Suppose that $\mathfrak{A}$ is a Banach algebra which is either weakly amenable or has a b.a.i.. Then for each $n \geq 0$, $\mathfrak{A}^n$ is $n$-weakly amenable if and only if $\mathfrak{A}$ is $n$-weakly amenable.
CHAPTER 4

The $n$-Weak Amenability of Module Extensions of Banach Algebras

Suppose that $\mathfrak{A}$ is a Banach algebra and $X$ is a Banach $\mathfrak{A}$-bimodule. The module extension algebra $\mathfrak{A} \oplus X$ is the $l_1$-direct sum of $\mathfrak{A}$ and $X$, with the algebra product defined as follows:

$$(a, x) \cdot (b, y) = (ab, ay + xb), \quad (a, b \in \mathfrak{A}, x, y \in X).$$

It is a Banach algebra. Some properties of this kind of algebra have been discussed in [3] and [17]. In this chapter we study the $n$-weak amenability of this kind of Banach algebra. Since $X$ is a complemented nilpotent ideal of $\mathfrak{A} \oplus X$, according to Theorem 1.3, $\mathfrak{A} \oplus X$ can never be an amenable Banach algebra unless $X = 0$. If $\mathfrak{A} \oplus X$ has both left and right approximate identities (for instance, when $\mathfrak{A}$ has both left and right approximate identities and they are also, respectively, left and right approximate identities for $X$), then $\mathfrak{A} \oplus X$ can not be pointwise approximately biprojective (see Definition 7.10 and Theorem 7.14). But $\mathfrak{A} \oplus X$ can be weakly amenable. It may even be permanently weakly amenable. In section 4.4 we will see that if $\mathfrak{A}$ is a weakly amenable commutative Banach algebra with a b.a.i., then $\mathfrak{A} \oplus \mathcal{A}_0$ is
permanently weakly amenable, where $A_0$ is identical to $\mathfrak{A}$ as a left $\mathfrak{A}$-module while the right module action on $A_0$ is trivial.

This chapter is organized as follows: In Section 4.1 we first discuss the $(\mathfrak{A} \oplus X)$-module $(\mathfrak{A} \oplus X)^{(n)}$, giving the formulas for the module actions. Then we will discuss various techniques for lifting derivations. In Section 4.2 we prove two theorems which give the necessary and sufficient conditions for $(\mathfrak{A} \oplus X)$ to be $n$-weakly amenable. Sections 4.3 and 4.4 deal with the special cases of $X = \mathfrak{A}$ and $X = X_0$, where $X_0$ denotes a Banach $\mathfrak{A}$-bimodule with the right module action trivial. We will encounter several interesting examples there. This chapter is also a foundation for Chapter 5, where we construct a counter-example to answer question 1.5 in Chapter 1 in the negative.

Sometimes we regard $\mathfrak{A} \oplus X$ as an $\mathfrak{A}$-bimodule with the natural module actions: $b \cdot (a, x) = (ba, bx)$ and $(a, x) \cdot b = (ab, xb)$ for $b \in \mathfrak{A}$, $(a, x) \in \mathfrak{A} \oplus X$. Since the dual module $(\mathfrak{A} \oplus X)^*$ is identical with $(0+X)^\perp+(\mathfrak{A}+0)^\perp$, where the sum is $l_\infty$-sum, and $(0+X)^\perp$ (resp. $(\mathfrak{A}+0)^\perp$) is isometrically isomorphic to $\mathfrak{A}^*$ (resp. $X^*$) as $\mathfrak{A}$-bimodules, for convenience in this thesis we just write

$$(\mathfrak{A} \oplus X)^* = \mathfrak{A}^* + X^*.$$ 

Similarly we will identify the $n$th dual $(\mathfrak{A} \oplus X)^{(n)}$ with $\mathfrak{A}^{(n)} + X^{(n)}$, where the sum is $l_\infty$-sum when $n$ is odd, and is $l_1$-sum when $n$ is even. For the
moment this identity is in the sense of \( \mathcal{A} \)-bimodules. We will study the \((\mathcal{A} \oplus X)\)-bimodule actions in the next section.

\section*{4.1. Lifting derivations}

In this section we give several lemmas concerning lifting derivations (resp. module morphisms) from \( \mathcal{A} \) (resp. \( X \)) into the modules \( \mathcal{A}^{(n)} \) or \( X^{(n)} \) to derivations from \( \mathcal{A} \oplus X \) into \( (\mathcal{A} \oplus X)^{(n)} \). We first discuss the \((\mathcal{A} \oplus X)\)-module actions on the \( n \)th dual module \( (\mathcal{A} \oplus X)^{(n)} \). Recall that, when \( n \) is an odd number, the notations \( xx^{(n)} \) and \( x^{(n)}x \) have been defined in Chapter 2 on page 12 for \( x \in X \).

\textbf{Lemma 4.1.} Suppose that \( X \) is a Banach \( \mathcal{A} \)-bimodule. Then the \((\mathcal{A} \oplus X)^{(n)}\)-bimodule actions on \( (\mathcal{A} \oplus X)^{(n)} \) are given by the following formulas

\begin{equation}
(a, x) \cdot (a^{(n)}, x^{(n)}) = \begin{cases} 
(aa^{(n)} + xx^{(n)}, ax^{(n)}), & \text{if } n \text{ is odd;} \\
(aa^{(n)}, ax^{(n)} + xa^{(n)}), & \text{if } n \text{ is even},
\end{cases}
\end{equation}

\begin{equation}
(a^{(n)}, x^{(n)}) \cdot (a, x) = \begin{cases} 
(a^{(n)}a + x^{(n)}x, x^{(n)}a), & \text{if } n \text{ is odd;} \\
(a^{(n)}a, a^{(n)}x + x^{(n)}a), & \text{if } n \text{ is even},
\end{cases}
\end{equation}

where \((a, x) \in \mathcal{A} \oplus X\), \((a^{(n)}, x^{(n)}) \in \mathcal{A}^{(n)} \oplus X^{(n)} = (\mathcal{A} \oplus X)^{(n)}\).

\textbf{Proof.} We use induction to give the proof. For \( n = 0 \) the equalities are just the definition of the algebra multiplications for \( \mathcal{A} \oplus X \). Assume that
they are true for \( n = 2m, m \geq 0 \). Then for \((a^{(2m)}, x^{(2m)}) \in (\mathfrak{A} \oplus X)^{(2m)}, (a^{(2m+1)}, x^{(2m+1)}) \in (\mathfrak{A} \oplus X)^{(2m+1)},\)

\[
\langle \langle a^{(2m)}, x^{(2m)} \rangle, (a, x) \cdot \langle a^{(2m+1)}, x^{(2m+1)} \rangle \rangle \\
= \langle \langle a^{(2m)}, x^{(2m)} \rangle \cdot (a, x), \langle a^{(2m+1)}, x^{(2m+1)} \rangle \rangle \\
= \langle \langle a^{(2m)} a, a^{(2m)} x + x^{(2m)} a \rangle, \langle a^{(2m+1)}, x^{(2m+1)} \rangle \rangle \\
= \langle a^{(2m)} a, a^{(2m+1)} \rangle + \langle a^{(2m)} x + x^{(2m)} a, x^{(2m+1)} \rangle \\
= \langle \langle a^{(2m)}, x^{(2m)} \rangle, (aa^{(2m+1)} + xa^{(2m+1)}), ax^{(2m+1)} \rangle,\]

and

\[
\langle \langle a^{(2m)}, x^{(2m)} \rangle, (a^{(2m+1)}, x^{(2m+1)}) \cdot (a, x) \rangle \\
= \langle \langle a, x \rangle \cdot \langle a^{(2m)}, x^{(2m)} \rangle, \langle a^{(2m+1)}, x^{(2m+1)} \rangle \rangle \\
= \langle \langle aa^{(2m)} , xa^{(2m)} + ax^{(2m)} \rangle, \langle a^{(2m+1)}, x^{(2m+1)} \rangle \rangle \\
= \langle \langle a^{(2m)}, x^{(2m)} \rangle, (a^{(2m+1)} a + x^{(2m+1)} a), x^{(2m+1)} a \rangle).\]

These show that the equalities hold for \( n = 2m + 1 \).
4.1. LIFTING DERIVATIONS

Now assume that they are true for \( n = 2m - 1, \ m \geq 1 \). Then for

\[
(a^{(2m-1)}, x^{(2m-1)}) \in (\mathfrak{A} \oplus X)^{(2m-1)} \text{ and } (a^{(2m)}, x^{(2m)}) \in (\mathfrak{A} \oplus X)^{(2m)},
\]

\[
\langle (a^{(2m-1)}, x^{(2m-1)}), (a, x) \cdot (a^{(2m)}, x^{(2m)}) \rangle
\]

\[
= \langle (a^{(2m-1)}, x^{(2m-1)}), (a, x), (a^{(2m)}, x^{(2m)}) \rangle
\]

\[
= \langle (a^{(2m-1)}a + x^{(2m-1)}a, x^{(2m-1)}a), (a^{(2m)}, x^{(2m)}) \rangle
\]

\[
= \langle a^{(2m-1)}, a^{(2m)} \rangle + \langle x^{(2m-1)}, xa^{(2m)} + ax^{(2m)} \rangle
\]

\[
= \langle (a^{(2m-1)}, x^{(2m-1)}), (aa^{(2m)}, xa^{(2m)} + ax^{(2m)}) \rangle,
\]

and

\[
\langle (a^{(2m-1)}, x^{(2m-1)}), (a^{(2m)}, x^{(2m)}) \cdot (a, x) \rangle
\]

\[
= \langle (a, x) \cdot (a^{(2m-1)}, x^{(2m-1)}), (a^{(2m)}, x^{(2m)}) \rangle
\]

\[
= \langle (aa^{(2m-1)} + xx^{(2m-1)}, ax^{(2m-1)}), (a^{(2m)}, x^{(2m)}) \rangle
\]

\[
= \langle (a^{(2m-1)}, x^{(2m-1)}), (a^{(2m)}a, ax^{(2m-1)} + x^{(2m-1)}a) \rangle.
\]

So the equalities hold also for \( n = 2m \).

By induction, we see that the equalities (4.1) and (4.2) hold for any integer \( n \geq 0 \). \( \square \)

We have known that for each integer \( m, X^{(2m)} \) is a Banach \( \mathfrak{A}^{(2m)} \)-bimodule. So the module extension Banach algebra \( \mathfrak{A}^{(2m)} \oplus X^{(2m)} \) makes sense. Also the \( 2m \)th dual algebra \( (\mathfrak{A} \oplus X)^{(2m)} \) has the same underlying space as \( \mathfrak{A}^{(2m)} \oplus X^{(2m)} \).
The distinction between these two algebras is in fact superficial as stated in the following remark.

**Remark 4.2.** The algebra \( \mathfrak{A}^{(2m)} \oplus X^{(2m)} \) is isometrically isomorphic to \((\mathfrak{A} \oplus X)^{(2m)}\).

**Proof.** Both have the underlying space \( A^{(2m)} \oplus X^{(2m)} \). Using induction, one sees easily that the algebra products defined on them are also the same.

Now we can give our lifting results.

**Lemma 4.3.** Suppose that \( m \geq 0 \) and \( D: \mathfrak{A} \rightarrow \mathfrak{A}^{(2m+1)} \) is a (continuous) derivation. Then \( \overline{D}: \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m+1)} \) defined by

\[
\overline{D}((a, x)) = (D(a), 0)
\]

is also a (continuous) derivation. \( D \) is inner if and only if \( \overline{D} \) is inner.

**Proof.** It is routine to check that \( \overline{D} \) is a (continuous) derivation if \( D \) is so.

If \( D \) is inner, then there is \( u \in \mathfrak{A}^{(2m+1)} \) such that \( D(a) = au - ua \). This leads to

\[
\overline{D}((a, x)) = (a, x) \cdot (u, 0) - (u, 0) \cdot (a, x),
\]

and so \( \overline{D} \) is inner.
4.1. LIFTING DERIVATIONS

If \( \overline{D} \) is inner, then there is \( (u, F) \in \mathcal{A}^{(2m+1)} \oplus X^{(2m+1)} \), such that

\[
(D(a), 0) = \overline{D}((a, 0)) = (a, 0) \cdot (u, F) - (u, F) \cdot (a, 0)
\]

\[
= (au - ua, aF - Fa)
\]

and so \( D(a) = au - ua \), for all \( a \in \mathcal{A} \). This shows that \( D \) is inner. \( \square \)

**Lemma 4.4.** Suppose that \( D: \mathcal{A} \to X^{(2m)} \) is a (continuous) derivation. Then \( \overline{D}: \mathcal{A} \oplus X \to (\mathcal{A} \oplus X)^{(2m)} \) defined by

\[
\overline{D}((a, x)) = (0, D(a))
\]

is also a (continuous) derivation. \( D \) is inner if and only if \( \overline{D} \) is.

**Proof.** The proof is similar to that of Lemma 4.3. \( \square \)

**Lemma 4.5.** Suppose that \( n \geq 0 \) and \( D: \mathcal{A} \to X^{(n)} \) is a (continuous) derivation. Then \( D^*: X^{(n+1)} \to \mathcal{A}^* \), the dual operator of \( D \), satisfies

\[
D^*(aF) = aD^*(F) - (D(a)F)_{|\mathcal{A}},
\]

\[
D^*(Fa) = D^*(F)a - (FD(a))_{|\mathcal{A}},
\]

for \( a \in \mathcal{A} \) and \( F \in X^{(n+1)} \).

**Proof.** For any \( b \in \mathcal{A} \),

\[
\langle b, D^*(aF) \rangle = \langle D(b)a, F \rangle = \langle D(ba) - bD(a), F \rangle
\]

\[
= \langle b, aD^*(F) - D(a)F \rangle,
\]
and so $D^\ast(aF) = aD^\ast(F) - (D(a)F)|_a$. A similar argument will give the second equality.

**Lemma 4.6.** Suppose that $k > 0$ is an integer and $D: \mathfrak{A} \rightarrow X^{(k)}$ is a (continuous) derivation. Then for $m \geq 0$, $D^{(2m+1)}: X^{(k+2m+1)} \rightarrow \mathfrak{A}^{(2m+1)}$, the $(2m+1)$th dual operator of $D$, satisfies

$$D^{(2m+1)}(aF) = aD^{(2m+1)}(F) - (D(a)F)|_{\mathfrak{A}^{(2m)}},$$

$$D^{(2m+1)}(Fa) = D^{(2m+1)}(F)a - (FD(a))|_{\mathfrak{A}^{(2m)}},$$

for $a \in \mathfrak{A}$ and $F \in X^{(k+2m+1)}$.

**Proof.** First, for $m = 0$, Lemma 4.5 gives the result. For $m > 0$, from [17, Proposition 1.7], $D^{(2m)}: \mathfrak{A}^{(2m)} \rightarrow X^{(k+2m)}$ is a (continuous) derivation. Then from Lemma 4.5,

$$D^{(2m+1)} = (D^{(2m)})^\ast : X^{(k+2m+1)} \rightarrow \mathfrak{A}^{(2m+1)}$$

satisfies

$$D^{(2m+1)}(uF) = uD^{(2m+1)}(F) - (D^{(2m)}(u)F)|_{\mathfrak{A}^{(2m)}},$$

and

$$D^{(2m+1)}(Fu) = D^{(2m+1)}(F)u - (FD^{(2m)}(u))|_{\mathfrak{A}^{(2m)}},$$

for $u \in \mathfrak{A}^{(2m)}$, $F \in X^{(k+2m+1)}$. In particular when $u = a \in \mathfrak{A}$, these give the formulas of the lemma. □
4.1. LIFTING DERIVATIONS

LEMMA 4.7. For any integer $2m + 1 > 0$, suppose that $D: \mathfrak{A} \to X^{(2m+1)}$ is a (continuous) derivation. Then $\overline{D}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ defined by

$$\overline{D}((a, x)) = (-D^{(2m+1)}(x), D(a)) \text{ for } (a, x) \in \mathfrak{A} \oplus X,$$

is a (continuous) derivation. Moreover,

1. if $\overline{D}$ is inner, then so is $D$;
2. if $D$ is inner, then a (continuous) derivation $\overline{D}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$

    can be set so that $\overline{D}((a, 0)) = 0$ for $a \in \mathfrak{A}$, and $\overline{D} - \overline{D}$ is inner.

PROOF. For $a, b \in \mathfrak{A}$ and $x, y \in X$, from Lemma 4.6 we have

$$\overline{D}((a, x) \cdot (b, y)) = \overline{D}((ab, ay + xb)) = \left( -D^{(2m+1)}(ay + xb), D(ab) \right)$$

$$= \left( -[aD^{(2m+1)}(y) - (D(a)y)|_{\mathfrak{A}^{(2m)}} + D^{(2m+1)}(x)b - (xD(b))|_{\mathfrak{A}^{(2m)}}], D(a)b + aD(b) \right)$$

$$= \left( -[aD^{(2m+1)}(y) - D(a)y + D^{(2m+1)}(x)b - xD(b)], D(a)b + aD(b) \right)$$

$$= \left( -aD^{(2m+1)}(y) + xD(b), aD(b) \right) + \left( -D^{(2m+1)}(x)b + D(a)y, D(a)b \right)$$

$$= (a, x) \cdot \left( -D^{(2m+1)}(y), D(b) \right) + \left( -D^{(2m+1)}(x), D(a) \right) \cdot (b, y)$$

$$= (a, x) \cdot \overline{D}((b, y)) + \overline{D}((a, x)) \cdot (b, y).$$

Therefore $\overline{D}$ is a (continuous) derivation.

If $\overline{D}$ is inner, then for some $u \in \mathfrak{A}^{(2m+1)}$, $F \in X^{(2m+1)}$,

$$\overline{D}((a, x)) = (a, x) \cdot (u, F) - (u, F) \cdot (a, x).$$
Then
\[(0, D(a)) = \overline{D}((a, 0)) = (a, 0) \cdot (u, F) - (u, F) \cdot (a, 0)\]
\[= (au - ua, aF - Fa).\]

So \(D(a) = aF - Fa\) for all \(a \in \mathfrak{A}\). Thus \(D\) is inner.

Conversely, if \(D\) is inner, then there is \(F \in X^{(2m+1)}\) such that \(D(a) = aF - Fa\)
for \(a \in \mathfrak{A}\). Let \(T: X \to \mathfrak{A}^{(2m+1)}\) be defined by
\[T(x) = -D^{(2m+1)}(x) - (xF - Fx) \quad (x \in X)\]
and \(\overline{T}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}\) be defined by
\[\overline{T}((a, x)) = (T(x), 0), \quad ((a, x) \in \mathfrak{A} \oplus X).\]

Then
\[(\overline{D} - \overline{T})((a, x)) = (xF - Fx, aF - Fa)\]
\[= (a, x) \cdot (0, F) - (0, F) \cdot (a, x)\]
for \((a, x) \in \mathfrak{A} \oplus X\). Therefore \(\overline{D} - \overline{T}\) is an inner derivation. This in turn
implies that \(\overline{T}\) is a (continuous) derivation. So \(\overline{D} = \overline{T}\) will satisfy all the
requirements. The proof is complete. \(\square\)

**Lemma 4.8.** Suppose that \(\Gamma: X \to \mathfrak{A}^{(2m+1)}\) is a continuous \(\mathfrak{A}\)-bimodule
morphism. Then \(D: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}\) defined by
\[D((a, x)) = (\Gamma(x), 0)\]
is a continuous derivation. $D$ is inner if and only if there is $F \in X^{(2m+1)}$ such that $aF - Fa = 0$ for $a \in \mathcal{A}$, and $\Gamma(x) = xF - Fx$.

**Proof.** It is easy to check that $D$ is a continuous derivation. If $D$ is inner, then there is $(u, F) \in \mathcal{A}^{(2m+1)} \oplus X^{(2m+1)}$, such that

$$(\Gamma(x), 0) = D((0, x)) = (0, x) \cdot (u, F) - (u, F) \cdot (0, x)$$

$$= (xF - Fx, 0).$$

This shows that $\Gamma(x) = xF - Fx$ for $x \in X$. Also

$$(0, 0) = D((a, 0)) = (a, 0) \cdot (u, F) - (u, F) \cdot (a, 0)$$

$$= (au - ua, aF - Fa).$$

We have $aF - Fa = 0$ for $a \in \mathcal{A}$.

Conversely, if there is $F \in X^{(2m+1)}$ such that $aF - Fa = 0$ for $a \in \mathcal{A}$, and $\Gamma(x) = xF - Fx$, then

$$D((a, x)) = (\Gamma(x), 0) = (xF - Fx, aF - Fa)$$

$$= (a, x) \cdot (0, F) - (0, F) \cdot (a, x),$$

showing that $D$ is inner.

**Lemma 4.9.** Suppose that $T: X \to X^{(2m+1)}$ is a continuous $\mathcal{A}$-bimodule morphism, satisfying $xT(y) + T(x)y = 0$ for all $x, y \in X$. Then

$$D: \mathcal{A} \oplus X \to (\mathcal{A} \oplus X)^{(2m+1)}$$
4.1. LIFTING DERIVATIONS

defined by

\[ D((a, x)) = (0, T(x)) \quad (a \in \mathcal{A}, x \in X) \]

is a continuous derivation. \( D \) is inner if and only if \( T = 0 \).

**Proof.** The first statement is easy to check. If \( D \) is inner, then for some \((u, F) \in \mathcal{A}^{(2m+1)} \oplus X^{(2m+1)}, \)

\[ (0, T(x)) = D((0, x)) = (0, x) - (u, F) - (u, F) - (0, x) \]

\[ = (xF - Fx, 0), \quad x \in X. \]

Therefore \( T = 0 \). The converse is trivial. \( \Box \)

**Lemma 4.10.** Suppose that \( T: X \to \mathcal{A}^{(2m)} \) is a (continuous) \( \mathcal{A} \)-bimodule morphism, satisfying \( xT(y) + T(x)y = 0 \) in \( X^{(2m)} \) for \( x, y \in X \). Then \( \overline{T}: \mathcal{A} \oplus X \to (\mathcal{A} \oplus X)^{(2m)} \) defined by

\[ \overline{T}((a, x)) = (T(x), 0) \]

is a (continuous) derivation. \( \overline{T} \) is inner if and only if \( T = 0 \).

**Proof.** The proof is similar to that of Lemma 4.9. \( \Box \)

**Lemma 4.11.** Suppose that \( T: X \to X^{(2m)} \) is a (continuous) \( \mathcal{A} \)-bimodule morphism. Then \( \overline{T}: \mathcal{A} \oplus X \to (\mathcal{A} \oplus X)^{(2m)} \) defined by

\[ \overline{T}((a, x)) = (0, T(x)) \]
4.2. ON n-WEAK AMENABILITY OF $\mathfrak{A} \otimes X$

is a continuous derivation. $\overline{T}$ is inner if and only if there is some $u \in \mathfrak{A}^{(2m)}$ such that $ua = au$ for $a \in \mathfrak{A}$, and $T(x) = xu - ux$ for all $x \in X$.

PROOF. The first statement is obviously true. If $\overline{T}$ is inner, then for some $(u, \phi) \in \mathfrak{A}^{(2m)} \oplus X^{(2m)}$,

$$(0, T(x)) = \overline{T}((0, x)) = (0, x) \cdot (u, \phi) - (u, \phi) \cdot (0, x)$$

$$= (0, xu - ux)$$

and

$$(0, 0) = \overline{T}((a, 0)) = (a, 0) \cdot (u, \phi) - (u, \phi) \cdot (a, 0)$$

$$= (au - ua, a\phi - \phi a).$$

Therefore, $T(x) = xu - ux$ and $au - ua = 0$.

For the converse, suppose that, for some $u \in \mathfrak{A}^{(2m)}$ satisfying $au - ua = 0$ for all $a \in \mathfrak{A}$, $T(x) = xu - ux$ for $x \in X$, then

$$\overline{T}((a, x)) = (0, T(x)) = (au - ua, xu - ux)$$

$$= (a, x) \cdot (u, 0) - (u, 0) \cdot (a, x).$$

So $\overline{T}$ is inner. \qed

4.2. On $n$-weak amenability of $\mathfrak{A} \otimes X$

Suppose that $\mathfrak{A}$ is a Banach algebra and $X$ is a Banach $\mathfrak{A}$-bimodule. In this section we are concerned with the conditions for the $n$-weak amenability
of the module extension Banach algebra $\mathfrak{A} \oplus X$. The discussion naturally splits into two cases of odd integers $n$ and even integers $n$. For odd $n$ we have the following main result.

**Theorem 4.12.** For $m \geq 0$, $\mathfrak{A} \oplus X$ is $(2m + 1)$-weakly amenable if and only if the following conditions hold:

1. $\mathfrak{A}$ is $(2m + 1)$-weakly amenable;
2. $\mathcal{H}^1(\mathfrak{A}, X^{(2m+1)}) = \{0\}$;
3. For any continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X \to \mathfrak{A}^{(2m+1)}$ there is $F \in X^{(2m+1)}$ such that $aF - Fa = 0$ for $a \in \mathfrak{A}$ and $\Gamma(x) = xF - Fx$ for $x \in X$;
4. For any continuous $\mathfrak{A}$-bimodule morphism $T: X \to X^{(2m+1)}$, if

$$xT(y) + T(x)y = 0$$

in $\mathfrak{A}^{(2m+1)}$ for all $x, y \in X$, then $T = 0$.

**Proof.** Denote by $\Delta_1$ the projection from $(\mathfrak{A} \oplus X)^{(2m+1)}$ onto $\mathfrak{A}^{(2m+1)}$; the kernel of $\Delta_1$ is $X^{(2m+1)}$. Let $\Delta_2 = I - \Delta_1$, the projection from $(\mathfrak{A} \oplus X)^{(2m+1)}$ onto $X^{(2m+1)}$. Both $\Delta_1$ and $\Delta_2$ are obviously $\mathfrak{A}$-bimodule morphisms. Let $\tau_1: \mathfrak{A} \to \mathfrak{A} \oplus X$ be the inclusion mapping (i.e. $\tau_1(a) = (a, 0)$). It is easy to see that $\tau_1$ is an algebra homomorphism.

We first prove the sufficiency. Suppose that conditions 1–4 hold. Suppose that $D: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous derivation. Then
4.2. ON n-WEAK AMENABILITY OF $\mathfrak{A} \oplus X$

$D \circ \tau_1: \mathfrak{A} \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous derivation. In fact,

$$D \circ \tau_1(ab) = D((ab, 0)) = D((a, 0) \cdot (b, 0)) = D((a, 0))b + aD((b, 0)) = (D \circ \tau_1(a))b + a(D \circ \tau_1(b)).$$

Then $\Delta_1 \circ D \circ \tau_1: \mathfrak{A} \to \mathfrak{A}^{(2m+1)}$ and $\Delta_2 \circ D \circ \tau_1: \mathfrak{A} \to X^{(2m+1)}$ are continuous derivations. By conditions 1 and 2, they are inner. This implies that $D \circ \tau_1$ is inner. From lemmas 4.3 and 4.7,

$$D \circ \tau_1 = \Delta_1 \circ D \circ \tau_1 + \Delta_2 \circ D \circ \tau_1 : \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$$

is a continuous derivation and there is a derivation $\overline{D}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$, satisfying $\overline{D}((a, 0)) = 0$ for $a \in \mathfrak{A}$, such that $D \circ \tau_1 - \overline{D}$ is inner. Also

$$(D - D \circ \tau_1)((a, 0)) = D((a, 0)) - D \circ \tau_1((a, 0)) = D \circ \tau_1(a) - D \circ \tau_1(a) = 0, \; (a \in \mathfrak{A}).$$

Let $\widehat{D} = D - D \circ \tau_1 + \overline{D}$. We have that $\widehat{D}$ is a continuous derivation from $\mathfrak{A} \oplus X$ into $(\mathfrak{A} \oplus X)^{(2m+1)}$ satisfying $\widehat{D}((a, 0)) = 0$ for $a \in \mathfrak{A}$. On the other hand,

$$\widehat{D}((0, ax)) = \widehat{D}((a, 0) \cdot (0, x)) = (a, 0) \cdot \widehat{D}((0, x)) = a\widehat{D}((0, x)) \; (a \in \mathfrak{A}, x \in X),$$
Denote by \( \tau_2 : X \to \mathfrak{A} \oplus X \) the inclusion mapping given by \( \tau_2(x) = (0, x) \), \( x \in X \). Then \( \overline{D} \circ \tau_2 : X \to (\mathfrak{A} \oplus X)^{(2m+1)} \) is a continuous \( \mathfrak{A} \)-bimodule morphism. From condition 3 there is \( F \in X^{(2m+1)} \) such that

\[
\Delta_1 \circ \overline{D} \circ \tau_2(x) = xF - Fx, \text{ and } aF - Fa = 0
\]

for \( x \in X \) and \( a \in \mathfrak{A} \). Since

\[
(0, 0) = \overline{D}((0, 0)) = \overline{D}((0, x) \cdot (0, y))
\]

\[
= \overline{D}((0, x)) \cdot (0, y) + (0, x) \cdot \overline{D}((0, y))
\]

\[
= ([\Delta_2 \circ \overline{D}((0, x))]y, 0) + (x[\Delta_2 \circ \overline{D}((0, y))], 0)
\]

\[
= ([\Delta_2 \circ \overline{D} \circ \tau_2(x)]y + x[\Delta_2 \circ \overline{D} \circ \tau_2(y)], 0),
\]

we have

\[
(\Delta_2 \circ \overline{D} \circ \tau_2(x))y + x(\Delta_2 \circ \overline{D} \circ \tau_2(y)) = 0
\]

for \( x, y \in X \). From condition 4, \( \Delta_2 \circ \overline{D} \circ \tau_2 = 0 \). So

\[
\overline{D}((a, x)) = \overline{D}((0, x)) = \overline{D} \circ \tau_2(x)
\]

\[
= (\Delta_1 \circ \overline{D} \circ \tau_2(x), \Delta_2 \circ \overline{D} \circ \tau_2(x))
\]

\[
= (xF - Fx, 0) = (a, x) \cdot (0, F) - (0, F) \cdot (a, x)
\]
This shows that $\widehat{D}$ is inner. Then $D = \widehat{D} + (D \circ \tau_l - \widehat{D})$ is inner. This proves that $\mathfrak{A} \oplus X$ is $(2m + 1)$-weakly amenable.

Necessity: Suppose that $\mathfrak{A} \oplus X$ is $(2m + 1)$-weakly amenable, and suppose that $D: \mathfrak{A} \to \mathfrak{A}^{(2m+1)}$ is a continuous derivation. Then, from Lemma 4.3, $D$ is inner. This shows that $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(2m+1)}) = \{0\}$. Similarly Lemma 4.7 implies that $\mathcal{H}^1(\mathfrak{A}, X^{(2m+1)}) = \{0\}$. Therefore conditions 1 and 2 hold.

Since any continuous derivation $D: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is inner, Lemma 4.8 gives condition 3, and Lemma 4.9 shows that condition 4 holds. The proof is complete.

For even integers $n$ we have the following theorem.

**Theorem 4.13.** For $m \geq 0$, $\mathfrak{A} \oplus X$ is $2m$-weakly amenable if and only if the following conditions hold:

1. For any continuous derivation $D: \mathfrak{A} \to \mathfrak{A}^{(2m)}$, if there is a continuous operator $T: X \to X^{(2m)}$ such that $T(ax) = D(a)x + aT(x)$ and $T(xa) = xD(a) + T(x)a$, for $a \in \mathfrak{A}$ and $x \in X$, then $D$ is inner;
2. $\mathcal{H}^1(\mathfrak{A}, X^{(2m)}) = \{0\}$;
3. Any continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X \to \mathfrak{A}^{(2m)}$ satisfying

$$x\Gamma(y) + \Gamma(x)y = 0$$

in $X^{(2m)}$, for $x, y \in X$, is trivial;
4.2. ON n-WEAK AMENABILITY OF $\mathfrak{A} \oplus X$

4. For any continuous $\mathfrak{A}$-bimodule morphism $T: X \to X^{(2m)}$, there exists $u \in \mathfrak{A}^{(2m)}$, such that $au = ua$ for $a \in \mathfrak{A}$, and $T(x) = xu - ux$ for $x \in X$.

**Proof.** Denote by $\tau_1$ and $\tau_2$ the natural inclusion mappings from, respectively, $\mathfrak{A}$ and $X$ into $\mathfrak{A} \oplus X$, and by $\Delta_1$ and $\Delta_2$ the natural projections from $(\mathfrak{A} \oplus X)^{(2m)}$ onto $\mathfrak{A}^{(2m)}$ and $X^{(2m)}$, respectively. These are $\mathfrak{A}$-bimodule morphisms.

We first prove the sufficiency. Suppose that the conditions 1-4 in Theorem 4.13 hold. Let $D: (\mathfrak{A} \oplus X) \to (\mathfrak{A} \oplus X)^{(2m)}$ be a continuous derivation. Then clearly $\Delta_1 \circ D \circ \tau_1: \mathfrak{A} \to \mathfrak{A}^{(2m)}$ and $\Delta_2 \circ D \circ \tau_1: \mathfrak{A} \to X^{(2m)}$ are continuous derivations.

**Claim 1:** $\Delta_1 \circ D \circ \tau_2: X \to \mathfrak{A}^{(2m)}$ is trivial.

Let $\Gamma = \Delta_1 \circ D \circ \tau_2$. To prove Claim 1, by condition 3 it suffices to show that $\Gamma$ is an $\mathfrak{A}$-bimodule morphism satisfying $x\Gamma(y) + \Gamma(x)y = 0$ in $X^{(2m)}$ for $x, y \in X$. In fact,

$$0 = D((0, 0)) = D((0, x) \cdot (0, y)) = D((0, x)) \cdot (0, y) + (0, x) \cdot D((0, y)) = (0, \Gamma(x)y) + (0, x\Gamma(y)).$$
Therefore \( x\Gamma(y) + \Gamma(x)y = 0 \). On the other hand

\[
\Gamma(ax) = \Delta_1 \circ D((0, ax)) = \Delta_1 \circ D((a, 0) \cdot (0, x))
\]

\[
= \Delta_1 (D((a, 0)) \cdot (0, x) + (a, 0) \cdot D((0, x)))
\]

\[
= \Delta_1 ((a, 0) \cdot D((0, x)))
\]

\[
= \Delta_1 (aD \circ \tau_2(x)) = a\Gamma(x).
\]

Similarly, \( \Gamma(xa) = \Gamma(x)a \) and so \( \Gamma \) is an \( \mathfrak{A} \)-bimodule morphism. This verifies Claim 1.

Now let \( T = \Delta_2 \circ D \circ \tau_2 \colon X \to X^{(2m)} \), and \( D_1 = \Delta_1 \circ D \circ \tau_1 \colon \mathfrak{A} \to \mathfrak{A}^{(2m)} \).

Claim 2: \( T(ax) = D_1(a)x + aT(x) \) and \( T(xa) = xD_1(a) + T(x)a \) for \( a \in \mathfrak{A}, x \in X \).

In fact, from Claim 1

\[
(0, T(ax)) = D((0, ax)) = D((a, 0) \cdot (0, x))
\]

\[
= D((a, 0)) \cdot (0, x) + (a, 0) \cdot D((0, x))
\]

\[
= (0, D_1(a)x) + a(0, T(x)) = (0, D_1(a)x + aT(x)),
\]

Similarly, \( (0, T(xa)) = (0, xD_1(a) + T(x)a) \), for \( a \in \mathfrak{A}, x \in X \). So Claim 2 is verified.

Therefore by condition 1, \( D_1 = \Delta_1 \circ D \circ \tau_1 \) is inner. Suppose that \( u \in \mathfrak{A}^{(2m)} \) satisfies \( D_1(a) = au - ua \) for \( a \in \mathfrak{A} \). Take \( T_1 \colon X \to X^{(2m)} \) specified by \( T_1(x) = xu - ux \) for \( x \in X \). Then \( T - T_1 \colon X \to X^{(2m)} \) is a continuous
4.2. ON n-WEAK AMENABILITY OF $\mathfrak{A} \oplus X$  

$\mathfrak{A}$-bimodule morphism. In fact, from Claim 2, for $a \in \mathfrak{A}$ and $x \in X$,

$$(T - T_1)(ax) = T(ax) - T_1(ax) = (D_1(a)x + aT(x)) - (axu - uax)$$

$$= (au - ua)x + aT(x) - (axu - uax)$$

$$= a(ux - xu) + aT(x) = a(T - T_1)(x).$$

Similarly, $T - T_1$ is a right $\mathfrak{A}$-module morphism.

From condition 4, there is a $v \in \mathfrak{A}^{(2m)}$ such that $av = va$ for $a \in \mathfrak{A}$, and

$$(T - T_1)(x) = xv - vx$$

for $x \in X$. From Lemma 4.11

$$\overline{T - T_1}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}$$

defined by

$$\overline{T - T_1}((a, x)) = (0, (T - T_1)(x)) \quad ((a, x) \in \mathfrak{A} \oplus X)$$

is an inner derivation.

Since $\Delta_2 \circ D \circ \tau_1: \mathfrak{A} \to X^{(2m)}$ is a continuous derivation, it is inner by condition 2. From Lemma 4.4

$$\overline{\Delta_2 \circ D \circ \tau_1}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}$$

defined by

$$\overline{\Delta_2 \circ D \circ \tau_1}((a, x)) = (0, \Delta_2 \circ D \circ \tau_1(a)), \quad ((a, x) \in \mathfrak{A} \oplus X),$$

is also inner. From all the discussion above combined, we have

$$D((a, x)) = (D_1(a), \Delta_2 \circ D \circ \tau_1(a) + T(x))$$

$$= \overline{\Delta_2 \circ D \circ \tau_1}((a, x)) + \overline{T - T_1}((a, x)) + (D_1(a), T_1(x)).$$
Since

\[(D_1(a), T_1(x)) = (au - ua, xu - ux)\]

\[= (a, x) \cdot (u, 0) - (u, 0) \cdot (a, x),\]

for \(a \in \mathfrak{A}, x \in X\), the operator \(D_T: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}\) defined by

\[D_T((a, x)) = (D_1(a), T_1(x))\]

is inner.

Hence as a sum of three inner derivations, \(D\) is inner. This shows that under conditions 1-4 of Theorem 4.13, \(\mathfrak{A} \oplus X\) is \(2m\)-weakly amenable. So the sufficiency is true.

Now we prove the necessity. Suppose that \(\mathfrak{A} \oplus X\) is \(2m\)-weakly amenable. Let \(D: \mathfrak{A} \rightarrow \mathfrak{A}^{(2m)}\) be a derivation with the property given in condition 1. Then \(\overline{D}: \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m)}\) defined by

\[\overline{D}((a, x)) = (D(a), T(x)), (a, x) \in \mathfrak{A} \oplus X,\]

is a derivation. Therefore, there are \(u \in \mathfrak{A}^{(2m)}\) and \(\phi \in X^{(2m)}\) such that

\[\overline{D}((a, x)) = (a, x) \cdot (u, \phi) - (u, \phi) \cdot (a, x)\]

\[= (au - ua, a\phi - \phi a + (xu - ux)).\]

This implies that \(D(a) = au - ua\) for \(a \in \mathfrak{A}\). Hence \(D\) is inner. So condition 1 holds. Also lemmas 4.4, 4.10 and 4.11 show, respectively, that conditions 2-4 hold. The proof is complete. \(\square\)
4.2. ON n-WEAK AMENABILITY OF $\mathfrak{A} \oplus X$

Condition 4 in Theorem 4.13 seems to be a very tough condition, if we note that the inclusion mapping from $X$ into $X^{(2m)}$ is an $\mathfrak{A}$-bimodule morphism. So there are not many $2m$-weakly amenable Banach algebras of the form $\mathfrak{A} \oplus X$. In the rest of the section we make some comments on the conditions in Theorem 4.12.

**Remark** 4.14. When $m = 0$, condition 3 in Theorem 4.12 is equivalent to

$$3^0. \text{ There is no non-zero continuous } \mathfrak{A}\text{-bimodule morphism } \Gamma: X \to \mathfrak{A}^*. $$

**Proof.** If $3^0$ holds, then choose $F = 0 \in X^*$. We see that condition 3 holds for $m = 0$. Conversely suppose that condition 3 holds for $m = 0$ and $\Gamma: X \to \mathfrak{A}^*$ is a continuous $\mathfrak{A}$-bimodule morphism. Then there is an $F \in X^*$ with $aF - Fa = 0$, such that $\Gamma(x) = xF - Fx$ for $x \in X$. So

$$\langle a, \Gamma(x) \rangle = \langle a, xF - Fx \rangle = \langle x, Fa - aF \rangle = 0, \quad a \in \mathfrak{A}, x \in X. $$

Therefore, $\Gamma(x) = 0$ for $x \in X$. This shows that $\Gamma = 0$.

For the general case, condition 3 in Theorem 4.12 is equivalent to:

$$3^m. \text{ For any continuous } \mathfrak{A}\text{-bimodule morphism } \Gamma: X \to \mathfrak{A}^{(2m+1)}, \text{ the range } \Gamma(X) \text{ is contained in } \mathfrak{A}^\perp \text{ and there is } G \in \mathfrak{A} \text{ and } \Gamma(x) = xG - Gx \text{ for } x \in X. $$

**Proof.** It suffices to prove that $3$ implies $3^m$. Suppose that $3$ holds, i.e. for some $F \in X^{(2m+1)}$ satisfying $aF - Fa = 0$, $\Gamma(x) = xF - Fx \ (x \in X)$.
Then clearly \( \Gamma(x) \in \mathfrak{A}^\perp \). Let \( F = \hat{f} + G \), where \( f \in X^* \), \( G \in X^\perp \). Since \( aG - Ga \in X^\perp \), we have \( a\hat{f} - \hat{f}a \in \iota(X^*) \cap X^\perp \), where \( \iota(X^*) \) denotes the image of \( X^* \) in \( X^{(2m+1)} \) under the canonical mapping. This implies that \( a\hat{f} - \hat{f}a = 0 \). Then \( aG - Ga = 0 \) in \( X^{(2m+1)} \). From the preceding remark the operator \( \Gamma_1: X \to \mathfrak{A}^* \) defined by \( \Gamma_1(x) = xf - fx \) is trivial. Therefore,

\[
\Gamma(x) = (xf - fx) + (xG - Gx) = xG - Gx.
\]

Here we have used the fact that \( x\hat{f} = (xf)^\wedge \) and \( \hat{f}x = (fx)^\wedge \) (see page 14).

**Proposition 4.15.** If condition 4 of Theorem 4.12 holds, then

\[
\text{cl}(\mathfrak{A}X + X\mathfrak{A}) = X.
\]

**Proof.** Assume conversely that \( \text{cl}(\mathfrak{A}X + X\mathfrak{A}) \neq X \). Take a non-zero element \( F \in X^* \cap (\mathfrak{A}X + X\mathfrak{A})^\perp \), and let \( T: X \to X^* \) be defined by

\[
T(x) = F(x)F.
\]

Since \( F|_{\mathfrak{A}X + X\mathfrak{A}} = 0 \), it is easy to see that \( T \) is a non-zero continuous \( \mathfrak{A} \)-bimodule morphism and \( \mathfrak{A}T(X) = T(X)\mathfrak{A} = \{0\} \). Also for \( x, y \in X \), \( xT(y) = T(x)y = 0 \) in \( \mathfrak{A}^* \) since \( T(X) \subset (\mathfrak{A}X)^\perp \cap (X\mathfrak{A})^\perp \). This shows that condition 4 of Theorem 4.12 does not hold for \( m = 0 \). So it does not hold for any \( m \geq 0 \).

**Remark 4.16.** For \( m = 0 \), condition 4 in Theorem 4.12 is equivalent to the following.
4.2. ON \( n \)-WEAK AMENABILITY OF \( \mathfrak{A} \oplus X \)

4°. \( \text{cl}(\mathfrak{A}X + X\mathfrak{A}) = X \) and there is no non-zero \( \mathfrak{A} \)-bimodule morphism 

\( T: X \to X^* \) satisfying \( \langle x, T(y) \rangle + \langle y, T(x) \rangle = 0 \), for \( x, y \in X \).

**Proof.** If condition 4 in Theorem 4.12 holds, the previous proposition implies that \( \text{cl}(\mathfrak{A}X + X\mathfrak{A}) = X \). If an \( \mathfrak{A} \)-bimodule morphism \( T: X \to X^* \) satisfies

\[
\langle x, T(y) \rangle + \langle y, T(x) \rangle = 0 \quad \text{for} \quad x, y \in X,
\]

then for any \( a \in \mathfrak{A} \),

\[
\langle a, xT(y) + T(x)y \rangle = \langle ax, T(y) \rangle + \langle y, T(ax) \rangle = 0.
\]

This shows that \( xT(y) + T(x)y = 0 \) for \( x, y \in X \). Therefore \( T = 0 \) and so 4° holds.

Conversely, if 4° holds, and \( T: X \to X^* \) is a continuous \( \mathfrak{A} \)-bimodule morphism satisfying \( xT(y) + T(x)y = 0 \) in \( \mathfrak{A}^* \), then for any \( x \in \mathfrak{A}X + X\mathfrak{A} \), \( x = ax_1 + x_2b \), and \( y \in X \), we have

\[
\langle x, T(y) \rangle + \langle y, T(x) \rangle = \langle a, x_1T(y) + T(x_1)y \rangle + \langle b, T(y)x_2 + yT(x_2) \rangle = 0.
\]

Since \( \text{cl}(\mathfrak{A}X + X\mathfrak{A}) = X \), this implies that \( \langle x, T(y) \rangle + \langle y, T(x) \rangle = 0 \) for all \( x, y \in X \). Hence \( T = 0 \) and so condition 4 of Theorem 4.12 holds for \( m = 0 \). \( \Box \)
4.3. The algebra $\mathfrak{A} \oplus \mathfrak{A}$

In this and the following sections we will investigate several concrete situations. This section is concerned mainly with the two cases $X = \mathfrak{A}$ and $X = \mathfrak{A}^*$ as Banach $\mathfrak{A}$-bimodules.

First we note that if $\mathfrak{A}$ is not amenable, then there is a Banach $\mathfrak{A}$-bimodule $X$, such that $\mathcal{H}(\mathfrak{A}, X^*) \neq \{0\}$. From Theorem 4.12, for this $X$, $\mathfrak{A} \oplus X$ is not weakly amenable. In fact, if $X = \mathfrak{A}^*$, $\mathfrak{A} \oplus X$ is never weakly amenable as stated in the following proposition.

**Proposition 4.17.** For any Banach algebra $\mathfrak{A}$, $\mathfrak{A} \oplus \mathfrak{A}^*$ is never $n$-weakly amenable for any $n \geq 0$.

**Proof.** From Proposition 3.1, it suffices to prove the proposition for the cases of $n = 1$ and $n = 2m$, where $m \geq 0$. Notice that Condition 3 does not hold, because the identity mapping from $X$ (which is $\mathfrak{A}^*$ in this case) onto $\mathfrak{A}^*$ is a non-zero continuous $\mathfrak{A}$-bimodule morphism. So from Remark 4.14 the statement is true for $n = 1$.

Suppose that $n = 2m$. Let $T: X \rightarrow X^{(2m)}$ be a continuous $\mathfrak{A}$-bimodule morphism characterized in Condition 4 of Theorem 4.13 with $X = \mathfrak{A}^*$, then $T(f) \in \mathfrak{A}^\perp$ for $f \in X$. In fact, for $a \in \mathfrak{A}$,

$$
\langle a, T(f) \rangle = \langle a, f \cdot u - u \cdot f \rangle = \langle af - fa, u \rangle
$$

$$
= \langle f, ua - au \rangle = 0.
$$
But when viewed as a subspace of $X^{(2m)}$, $X$ does not annihilate $\mathfrak{A}$, where the latter is viewed as a subspace of $X^{(2m-1)}$. This shows that, as an $\mathfrak{A}$-bimodule morphism, the inclusion mapping from $X$ into $X^{(2m)}$ does not satisfy Condition 4 of Theorem 4.13. Therefore $\mathfrak{A} \oplus \mathfrak{A}^*$ is not $2m$-weakly amenable. \qed

Now we consider the case $X = \mathfrak{A}$. To avoid any confusion, from now on when we regard $\mathfrak{A}$ as an $\mathfrak{A}$-bimodule, we will use the notation $A$ instead of $\mathfrak{A}$. Similarly, we use the notation $A^{(n)}$ to denote the $\mathfrak{A}$-bimodule $\mathfrak{A}^{(n)}$. In this case, since condition 4 in Theorem 4.13 never holds for any integer $m$ (the canonical embedding is a non-zero morphism), we have that $\mathfrak{A} \oplus A$ is never $2m$-weakly amenable for any $m \geq 0$. If $\mathfrak{A}$ is commutative, the same reason yields a little bit more extended result. Recall that an $\mathfrak{A}$-bimodule $X$ is symmetric if $ax = xa$, for $a \in \mathfrak{A}$, $x \in X$.

**Proposition 4.15.** Suppose that $\mathfrak{A}$ is a commutative Banach algebra. Then for any non-zero symmetric $\mathfrak{A}$-bimodule $X$, $\mathfrak{A} \oplus X$ is not $2m$-weakly amenable.

**Proof.** If $X$ is symmetric, then we will have $xu = ux$ for $u \in \mathfrak{A}^{(2m)}$ and $x \in X$. Since the inclusion mapping is a non-trivial $\mathfrak{A}$-bimodule morphism, Condition 4 in Theorem 4.13 does not hold. \qed

But $\mathfrak{A} \oplus A$ can be weakly amenable. Before giving an example for this let us discuss some relations concerning corresponding elements of $A^{(n)}$ and $\mathfrak{A}^{(n)}$. Suppose that $\phi \in A^{(n)}$. We denote the same element in $\mathfrak{A}^{(n)}$ by $\phi$. 


LEMMA 4.19. Suppose that $\mathfrak{A}$ is a Banach algebra and let $m \geq 0$. Then

for $\phi, \psi \in A^{(2m)}$ and $F \in A^{(2m+1)}$,

$$(\phi \psi)\sim = \phi \bar{\psi} = (\phi \bar{\psi})\sim,$$

$$\phi F = (\phi F)\sim = \phi \bar{F}, \quad F \phi = (F \phi)\sim = \bar{F} \phi.$$

PROOF. It is easy to check that these equalities hold for $m = 0$. Assume that they hold for $m = k$. Then for $m = k + 1$, letting $\phi, \psi \in A^{(2k+2)}$ and $f \in A^{(2k+1)}$, we have

$$\langle \tilde{f}, (\tilde{\phi} \psi)\sim \rangle = \langle f, \bar{\phi} \psi \rangle = \langle \psi f, \bar{\phi} \rangle.$$

Here $\psi f \in \mathfrak{A}^{(2k+1)}$. For $x \in A^{(2k)}$,

$$\langle \tilde{x}, \psi f \rangle = \langle f \tilde{x}, \psi \rangle = \langle (f \tilde{x})\sim, \bar{\psi} \rangle = \langle f \tilde{x}, \psi \rangle = \langle \tilde{x}, \bar{\psi} \tilde{f} \rangle.$$

This shows that $\psi f = \bar{\psi} \tilde{f}$. So

$$\langle \tilde{f}, (\tilde{\phi} \psi)\sim \rangle = \langle \bar{\psi} \tilde{f}, \bar{\phi} \rangle = \langle \tilde{f}, \bar{\phi} \psi \rangle.$$

Hence $$(\phi \psi)\sim = \phi \bar{\psi}, \phi, \psi \in A^{(2k+2)}$$. Similarly $\phi \psi = (\phi \bar{\psi})\sim$. These in turn verify the following equalities.

$$\langle \bar{\psi}, \phi F \rangle = \langle \bar{\psi} \phi, F \rangle = \langle \bar{\psi} \phi, \bar{F} \rangle = \langle \bar{\psi}, \phi \bar{F} \rangle,$$
where \( \phi, \psi \in A^{(2k+2)}, F \in A^{(2k+3)} \). So \( \phi F = \tilde{\phi} \tilde{F} \). Also
\[
\langle \tilde{\psi}, (\tilde{\phi} F) \rangle = \langle \psi, \phi F \rangle = \langle \psi \phi, F \rangle
\]
\[
= \langle (\psi \phi), \tilde{F} \rangle = \langle \psi \phi, \tilde{F} \rangle = \langle \tilde{\psi}, \tilde{\phi} \tilde{F} \rangle.
\]
Hence \((\tilde{\phi} F) \sim = \tilde{\phi} \tilde{F} \). So we have proved \( \phi F = (\tilde{\phi} F) \sim = \tilde{\phi} \tilde{F} \) for the case \( m = k + 1 \). A similar proof will give \( F \phi = (F \tilde{\phi}) \sim = \tilde{F} \tilde{\phi} \). This completes the proof. \( \square \)

A special case of Lemma 4.19 is the following group of equalities which we will use in the proof of the next theorem.

\[
(a \phi) \sim = a \tilde{\phi}, \quad (\phi a) \sim = \tilde{\phi} a,
\]
\[
x F = (\tilde{x} F) \sim = \tilde{x} \tilde{F}, \quad F x = (F \tilde{x}) \sim = \tilde{F} \tilde{x},
\]
for \( a \in \mathfrak{A}, x \in A, \phi \in A^{(2m)} \) and \( F \in A^{(2m+1)} \). From these equalities, we also see that for \( X = A \) and \( m \geq 0 \) the conditions 3 and 4 in Theorem 4.12 hold if and only if there is no non-zero \( \mathfrak{A} \)-bimodule morphism \( T \) from \( A \) into \( A^{(2m+1)} \). Also we note that in the case \( X = A \) conditions 1 and 2 of Theorem 4.12 are the same.

**THEOREM 4.20.** Suppose that \( \mathfrak{A} \) is a Banach algebra.

1. if \( \text{span}\{ab – ba; a, b \in \mathfrak{A}\} \) is not dense in \( \mathfrak{A} \), then \( \mathfrak{A} \oplus A \) is not weakly amenable;

2. if \( \text{span}\{ab – ba; a, b \in \mathfrak{A}\} \) is dense in \( \mathfrak{A} \), then \( \mathfrak{A} \oplus A \) is weakly amenable, provided \( \mathfrak{A} \) is weakly amenable and has a b.a.i.
Proof. By condition 1 of Theorem 4.12 without loss of generality we can assume \( \mathfrak{A} \) is weakly amenable for both cases. If \( \text{span}\{ab - ba; \ a, b \in \mathfrak{A}\} \) is not dense in \( \mathfrak{A} \), then there is \( f \in \mathfrak{A}^* \) such that \( f \neq 0 \) and \( \langle ab - ba, f \rangle = 0 \) for \( a, b \in \mathfrak{A} \). So \( af = fa \) for \( a \in \mathfrak{A} \). Then \( T: A \to \mathfrak{A}^* \) defined by

\[
T(x) = \hat{x}f = f\hat{x},
\]

is an \( \mathfrak{A} \)-bimodule morphism. According to Proposition 3.2, \( \mathfrak{A}^2 \), the linear span of all product elements \( ab, a, b \in \mathfrak{A} \), is dense in \( \mathfrak{A} \). So there are \( a, b \in \mathfrak{A} \) such that \( \langle ab, f \rangle \neq 0 \). This implies that \( T \neq 0 \). Therefore in this case \( \mathfrak{A} \oplus A \) is not weakly amenable due to

If \( \text{span}\{ab - ba; \ a, b \in \mathfrak{A}\} \) is dense in \( \mathfrak{A} \), and \( \mathfrak{A} \) has a b.a.i., say \( (e_i) \), then for any given continuous \( \mathfrak{A} \)-bimodule morphism \( T: A \to \mathfrak{A}^* \), we have

\[
T(a) = af = fa,
\]

where \( f \) is a weak* cluster point of \( (T(e_i)) \). Therefore \( f(ab - ba) = 0 \) for all \( a, b \in \mathfrak{A} \). This shows that \( f = 0 \) and hence \( T = 0 \). So conditions 3 and 4 in Theorem 4.12 hold for \( m = 0 \). The other two conditions hold automatically for \( m = 0 \). So by using Theorem 4.12 we see that the second statement of the theorem is true.

From the first statement of Theorem 4.20 we have the following corollary immediately:

**Corollary 4.21.** For any commutative Banach algebra \( \mathfrak{A} \), \( \mathfrak{A} \oplus A \) is never weakly amenable.
Let $\mathcal{H}$ be an infinite dimensional Hilbert space. According to a classical result due to Halmos, every element in $B(\mathcal{H})$ can be written as a sum of two commutators (Theorem 8 of [37]; for further results about commutators in $C^*$-algebras we refer to [5]–[8], [38] and [59]). Together with the fact that $B(\mathcal{H})$ has an identity and as a $C^*$-algebra is weakly amenable ([36]), from Theorem 4.20 we see that $B(\mathcal{H}) \oplus B(\mathcal{H})$ is weakly amenable. Later in this section we will see that it is actually $(2m+1)$-weakly amenable for all $m \geq 0$.

**Proposition 4.22.** In condition 2 of Theorem 4.20 if we further assume that

$$V = \text{span}\{au - ua; u \in A^{**}, a \in A\}$$

is not dense in $A^{**} A$ (if $A$ has an identity this is equivalent to saying that $V$ is not dense in $A^{**}$), then $A \oplus A$ is not 3-weakly amenable.

**Proof.** In fact, in this case $A^{**} A \not\subseteq cl(V)$, since otherwise it would follow that both $A A^{**}$ and $A^{**} A$ are in $cl(V)$ and then $cl(V) \supseteq A A^{**} + A^{**} A$, which is a contradiction to the assumption that $V$ is not dense in $A A^{**} + A^{**} A$.

Hence from the Hahn-Banach Theorem, there is $F \in A^{***}$, such that $F|_V = 0$ but $F \neq 0$ on $A A^{**}$. This implies that $aF = Fa$ for all $a \in A$ and $aF \neq 0$ for some $a \in A$. Then define $T: A \to A^{***}$ by

$$T(x) = \tilde{a}F(= F\tilde{x}).$$
4.3. THE ALGEBRA $\mathfrak{A} \oplus \mathfrak{A}$

$T$ is a non-zero continuous $\mathfrak{A}$-bimodule morphism from $A$ into $\mathfrak{A}^{***}$. Therefore condition 3 in Theorem 4.12 does not hold for $m = 1$. Thus $\mathfrak{A} \oplus A$ is not 3-weakly amenable. \hfill \Box

Concerning question 1.5 in Chapter 1, Theorem 4.20 and Proposition 4.22 suggest that we might be able to find a counter-example in the Banach algebras of the form $\mathfrak{A} \oplus A$. Unfortunately, $B(H)$ can not be a candidate. We will see this after the next two lemmas. The following lemma is basically in [37, Theorem 8], but we have highlighted some of its features which are important for our purposes.

**Lemma 4.23.** Suppose that $\mathcal{H}$ is an infinite dimensional Hilbert space. There are two elements $Q_0$ and $S_0$ in $B(\mathcal{H})$ such that for each $B \in B(\mathcal{H})$ there exist $P_B \in B(\mathcal{H})$ and $R_B \in B(\mathcal{H})$ such that $\|P_B\| \leq \|B\|$, $\|R_B\| \leq \|B\|$ and

$$B = (P_B \circ Q_0 - Q_0 \circ P_B) + (R_B \circ S_0 - S_0 \circ R_B).$$

**Proof.** First for an infinite dimensional Hilbert space $\mathcal{H}$, there exists an isometry $\eta: \mathcal{H} \to \sum_{i=1}^{\infty} \mathcal{H}_i$, where $\sum_{i=1}^{\infty} \mathcal{H}_i$ denotes the $l_2$ direct sum, and each $\mathcal{H}_i$ is a copy of $\mathcal{H}$. 
Let $Q: \mathcal{H} \to \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ and $S: \bigoplus_{i=1}^{\infty} \mathcal{H}_i \to \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ be the bounded operators given by the infinite matrices

\[
Q = \begin{pmatrix} I \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ I & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}.
\]

Let $Q_0 = \eta^{-1} \circ Q$, $S_0 = \eta^{-1} \circ S \circ \eta$. Then $Q_0, S_0 \in B(\mathcal{H})$. For any element $B \in B(\mathcal{H})$, let $P: \bigoplus_{i=1}^{\infty} \mathcal{H}_i \to \mathcal{H}$ and $R: \bigoplus_{i=1}^{\infty} \mathcal{H}_i \to \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ be the bounded operators given by the infinite matrices

\[
P = \begin{pmatrix} B \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & B & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & \cdots \\ 0 & 0 & 0 & B & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}.
\]

Take $P_B = P \circ \eta$ and $R_B = \eta^{-1} \circ R \circ \eta$. Then $P_B, R_B \in B(\mathcal{H})$ and $\|P_B\| \leq \|B\|$, $\|R_B\| \leq \|B\|$. One can check easily that

\[
B = (P_B \circ Q_0 - Q_0 \circ P_B) + (R_B \circ S_0 - S_0 \circ R_B).
\]

\[\square\]

The following result on the 2nth dual of $B(\mathcal{H})$ seems not to be previously known.
4.3. THE ALGEBRA $\mathfrak{A} \otimes \mathfrak{A}$

**Lemma 4.24.** For any integer $n \geq 0$,

$$B(\mathcal{H})^{(2n)} = \text{span}\{au - ua; \ a \in B(\mathcal{H}), u \in B(\mathcal{H})^{(2n)}\}.$$  

**Proof.** By taking weak* limits and using induction, immediately the result follows from the preceding lemma. $\square$

**Proposition 4.25.** For any $m \geq 0$, $B(\mathcal{H}) \oplus B(\mathcal{H})$ is $(2m + 1)$-weakly amenable but is not $2m$-weakly amenable.

**Proof.** First as a $C^*$-algebra $B(\mathcal{H})$ is permanently weakly amenable. So conditions 1 and 2 of Theorem 4.12 hold for $X = \mathfrak{A} = B(\mathcal{H})$ and $m \geq 0$. To show that conditions 3 and 4 also hold it suffices to prove that any continuous $B(\mathcal{H})$-bimodule morphism $T$ from $B(\mathcal{H})$ into $B(\mathcal{H})^{(2m+1)}$ is trivial.

In fact, letting $e$ be the identity of $B(\mathcal{H})$ and $F = T(e)$, we have $T(a) = aF = Fa$ for any $a \in B(\mathcal{H})$. Therefore for any $u \in B(\mathcal{H})^{(2m)}$, $\langle au - ua, F\rangle = 0$. From Lemma 4.24, this implies that $F = 0$. Hence $T = 0$.

The above proves that $B(\mathcal{H}) \oplus B(\mathcal{H})$ is $(2m + 1)$-weakly amenable for $m \geq 0$.

Now we prove that condition 4 of Theorem 4.13 does not hold for $m \geq 0$. From Lemma 4.19, when $X = B(\mathcal{H})$, any $B(\mathcal{H})$-bimodule morphism $T$: $X \to X^{(2m)}$ satisfying condition 4 of Theorem 4.13 must be trivial. But the inclusion mapping from $X$ into $X^{(2m)}$ is a non-trivial $B(\mathcal{H})$-bimodule morphism. This shows that condition 4 of Theorem 4.13 does not hold. So $B(\mathcal{H}) \oplus B(\mathcal{H})$ is not $2m$-weakly amenable for any $m \geq 0$. The proof is complete. $\square$
Remark 4.26. Denote by $K(\mathcal{H})$ the compact operators on $\mathcal{H}$. Using Theorem 1 of [59] one can also prove that $K(\mathcal{H}) \oplus K(\mathcal{H})$ and $B(\mathcal{H}) \oplus K(\mathcal{H})$ are $(2m+1)(\text{but not } 2m)$-weakly amenable. With this list, it is interesting to recall Proposition 4.15 to see that $K(\mathcal{H}) \oplus B(\mathcal{H})$ is not weakly amenable.

4.4. The algebra $\mathfrak{A} \oplus X_0$

In this section we consider the case that one sided (precisely, right) module action on $X$ is trivial. We denote this kind of $\mathfrak{A}$-bimodules specifically by $X_0$. First we observe that when $X = X_0$ conditions 3 and 4 in Theorem 4.12 are reduced, respectively, to

3'. For any continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X_0 \to \mathfrak{A}^{(2m+1)}$ there is $F \in X_0^{(2m+1)}$ such that $Fa = 0$ for $a \in \mathfrak{A}$ and $\Gamma(x) = xF$ for $x \in X_0$;

4'. $\mathfrak{A}X_0$ is dense in $X_0$.

PROOF. The equivalence of 3 and 3' in this case is clear. If 4 holds for $X = X_0$, then Proposition 4.15 says that $\mathfrak{A}X_0 (= \mathfrak{A}X_0 + X_0\mathfrak{A})$ is dense in $X_0$. So 4' holds. Conversely, if 4' holds, then any $\mathfrak{A}$-bimodule morphism $T: X_0 \to X_0^{(2m+1)}$ is trivial, since the left $\mathfrak{A}$-module action on $X_0^{(2m+1)}$ is trivial. Therefore 4 holds. \qed

Also conditions 1, 3 and 4 in Theorem 4.13 are reduced, respectively, to

1''. For any continuous derivation $D: \mathfrak{A} \to \mathfrak{A}^{(2m)}$, if there is a continuous operator $T: X_0 \to X_0^{(2m)}$ such that $T(ax) = D(a)x + aT(x)$ for $a \in \mathfrak{A}$ and $x \in X_0$, then $D$ is inner;
3\textsuperscript{\textordmasculine}{\textordmasculine}. Any continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X_0 \to \mathfrak{A}^{(2m)}$ satisfying 
\[\Gamma(x) y = 0 \text{ in } X_0^{(2m)}\] is trivial;

4\textsuperscript{\textordmasculine}{\textordmasculine}. For any continuous $\mathfrak{A}$-bimodule morphism $T: X_0 \to X_0^{(2m)}$, there exists $u \in \mathfrak{A}^{(2m)}$, such that $au = ua$ for $a \in \mathfrak{A}$ and $T(x) = ux$ for $x \in X_0$.

Suppose now that $\mathfrak{A}$ has a b.a.i., then from Corollary 3.7, condition 2 in Theorem 4.12 always holds for $X = X_0$. This verifies the following theorem.

**Theorem 4.27.** Suppose that $\mathfrak{A}$ is a $(2m + 1)$-weakly amenable Banach algebra with a b.a.i.. Then $\mathfrak{A} \oplus X_0$ is $(2m + 1)$-weakly amenable if and only if conditions 3\textsuperscript{\textordmasculine}{\textordmasculine} and 4\textsuperscript{\textordmasculine} hold.

**Corollary 4.28.** Under the assumptions of Theorem 4.27, if in addition $\mathfrak{A}^{(2m)} = \mathfrak{A}^{(2m)}$, then $\mathfrak{A} \oplus X_0$ is $(2m + 1)$-weakly amenable if and only if $\mathfrak{A} X_0$ is dense in $X_0$.

**Proof.** If $\mathfrak{A}^{(2m)} = \mathfrak{A}^{(2m)}$, then there is no non-zero continuous $\mathfrak{A}$-bimodule morphism $T: X_0 \to \mathfrak{A}^{(2m+1)}$, since

\[\langle au, T(x) \rangle = \langle u, T(xa) \rangle = 0 \quad (a \in \mathfrak{A}, u \in \mathfrak{A}^{(2m)}).\]

So condition 3\textsuperscript{\textordmasculine}{\textordmasculine} holds automatically.

Especially for $m = 0$, the above corollary gives:
4.4. THE ALGEBRA $\mathcal{A} \oplus X_0$

**Corollary 4.29.** Suppose that $\mathcal{A}$ is a weakly amenable Banach algebra with a b.a.i.. Then $\mathcal{A} \oplus X_0$ is weakly amenable if and only if $\mathcal{A}X_0$ is dense in $X_0$.

A dual result to Corollary 4.29 is as follows.

**Corollary 4.30.** Suppose that $\mathcal{A}$ is a weakly amenable Banach algebra with a b.a.i., and $\mathcal{Y}$ is a Banach $\mathcal{A}$-bimodule with the left module action trivial. Then $\mathcal{A} \oplus \mathcal{Y}$ is weakly amenable if and only if $\mathcal{A}\mathcal{Y}$ is dense in $\mathcal{Y}$.

The next theorem concerns the question: when weak amenability implies $(2m + 1)$-weak amenability?

**Theorem 4.31.** Suppose that $\mathcal{A}$ is an ideal of $\mathcal{A}^{**}$. Then the following statements are equivalent.

1. $\mathcal{A} \oplus X_0$ is $(2m + 1)$-weakly amenable ($m \geq 1$);
2. $\mathcal{A} \oplus X_0$ is weakly amenable and $\mathcal{H}^1(\mathcal{A}, X_0^{(2m+1)}) = \{0\}$.

**Proof.** It suffices to prove that 2 implies 1. If $\mathcal{A} \oplus X_0$ is weakly amenable, then $\mathcal{A}$ is weakly amenable and $3^0$ as well as $4'_0$ hold. From Theorem 3.5, $\mathcal{A}$ is $(2m + 1)$-weakly amenable.

Let $\Gamma: X_0 \to \mathcal{A}^{(2m+1)}$ be a continuous $\mathcal{A}$-bimodule morphism. Notice that $\mathcal{A}^{(2m+1)} = \mathcal{A}^* + \mathcal{A}^\perp$ as $\mathcal{A}$-bimodules. Let $P: \mathcal{A}^{(2m+1)} \to \mathcal{A}^\perp$ be the projection with the kernel $\mathcal{A}^*$. Then $P$ is an $\mathcal{A}$-bimodule morphism. $P \circ \Gamma = 0$ since
for $ax \in \mathcal{A}X_0$ and $u \in \mathcal{A}^{(2m)}$

$$\langle u, P \circ \Gamma(ax) \rangle = \langle ua, P \circ \Gamma(x) \rangle = 0.$$  

Here we have used the condition that $\mathcal{A}$ is an ideal of $\mathcal{A}^{**}$ and Lemma 3.3. So $\Gamma$ is actually an $\mathcal{A}$-bimodule morphism from $X_0$ into $\mathcal{A}^*$ and hence is 0 due to Remark 4.14. This shows that $3_0'$ holds for all $m$. Hence from Theorem 4.12, $\mathcal{A} \oplus X_0$ is $(2m + 1)$-weakly amenable if $\mathcal{H}^1(\mathcal{A}, X_0^{(2m+1)}) = \{0\}$. \hfill \square

**Remark 4.32.** Theorem 4.31 can be viewed as an extension of Theorem 3.5 in the sense that Theorem 3.5 deals with the case $X_0 = \{0\}$.

From Corollary 3.7, Theorem 4.31 yields immediately the following.

**Corollary 4.33.** Let $m \geq 1$ and suppose that $\mathcal{A}$ is an ideal of $\mathcal{A}^{**}$ and has a b.a.i.. Then $\mathcal{A} \oplus X_0$ is $(2m + 1)$-weakly amenable if and only if it is weakly amenable.

View $\mathcal{A}$ as a left $\mathcal{A}$-module and then impose a trivial right $\mathcal{A}$-module action on it. This results in a Banach $\mathcal{A}$-bimodule. We denote it by $A_0$. Similar to Lemma 4.19, one can check that the following equalities hold

$$(u\phi)^\sim = u\bar{\phi}, \quad \phi u = 0, \quad \phi F = \bar{\phi} \bar{F},$$

$$F\phi = 0, \quad uF = 0, \quad (Fu)^\sim = \bar{F} u,$$

where $u \in \mathcal{A}^{(2m)}$, $\phi \in A_0^{(2m)}$, $F \in A_0^{(2m+1)}$ ($m \geq 0$).
**Proposition 4.34.** Let \( m \geq 0 \) and suppose that \( \mathfrak{A} \) is a \((2m + 1)\)-weakly amenable Banach algebra with a b.a.i.. Then \( \mathfrak{A} \oplus A_0 \) is \((2m + 1)\)-weakly amenable.

**Proof.** Condition 4' holds automatically. From Theorem 4.27, it suffices to show that condition 3' holds. In fact, let \( (x_\alpha) \subset A_0 \) such that \( (\bar{x}_\alpha) \) is a b.a.i. for \( \mathfrak{A} \). Then for any \( x \in A_0 \), we have \( x = \lim \bar{x}x_\alpha \). Let \( \bar{F} \) be a weak* cluster point of \( (\Gamma(x_\alpha)) \), where \( F \in A_0^{(2m+1)} \), then

\[
\Gamma(x) = \bar{x} \lim \Gamma(x_\alpha) = \bar{x} \bar{F} = xF.
\]

On the other hand

\[
0 = \lim \Gamma(x_\alpha a) = Fa \quad (a \in \mathfrak{A}).
\]

So condition 3' is fulfilled. The proof is complete. \( \square \)

Concerning \( 2m \)-weak amenability, we have the following.

**Proposition 4.35.** Let \( m \geq 1 \) and suppose that \( \mathfrak{A} \) is a commutative \( 2m \)-weakly amenable Banach algebra with a b.a.i.. Then \( \mathfrak{A} \oplus A_0 \) is also \( 2m \)-weakly amenable.

**Proof.** It suffices to prove that conditions 3'' and 4'' hold. Suppose that an \( \mathfrak{A} \)-bimodule morphism \( \Gamma: A_0 \rightarrow \mathfrak{A}^{(2m)} \) satisfies \( \Gamma(x)y = 0 \) in \( A_0^{(2m)} \), \( x, y \in A_0 \). Then

\[
0 = (\Gamma(x)y)^\sim = \Gamma(x)\bar{y} = \bar{y}\Gamma(x) = \Gamma(\bar{y}x) \quad (x, y \in A_0).
\]
This implies that $\Gamma(ax) = 0$ for $a \in \mathfrak{A}$, $x \in A_0$, and so $\Gamma(x) = 0$ for all $x \in A_0$ (notice that $\mathfrak{A}A_0 = A_0$). Therefore $\Gamma = 0$. Condition $3_0'$ holds.

For checking condition $4_0''$, assume that $T: A_0 \rightarrow A_0^{(2m)}$ is a continuous $\mathfrak{A}$-bimodule morphism. Let $v$ be a weak* cluster point of $(T(x_i))$, where $(\tilde{x}_i)$ is a b.a.i. for $\mathfrak{A}$, and let $u = \tilde{v}$. Then

$$T(x) = \lim T(\tilde{x}x_i) = \tilde{x}v.$$ 

While

$$(\tilde{x}v)^* = \tilde{x}\tilde{v} = \tilde{x}u = ux = (ux)^*.$$ 

Hence $T(x) = ux$. On the other hand, $ua = au$ since $\mathfrak{A}$ is commutative. The proof is complete. \hfill \Box

**Remark 4.36.** In the preceding proposition, if $\mathfrak{A}$ does not have an identity, then $\mathfrak{A} \oplus A_0$ is not 0-weakly amenable. In fact, in this case $4_0''$ does not hold for $m = 0$, because for the identity mapping $i: A_0 \rightarrow A_0$, the existence of $u \in \mathfrak{A}$ such that $au = ua$ for $a \in \mathfrak{A}$ and $\iota(x) = ux$ for $x \in A_0$ means that $\mathfrak{A}$ has an identity $u$. So if $\mathfrak{A}$ is a commutative weakly amenable Banach algebra with a b.a.i. but with no identity, then $\mathfrak{A} \oplus A_0$ is a permanently weakly amenable Banach algebra without being 0-weakly amenable. Another known example with this property is $L^1(G)$ with $G$ an infinite, compact, non-abelian group (see [17]).
Although we already have an example of a Banach algebra which is 
\((2m + 1)\)-weakly amenable but not \(2m\)-weakly amenable (Proposition 4.25; another known example is the nuclear algebra \(\mathcal{N}(E)\) with \(E\) a reflexive Banach space having the approximation property [17, Corollary 5.4]), we end this section by giving one more example of a weakly amenable Banach algebra which is not \(2\)-weakly amenable.

Suppose that \(\mathfrak{A}\) is a weakly amenable Banach algebra with a b.a.i. and satisfying that \(\mathfrak{A}\mathfrak{A}^* \neq \mathfrak{A}^*\mathfrak{A}\) (an example is \(\mathfrak{A} = L^1(G)\) with \(G\) a non-SIN locally compact group; see [58] and [55] for the reference of SIN groups, and [43, Theorem 32.44] as well as [56] for the property we need). Without loss of generality assume \(\mathfrak{A}\mathfrak{A}^* \not\subset \mathfrak{A}^*\mathfrak{A}\).

**Example 4.37.** For the above Banach algebra \(\mathfrak{A}\), \(\mathfrak{A} \oplus A_0\) is weakly amenable but is not \(2\)-weakly amenable.

**Proof.** From Proposition 4.34, \(\mathfrak{A} \oplus A_0\) is weakly amenable. We show that condition \(3''\) does not hold for \(m = 1\). Take \(\phi \in \mathfrak{A}^{**}\) satisfying \(\phi|_{\mathfrak{A}\mathfrak{A}^*} = 0\) but \(\phi|_{\mathfrak{A}^*\mathfrak{A}} \neq 0\) (notice that, from Theorem 1.1, \(\mathfrak{A}\mathfrak{A}^*\) is closed in \(\mathfrak{A}^*\)). Then \(\phi a = 0\) for all \(a \in \mathfrak{A}\) and \(a\phi \neq 0\) for some \(a \in \mathfrak{A}\). Let \(T: A_0 \to \mathfrak{A}^{**}\) be defined by

\[
T(x) = \tilde{x}\phi.
\]
It is easily verified that $T$ is a continuous $\mathfrak{A}$-bimodule morphism and $T \neq 0$.

Since

$$(T(x)y)^\sim = T(x)\tilde{y} = (\tilde{x}\phi)\tilde{y} = \tilde{x}(\phi\tilde{y}) = 0,$$

we have $T(x)y = 0$ for all $x, y \in A_0$. But $T \neq 0$, and so condition $3''$ is not satisfied for $m = 1$. Consequently $\mathfrak{A}$ is not 2-weakly amenable. \qed
CHAPTER 5

Weak Amenability does not imply 3-Weak Amenability

In this chapter we answer the open question: Does weak amenability imply 3-weak amenability? We first investigate the conditions of the weak amenability for Banach algebras of the form $\mathfrak{A} \oplus (X_1 + X_2)$. Then we give an example of a weakly amenable Banach algebra of this form which is not 3-weakly amenable.

5.1. Weak amenability of the algebra $\mathfrak{A} \oplus (X_1 + X_2)$

Suppose that $X_1$ and $X_2$ are two Banach $\mathfrak{A}$-bimodules. We denote by $X_1 + X_2$ the direct module sum of $X_1$ and $X_2$, i.e. the $l_1$ direct sum of $X_1$ and $X_2$ with the module actions given by

$$a(x_1, x_2) = (ax_1, ax_2), \quad (x_1, x_2)a = (x_1a, x_2a).$$

For this module it is not hard to check the following equality:

$$(x_1, x_2) \cdot (f_1^*, f_2^*) = x_1 f_1^* + x_2 f_2^* \quad ((x_1, x_2) \in X_1 + X_2, (f_1^*, f_2^*) \in (X_1 + X_2)^*).$$

In this section we study the weak amenability of the module extension Banach algebra

$$\mathfrak{A} \oplus (X_1 + X_2).$$
5.1. WEAK AMENABILITY OF THE ALGEBRA $\mathcal{A} \oplus (X_1 \oplus X_2)$

PROPOSITION 5.1. Suppose that $\mathcal{A} \oplus X_1$ and $\mathcal{A} \oplus X_2$ are weakly amenable. Then the following are equivalent.

(i) $\mathcal{A} \oplus (X_1 \oplus X_2)$ is weakly amenable;

(ii) there is no non-zero continuous $\mathcal{A}$-bimodule morphism $\gamma: X_1 \to X_2^*$;

(iii) there is no non-zero continuous $\mathcal{A}$-bimodule morphism $\eta: X_2 \to X_1^*$.

PROOF. We first prove that (i) implies (ii). Assume that $\mathcal{A} \oplus (X_1 \oplus X_2)$ is weakly amenable. Suppose that $\gamma: X_1 \to X_2^*$ is a continuous $\mathcal{A}$-bimodule morphism. Let $T: X_1 \oplus X_2 \to (X_1 \oplus X_2)^*$ be the continuous $\mathcal{A}$-bimodule morphism formulated by

$$T((x_1, x_2)) = (\gamma^*(x_2), \gamma(x_1)), \quad (x_1, x_2) \in X_1 \oplus X_2.$$ 

Then for $(x_1, x_2), (y_1, y_2) \in X_1 \oplus X_2$, and $a \in \mathcal{A},$

$$\langle a, (x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) \rangle$$

$$= \langle a, -x_1 \gamma^*(y_2) + x_2 \gamma(y_1) \rangle + \langle a, -\gamma^*(x_2) y_1 + \gamma(x_1) y_2 \rangle$$

$$= \langle a, -\gamma(x_1) y_2 + x_2 \gamma(y_1) \rangle + \langle a, -x_2 \gamma(y_1) + \gamma(x_1) y_2 \rangle$$

$$= 0$$

So $(x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) = 0$. Then from condition 4 of Theorem 4.12, $T = 0$. Thus $\gamma = 0$, showing that (ii) holds.

To prove that (ii) implies (iii), suppose that $\eta: X_2 \to X_1^*$ is a continuous $\mathcal{A}$-bimodule morphism. Then $\gamma: X_1 \to X_2^*$ defined by $\gamma = \eta^*|_{X_1}$ is a
continuous $\mathfrak{A}$-bimodule morphism. Therefore $\gamma = 0$, leading to that $\eta^* = 0$ since $\eta^*$ is wk*-wk* continuous and $X_1$ is weak* dense in $X_1^{**}$. Thus $\eta = 0$, showing that (iii) holds. Similarly, one can prove that (iii) implies (ii).

Finally, we prove that (ii) + (iii) implies (i). Because $\mathfrak{A} \oplus X_1$ and $\mathfrak{A} \oplus X_2$ are weakly amenable, condition 1-3 of Theorem 4.12 hold automatically for $X = X_1 \oplus X_2$ and $m = 0$. So we only need to show that condition 4 also holds. Suppose that $T: X \to X^*$ is a continuous $\mathfrak{A}$-bimodule morphism satisfying

$$(x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) = 0, \quad ((x_1, x_2), (y_1, y_2) \in X).$$

Let $P_i: X^* \to X_i^*$ be the natural projections and $\tau_i: X_i \to X$ be the natural embeddings, $i = 1, 2$. Then, by taking $x_2 = y_2 = 0$ and $x_1 = y_1 = 0$ separately, we have

$$x_1 \cdot P_1 \circ T \circ \tau_1(y_1) + P_1 \circ T \circ \tau_1(x_1) \cdot y_1 = 0,$$

$$x_2 \cdot P_2 \circ T \circ \tau_2(y_2) + P_2 \circ T \circ \tau_2(x_2) \cdot y_2 = 0,$$

for all $x_i, y_i \in X_i, i = 1, 2$. So we have $P_i \circ T \circ \tau_i = 0$, by applying condition 4 of Theorem 4.12 to the weakly amenable Banach algebras $\mathfrak{A} \oplus X_i$, $i = 1, 2$. Furthermore, (ii) and (iii) imply that $P_1 \circ T \circ \tau_2: X_2 \to X_1^*$ and $P_2 \circ T \circ \tau_1: X_1 \to X_2^*$ are trivial. These show that $T = 0$, i.e. condition 4 of Theorem 4.12 holds. Consequently (i) is true. This completes the proof. \[\square\]
Corollary 5.2. If $\mathfrak{A} \oplus (X_1 + X_2)$ is weakly amenable if and only if both $\mathfrak{A} \oplus X_1$ and $\mathfrak{A} \oplus X_2$ are weakly amenable and one of the conditions (ii) and (iii) in Proposition 5.1 holds.

Proof. If $\mathfrak{A} \oplus (X_1 + X_2)$ is weakly amenable, then conditions 1-4 of Theorem 4.12 hold for this algebra. It follows that these conditions also hold for the algebras $\mathfrak{A} \oplus X_1$ and $\mathfrak{A} \oplus X_2$. So they are also weakly amenable. The rest is clear from Proposition 5.1.

5.2. Weak amenability is not 3-weak amenability

Now we can construct an example of a weakly amenable Banach algebra which is not 3-weakly amenable. In this section $\mathcal{H}$ denotes an infinite dimensional separable Hilbert space. $B(\mathcal{H})$ denotes the Banach algebra of all the bounded operators on $\mathcal{H}$ and $K(\mathcal{H})$ the ideal of all the compact operators on $\mathcal{H}$. It is well known that with Arens product as multiplication, $K(\mathcal{H})^{**} = B(\mathcal{H})$ (see [57, pp. 103] for details).

Lemma 5.3. There is an element $a_0 \in B(\mathcal{H})$ such that $a_0 \notin K(\mathcal{H})$, $a_0$ is not right invertible in $B(\mathcal{H})$ and $a_0 K(\mathcal{H})$ is dense in $K(\mathcal{H})$.

Proof. Let $(e_i)_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$, and let $a_0 \in B(\mathcal{H})$ be defined by

$$a_0(e_i) = \begin{cases} 
\frac{1}{i} e_i & \text{if } i \text{ is even}; \\
 e_i & \text{if } i \text{ is odd}.
\end{cases}$$
5.2. WEAK AMENABILITY IS NOT 3-WEAK AMENABILITY

Clearly \( a_0 \notin K(\mathcal{H}) \). Also \( a_0 \) is neither right nor left invertible because any one-sided inverse of \( a_0 \) must satisfy

\[
a_0^{-1}(e_i) = \begin{cases} 
  ie_i & \text{if } i \text{ is even;} \\
  e_i & \text{if } i \text{ is odd},
\end{cases}
\]

which can not be a bounded operator.

For any \( n \geq 1 \), denote by \( V_n \) the subspace of \( \mathcal{H} \) generated by \( \{e_1, e_2, \ldots, e_n\} \), and let \( P_n \) be the orthogonal projection from \( \mathcal{H} \) to \( V_n \). Then from Corollary II.4.5 of [14], for any \( k \in K(\mathcal{H}) \) and \( \varepsilon \geq 0 \), there is \( n = n(k, \varepsilon) \), such that

\[
\|P_n \circ k - k\| < \varepsilon.
\]

For this \( n = n(k, \varepsilon) \), let \( b_n \in B(\mathcal{H}) \) be defined by

\[
b_n(e_i) = \begin{cases} 
  ie_i & \text{if } i \leq n \text{ and } i \text{ is even;} \\
  e_i & \text{if } i \leq n \text{ and } i \text{ is odd;} \\
  0 & \text{for } i \geq n.
\end{cases}
\]

Then \( a_0 \circ b_n = P_n \) and \( a_0 \circ b_n \circ P_n = P_n^2 = P_n \). Let \( k_n = b_n \circ P_n \circ k \). Then \( k_n \in K(\mathcal{H}) \), and \( a_0 \circ k_n = P_n \circ k \). Also

\[
\|a_0 \circ k_n - k\| = \|P_n \circ k - k\| < \varepsilon.
\]

Since \( k \in K(\mathcal{H}) \) and \( \varepsilon \geq 0 \) are arbitrarily given, this shows that \( a_0 K(\mathcal{H}) \) is dense in \( K(\mathcal{H}) \). The proof is complete. \qed
For the element $a_0$ in the above lemma, $a_0B(\mathcal{H})$ is a proper right ideal of $B(\mathcal{H})$ since the identity $1 \notin a_0B(\mathcal{H})$. The closure of $a_0B(\mathcal{H})$ is also a proper right ideal of $B(\mathcal{H})$ ([4, pp. 46]). So there is $F \in B(\mathcal{H})^*$ such that $F \neq 0$ but $Fa_0 = 0$. Then $FB(\mathcal{H}) \neq \{0\}$ is a right $B(\mathcal{H})$-submodule of $B(\mathcal{H})^*$. Take

$$X_0 = (K(\mathcal{H}))_0, \text{ and } \circ Y = \text{cl}(FB(\mathcal{H})).$$

Then we have the following example.

**Example 5.4.** $B(\mathcal{H}) \oplus (X_0 + \circ Y)$ is weakly amenable but not 3-weakly amenable.

**Proof.** Clearly we have $B(\mathcal{H})X_0 = X_0$ and $\circ YB(\mathcal{H}) = \circ Y$. By corollaries 4.29 and 4.30 each of the Banach algebras $B(\mathcal{H}) \oplus X_0$ and $B(\mathcal{H}) \oplus \circ Y$ is weakly amenable.

Suppose that $T: \circ Y \rightarrow X_0^*$ is a continuous $B(\mathcal{H})$-bimodule morphism. We prove that $T$ is trivial. Let $f = T(F)$. Then

$$fa_0 = T(Fa_0) = 0$$

and so

$$\langle a_0K(\mathcal{H}), f \rangle = \{0\}.$$ 

Therefore $f = 0$ since $a_0K(\mathcal{H})$ is dense in $K(\mathcal{H})$. This shows that $T(F) = 0$ and hence $T(FB(\mathcal{H})) = \{0\}$. Thus $T = 0$. From Corollary 5.2, $B(\mathcal{H}) \oplus (X_0 + \circ Y)$
is weakly amenable. Next we show that $B(\mathcal{H}) \oplus (X_0 \oplus \mathcal{Y})$ fails condition 4 of Theorem 4.12 for $m = 1$. Since

$$(X_0)^{***} = \vartheta(K(\mathcal{H})^{***}) = \vartheta(B(\mathcal{H})^*) \supset \vartheta \mathcal{Y},$$

we see that there exists a non-zero $B(\mathcal{H})$-bimodule morphism from $\vartheta \mathcal{Y}$ into $(X_0)^{***}$ (e.g. the inclusion mapping). Let $\tau: \vartheta \mathcal{Y} \to (X_0)^{***}$ be such a morphism, and let $\Delta: (X_0)^{***} \to (X_0)^*$ be the projection with the kernel $X_0^\perp$. By taking $T = \Delta \circ \tau: \vartheta \mathcal{Y} \to X_0^*$, and using the result of the preceding paragraph, we have that $T = 0$. So we have

$$\langle x, \tau(y) \rangle = 0, \quad (y \in \vartheta \mathcal{Y}, \, x \in X_0).$$

Now let $\Gamma: X_0 \oplus \vartheta \mathcal{Y} \to (X_0 \oplus \vartheta \mathcal{Y})^{***}$ be the continuous $B(\mathcal{H})$-bimodule morphism defined by

$$\Gamma((x, y)) = (\tau(y), 0).$$

Then for $(x_1, y_1), (x_2, y_2) \in X_0 \oplus \vartheta \mathcal{Y}, \, u \in B(\mathcal{H})^{**}$,

$$\langle u, (x_1, y_1) \cdot \Gamma((x_2, y_2)) + \Gamma((x_1, y_1)) \cdot (x_2, y_2) \rangle$$

$$= \langle (u \cdot x_1, 0), (\tau(y_2), 0) \rangle + \langle (0, y_2 \cdot u), (\tau(y_1), 0) \rangle$$

$$= \langle u \cdot x_1, \tau(y_2) \rangle = \langle ux_1, \tau(y_2) \rangle = 0.$$

Here we used the fact $u \cdot x_1 = ux_1 \in X_0$ (see Remark 3.4). So

$$(x_1, y_1) \cdot \Gamma((x_2, y_2)) + \Gamma((x_1, y_1)) \cdot (x_2, y_2) = 0.$$
for all \((x_1, y_1), (x_2, y_2) \in X_0 +_o Y\). But \(\Gamma \neq 0\), showing that condition 4 of Theorem 4.12 does not hold for \(m = 1\). Consequently, \(B(\mathcal{H}) \oplus (X_0 +_o Y)\) is not 3-weakly amenable.
CHAPTER 6

Contractible and Reflexive Amenable Banach Algebras

It is a simple fact that any finite dimensional semi-simple Banach algebra is contractible and, of course, (reflexive) amenable (see Assertion I.3.68 and Theorem VII.1.74 in [39]; for the definition of semi-simple Banach algebra see Definition 6.8 on page 83). The converse is a challenging open problem. We have mentioned the related references and made a brief comment on the known results concerning this problem in Chapter 1. This chapter includes two parts. In Section 1 we explore Taylor's method developed in [67]. In Section 2 we treat the above problem from the viewpoint of some properties of maximal ideals.

6.1. Approximation property and contractible Banach algebras

Let us first recall some basic concepts of functional analysis.

DEFINITION 6.1. A Banach space $X$ is said to have the (compact) approximation property, AP (resp. CAP) in short, if for each compact set $K \subseteq X$ and each $\varepsilon > 0$ there is a finite rank (resp. compact) continuous operator $T$ on $X$ so that $\|Tx - x\| < \varepsilon$ for all $x \in K$. If $T$ can be chosen so that $\|T\| \leq M$ for some constant $M > 0$ independent of $K$ and $\varepsilon$, then $X$
is said to have the \textit{bounded (compact) approximation property}, BAP (resp. BCAP) in short.

Briefly speaking, \( X \) has the AP if the identity operator on it can be uniformly approximated on every compact subset by finite rank continuous operators, and it has the CAP if can be uniformly approximated on every compact subset by compact operators. If the norm of the operators can be chosen to be bounded by a fixed positive number, then \( X \) has the BAP (resp. BCAP). In Chapter 7 we will discuss more aspects of the approximation property. Here we just point out that all Banach spaces with a basis enjoy the BAP and the only naturally occurring Banach space known to lack this property is \( B(\mathcal{H}) \) with \( \mathcal{H} \) an infinite dimensional Hilbert space. We refer to [53] and [57] for more details about these concepts.

It is not hard to see that \( X \) has the BAP (BCAP) if and only if there exists a number \( M > 0 \) and a net \( \{\varphi_\lambda\} \) of finite rank (resp. compact) operators on \( X \) such that \( \text{||}\varphi_\lambda\text{||} \leq M \) for all \( \lambda \) and \( \varphi_\lambda(x) \to x \) for all \( x \in X \).

Suppose that \( X \) and \( Y \) are Banach spaces. We denote the projective tensor product of \( X \) and \( Y \) by \( X \hat{\otimes} Y \), and denote the elementary tensor of \( x \in X \) and \( y \in X \) by \( x \otimes y \); we refer to [63] for the details about this kind of product space. Suppose that \( \mathfrak{A} \) is a Banach algebra. Then \( \mathfrak{A} \hat{\otimes} \mathfrak{A} \) is naturally a Banach \( \mathfrak{A} \)-bimodule with the module multiplications specified by

\[
a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in \mathfrak{A}).
\]
We use \( \pi \) to denote the bounded \( \mathfrak{A} \)-bimodule morphism from \( \mathfrak{A} \otimes \mathfrak{A} \) into \( \mathfrak{A} \) specified by \( \pi(a \otimes b) = ab, \ a, b \in \mathfrak{A} \).

**Definition 6.2.** An element \( m \in \mathfrak{A} \otimes \mathfrak{A} \) is a **diagonal** for \( \mathfrak{A} \) if for all \( a \in \mathfrak{A} \), \( a \cdot m = m \cdot a \), and \( \pi(m) \) is an identity of \( \mathfrak{A} \). A bounded net \( (m_\alpha) \) of \( \mathfrak{A} \otimes \mathfrak{A} \) is an **approximate diagonal** for \( \mathfrak{A} \) if \( am_\alpha - m_\alpha a \to 0 \) and \( \pi(m_\alpha)a \to a \) in norm for all \( a \in \mathfrak{A} \).

It is a well-known fact that \( \mathfrak{A} \) is contractible if and only if there is a diagonal for it [39, Assertion VII.1.72], and \( \mathfrak{A} \) is amenable if and only if there is an approximate diagonal for it [47, Theorem 1.3]. From these results we notice that a contractible Banach algebra has an identity, which will be denoted by \( 1 \), and an amenable Banach algebra has a b.a.i.

Taylor proved in [67] that a contractible Banach algebra with the BAP is finite dimensional. His method can actually yield more as shown in the following theorem.

**Theorem 6.3.** Suppose that \( \mathfrak{A} \) is a contractible Banach algebra. If \( \mathfrak{A} \) has the BCAP, then it is finite dimensional.

**Proof.** Suppose that \( u = \sum_{i=1}^\infty a_i \otimes b_i \) is a diagonal for \( \mathfrak{A} \), where \( a_i, b_i \in \mathfrak{A} \) satisfy \( \sum_{i=1}^\infty \|a_i\|\|b_i\| < \infty \). Suppose that \( \{(\varphi_\lambda) : \lambda \in \Lambda\} \) is a net of compact operators on \( \mathfrak{A} \) such that \( \|\varphi_\lambda\| < M, \ \lambda \in \Lambda \), for some \( M > 0 \), and \( \varphi_\lambda(x) \to x \)
for all $x \in X$. For each $\lambda \in \Lambda$, define $\Phi_\lambda: \mathcal{A} \to \mathcal{A}$ by

$$
\Phi_\lambda(a) = \sum_{i=1}^{\infty} a_i \varphi_\lambda(b_i a) \quad (a \in \mathcal{A}).
$$

Then $\Phi_\lambda$ is a bounded operator on $\mathcal{A}$. Denote by $\phi_{\lambda,n}$ the $n$th partial sum of the series in (6.1), i.e.

$$
\phi_{\lambda,n}(a) = \sum_{i=1}^{n} a_i \varphi_\lambda(b_i a) \quad (a \in \mathcal{A}).
$$

Then $\phi_{\lambda,n}$ is a compact operator on $\mathcal{A}$. For fixed $\lambda$ we have

$$
\lim_{n} \phi_{\lambda,n} = \Phi_\lambda,
$$

in the uniform topology of operators. This shows that $\Phi_\lambda$ is compact. For any $x \in \mathcal{A}$, we have

$$
\lim_{\lambda} \Phi_\lambda(a) = \sum_{i=1}^{\infty} a_i (\lim_{\lambda} \varphi_\lambda(b_i a)) = \sum_{i=1}^{\infty} a_i b_i a = \pi(u)a = a.
$$

Here the first equality holds because the series converges uniformly. So $\Phi_\lambda$ converges pointwise to $I_\mathcal{A}$, the identity mapping on $\mathcal{A}$. On the other hand the bilinear mapping $(a, b) \mapsto a \varphi_\lambda(b)$ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ defines a bounded operator $\Psi_\lambda: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ satisfying $\Psi_\lambda(a \otimes b) = a \varphi_\lambda(b)$. We have

$$
\Phi_\lambda(a) = \Psi_\lambda(u \cdot a), \quad (a \in \mathcal{A}).
$$

Since $a \cdot u = u \cdot a$ we have for any $a \in \mathcal{A}$ that

$$
\Phi_\lambda(a) = \Psi_\lambda(a \cdot u) = \sum_{i=1}^{\infty} a a_i \varphi_\lambda(b_i) = a \sum_{i=1}^{\infty} a_i \varphi_\lambda(b_i) = a \Phi_\lambda(1).
$$
6.2. MAXIMAL IDEALS AND DIMENSIONS

It follows that $\Phi_\Lambda$ converges to $I_\mathfrak{A}$ in the uniform norm of operators (notice that $\mathfrak{A}$ has an identity). This shows that $I_\mathfrak{A}$ is compact. Consequently $\mathfrak{A}$ is finite dimensional. \qed

The same method can also prove the following result. We omit the proof to avoid redundancy.

**Proposition 6.4.** Suppose that $\mathfrak{A}$ is a contractible Banach algebra. If $\mathfrak{A}$ has a diagonal $u$ which has a finite sum representation $u = \sum_{i=1}^{n} a_i \otimes b_i$, then $\mathfrak{A}$ is finite dimensional.

**Remark 6.5.** It is known that amenable C*-algebras have the BAP ([10]). So contractible C*-algebras are finite dimensional ([65]).

**Remark 6.6.** A Banach space having a basis has the BAP. So a contractible Banach algebra with the underlying space having a basis is finite dimensional. Especially, a contractible Banach algebra with the underlying space isomorphic to $\ell^p(S)$ ($p \geq 1$) with $S$ a non-empty set is finite dimensional.

### 6.2. Maximal ideals and dimensions

If the underlying Banach space of an amenable Banach algebra is reflexive, we call it a reflexive Banach algebra. We note that both contractible and reflexive amenable Banach algebras have an identity. In this section an identity will simply be denoted by $1$. Suppose that $X$, $Y$, $Z$ are left (right)
Banach $\mathfrak{A}$-modules and $f: X \to Y$, $g: Y \to Z$ are continuous left (resp. right) $\mathfrak{A}$-module morphisms. The short sequence

$$
\sum : \ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0
$$

is exact if $f$ is injective, $g$ is surjective and $\text{Im}(f) = \text{Ker}(g)$. The exact sequence $\sum$ is *admissible* if $f$ has a continuous left inverse; and $\sum$ *splits* if $f$ has a continuous $\mathfrak{A}$-module morphism left inverse. We need the following known result.

**Lemma 6.7 (Theorem 6.1 in [16]).** Suppose that $\mathfrak{A}$ is a contractible Banach algebra. Then every admissible short exact sequence of left or right Banach $\mathfrak{A}$-modules splits.

We need also to recall some other general concepts. Suppose that $\mathfrak{A}$ is a Banach algebra. A proper left, right, or two-sided ideal of $\mathfrak{A}$ is *maximal* if there is no other proper left, right, or two-sided ideal of $\mathfrak{A}$ containing it, and is *minimal* if it contains no left, right, or two-sided ideal of $\mathfrak{A}$ other than $\{0\}$ and itself. In a Banach algebra with an identity a maximal left, right, or two-sided ideal is always closed.

For $a \in \mathfrak{A}$, the limit

$$
r(a) = \lim_{n \to \infty} \|a^n\|^\frac{1}{n}
$$

always exists [4, Proposition 2.8]. $r(a)$ is called the *spectral radius* of $a$. The *radical* of an algebra is defined algebraically as the intersection of all modular
maximal left ideals (when the algebra has an identity, it is the intersection of all maximal left ideals). However for Banach algebras we can adopt

**Definition 6.8.** The *radical* of a Banach algebra $\mathfrak{A}$ is

$$rad(\mathfrak{A}) = \{ q \in \mathfrak{A}; \ r(aq) = 0 \text{ for all } a \in \mathfrak{A} \}.$$ 

If $rad(\mathfrak{A}) = \{0\}$, then $\mathfrak{A}$ is said to be *semi-simple*.

A Banach algebra is called *semi-prime* if $\{0\}$ is the only ideal $J$ of it with $J^2 = \{0\}$. The following basic facts are known:

(i) Semi-simple Banach algebras are semi-prime [4, Proposition 30.5];

(ii) For a Banach algebra $\mathfrak{A}$, $rad(\mathfrak{A})$ is a closed ideal of $\mathfrak{A}$;

(iii) $\mathfrak{A}/rad(\mathfrak{A})$ is always semi-simple.

A *division algebra* is an algebra with an identity in which any non-zero element has an inverse. It is known that a normed division algebra is always finite dimensional (see [4, Section 14]). A non-zero element $e$ of a Banach algebra $\mathfrak{A}$ is a *minimal idempotent* if $e$ is an idempotent (i.e. $e^2 = e$) and $e\mathfrak{A}e$ is a division algebra. A well-known fact concerning minimal idempotents is that for any two minimal idempotents $e_1$ and $e_2$ in $\mathfrak{A}$, the dimension of $e_1\mathfrak{A}e_2$ is either 0 or 1, provided $\mathfrak{A}$ is semi-prime [4, Theorem 31.6]. There is a link between minimal idempotents and minimal left (right) ideals described in the following lemma.
6.2. MAXIMAL IDEALS AND DIMENSIONS

LEMMA 6.9 (Lemma 30.2 in [4]). Suppose that $L$ is a minimal left (right) ideal of $\mathfrak{A}$ such that $L^2 \neq \{0\}$. Then there exists an idempotent $e \in L$ such that $\mathfrak{A}e = L$ (resp. $e\mathfrak{A} = L$); and each idempotent $e$ with $\mathfrak{A}e = L$ (resp. $e\mathfrak{A} = L$) is minimal.

For a subset $E$ of $\mathfrak{A}$, the left annihilator and the right annihilator of $E$ are, respectively, the following sets

$$\text{lan}(E) = \{a \in \mathfrak{A}; aE = \{0\}\},$$
$$\text{ran}(E) = \{a \in \mathfrak{A}; Ea = \{0\}\}.$$

For an element $a \in \mathfrak{A}$, $\text{lan}\{a\}$ and $\text{ran}\{a\}$ will be simply denoted by $\text{lan}(a)$ and $\text{ran}(a)$ respectively. It is easy to see the following.

LEMMA 6.10. Suppose that $\mathfrak{A}$ is a Banach algebra having an identity $1$. Then for any $a \in \mathfrak{A}$,

(i) $\text{ran}(\mathfrak{A}(1-a)) = \text{ran}(1-a) = \{x \in \mathfrak{A}; ax = x\},$

(ii) $\text{lan}((1-a)\mathfrak{A}) = \text{lan}(1-a) = \{x \in \mathfrak{A}; xa = x\}.$

LEMMA 6.11. Suppose that $\mathfrak{A}$ is a contractible or a reflexive amenable Banach algebra. Let $L$ be a closed proper left (right) ideal of $\mathfrak{A}$. If $L$ is complemented in $\mathfrak{A}$, then $\text{ran}(L)$ (resp. $\text{lan}(L)$) contains an idempotent $e$ such that the following hold.

(i) $\text{ran}(L) = e\mathfrak{A}$ (resp. $\text{lan}(L) = \mathfrak{A}e$);

(ii) $L = \mathfrak{A}(1-e)$ (resp. $L = (1-e)\mathfrak{A}$);
(iii) If in addition, \( L \) is a maximal left (resp. right) ideal, then \( e \) is a minimal idempotent and \( \mathfrak{A} e \) (resp. \( e \mathfrak{A} \)) is a minimal left (resp. right) ideal.

**Proof.** We prove the case when \( L \) is a left ideal. To prove (i) and (ii) we can assume \( L \neq \{0\} \), for otherwise \( e = 1 \) satisfies the requirements. First we show that \( L \) contains a right identity \( \mu \). Then it is clear that \( \mu \) is an idempotent and \( L = \mathfrak{A} \mu \).

If \( \mathfrak{A} \) is contractible, we consider the following exact short sequence of left \( \mathfrak{A} \)-modules:

\[
\sum : \quad 0 \longrightarrow L \overset{i}{\longrightarrow} \mathfrak{A} \overset{q}{\longrightarrow} \mathfrak{A}/L \longrightarrow 0,
\]

where \( i \) is the inclusion mapping and \( q \) is the quotient mapping. Since \( L \) is complemented, \( \sum \) is admissible. From Lemma 6.7 \( \sum \) is a splitting sequence, i.e. there is a left \( \mathfrak{A} \)-module morphism \( \delta : \mathfrak{A} \rightarrow L \), such that \( \delta \circ i = I_L \), the identity operator on \( L \). Then \( \mu = \delta(1) \) is obviously a right identity of \( L \).

If \( \mathfrak{A} \) is reflexive and amenable, then according to Theorem 1.3 \( L \) contains a right b.a.i., say \( (l_\alpha) \). Since \( \mathfrak{A} \) is reflexive, as a closed subspace of \( \mathfrak{A} \), \( L \) is also reflexive. Then a weak* cluster point \( \mu \) of \( (l_\alpha) \) in \( L \) is a right identity of \( L \).

Now let \( e = 1 - \mu \). Then \( e \in \text{ran}(L) \), \( e \neq 0 \) and \( e \) is also an idempotent. Since \( L = \mathfrak{A} \mu = \mathfrak{A}(1-e) \), from Lemma 6.10, \( \text{ran}(L) = \{x \in \mathfrak{A}; ex = x\} = e \mathfrak{A} \). This proves the first two statements of this lemma.
Now suppose that $L$ is a maximal left ideal. Then since $\mathfrak{A}$ is the direct sum of $\mathfrak{A}(1 - e)$ and $\mathfrak{A}e$ we have that $\mathfrak{A}e$ is a minimal left ideal of $\mathfrak{A}$. By Lemma 6.9, $e$ is a minimal idempotent.

**Lemma 6.12.** Suppose that $\mathfrak{A}$ is a contractible or a reflexive amenable Banach algebra. If $\mathfrak{A}/\text{rad}(\mathfrak{A})$ is finite dimensional, then so is $\mathfrak{A}$ and $\mathfrak{A}$ is semi-simple.

**Proof.** If $\mathfrak{A}/\text{rad}(\mathfrak{A})$ is finite dimensional, then $\text{rad}(\mathfrak{A})$ is a complemented closed ideal of $\mathfrak{A}$. From Lemma 6.11 $\text{rad}(\mathfrak{A})$ contains an idempotent which is non-zero if $\text{rad}(\mathfrak{A}) \neq \{0\}$. But $\text{rad}(\mathfrak{A})$ can never have a non-zero idempotent from the definition. So $\text{rad}(\mathfrak{A}) = \{0\}$.

It is known that if a Banach algebra is contractible or amenable, then its continuous algebra homomorphic image is also contractible or amenable (see [39, Proposition VII.1.71] and [49, Proposition 5.3]). This leads to part of the following lemma.

**Lemma 6.13.** Suppose that $\mathfrak{A}$ is a Banach algebra. Then the following statements hold.

(i) If $\mathfrak{A}$ is contractible, or reflexive and amenable, then so is $\mathfrak{A}/\text{rad}(\mathfrak{A})$;

(ii) An ideal $M \subset \mathfrak{A}/\text{rad}(\mathfrak{A})$ is a maximal ideal if and only if $q^{-1}(M)$ is a maximal ideal in $\mathfrak{A}$, where $q: \mathfrak{A} \to \mathfrak{A}/\text{rad}(\mathfrak{A})$ is the quotient mapping;
(iii) If \( L \) is a maximal left (right) ideal of \( \mathfrak{A} \) containing \( \text{rad}(\mathfrak{A}) \), then \( q(L) \) is a maximal left (resp. right) ideal of \( \mathfrak{A}/\text{rad}(\mathfrak{A}) \). If \( L \) is complemented in \( \mathfrak{A} \), so is \( q(L) \) in \( \mathfrak{A}/\text{rad}(\mathfrak{A}) \).

**Proof.** The first statement is clear. Checking of the second one is also routine. Suppose that \( L \) is a left (right) ideal of \( \mathfrak{A} \) and \( \text{rad}(\mathfrak{A}) \subset L \). Then it is easily verified that \( L = q^{-1}(q(L)) \). So \( q(L) \) is maximal in \( \mathfrak{A}/\text{rad}(\mathfrak{A}) \) whenever \( L \) is maximal in \( \mathfrak{A} \). If \( L \) is complemented in \( \mathfrak{A} \), then there is a closed complement, say \( J \), of \( L \) in \( \mathfrak{A} \). The image \( q(J) \) is also closed and

\[ q(L) + q(J) = \mathfrak{A}/\text{rad}(\mathfrak{A}). \]

If \( m \in q(L) \cap q(J) \), then for some \( j \in J \) and \( l \in L \), \( m = q(j) = q(l) \). Hence \( j - l \in \text{rad}(\mathfrak{A}) \subset L \). So \( j \in L \). This shows that \( j = 0 \) and hence \( m = 0 \). Therefore \( q(J) \) is a complement of \( q(L) \) in \( \mathfrak{A}/\text{rad}(\mathfrak{A}) \). Thus \( q(L) \) is complemented in \( \mathfrak{A}/\text{rad}(\mathfrak{A}) \).

**Theorem 6.14.** Suppose that \( \mathfrak{A} \) is a contractible or a reflexive amenable Banach algebra. If each maximal ideal of \( \mathfrak{A} \) is contained in either a maximal left ideal or a maximal right ideal of \( \mathfrak{A} \) which is complemented in \( \mathfrak{A} \), then \( \mathfrak{A} \) is finite dimensional.

**Proof.** From Lemmas 6.12 and 6.13 we can assume that \( \mathfrak{A} \) is semi-simple. We can also assume that \( \mathfrak{A} \) is not a division algebra. Then \( \mathfrak{A} \) contains at least one maximal ideal and hence has at least one maximal
left or maximal right ideal which is complemented in \( \mathfrak{a} \). By Lemma 6.11, \( \mathfrak{a} \) has at least one minimal idempotent. Let \( E \) be the set of all minimal idempotents of \( \mathfrak{a} \), and let \( J \) be the ideal generated by \( E \) (called the socle of \( \mathfrak{a} \)). We prove \( J = \mathfrak{a} \).

If \( J \neq \mathfrak{a} \), then there is a maximal ideal \( M \) containing \( J \), since \( \mathfrak{a} \) has an identity. Then by assumption, there is either a maximal left ideal or a maximal right ideal of \( \mathfrak{a} \) which contains \( M \) and is complemented in \( \mathfrak{a} \). Assume the former is true and the corresponding left ideal is \( L \). Then from Lemma 6.11, \( \text{ran}(L) \neq \{0\} \) and contains a minimal idempotent \( e \) such that \( L = \mathfrak{a}(1 - e) \). So we would have \( Ee = \{0\} \) and \( e \in E \). This implies that \( e = e^2 = 0 \), a contradiction.

Therefore \( J = \mathfrak{a} \). Then the identity \( 1 \) of \( \mathfrak{a} \) can be expressed as

\[
1 = \sum_{i=1}^{n} a_i e_i b_i,
\]

where \( e_i, i = 1, 2, ..., n \), are minimal idempotents, and \( a_i, b_i \in \mathfrak{a} \). We then have

\[
\mathfrak{a} = 1 \cdot \mathfrak{a} \cdot 1 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i e_i b_i \mathfrak{a} a_j e_j b_j
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i (e_i \mathfrak{a} e_j) b_j.
\]

Since each space \( e_i \mathfrak{a} e_j \) has a dimension of at most one, each subspace \( a_i (e_i \mathfrak{a} e_j) b_j \) has a dimension of at most one. It follows that \( \mathfrak{a} \) is finite dimensional. This completes the proof. \( \square \)
Recall that a simple algebra is an algebra which has no proper ideals other than the zero ideal.

**Corollary 6.15.** Suppose that $𝒜$ is a contractible or a reflexive amenable Banach algebra. If $𝒜$ is also a simple algebra and has a maximal left or right ideal which is complemented in $𝒜$, then $𝒜$ has a finite dimension.

**Proof.** In this case, $\{0\}$ is the only maximal ideal and is contained in a maximal left or right ideal which is complemented in $𝒜$ by the assumption. \qed

**Corollary 6.16.** Suppose that $𝒜$ is a contractible or a reflexive amenable Banach algebra. If every maximal ideal in $𝒜$ is either a maximal left or a maximal right ideal, then $𝒜$ has a finite dimension.

**Proof.** For each maximal ideal $M$, $𝒜/M$ is a simple Banach algebra. Since $M$ itself is a maximal left or maximal right ideal in $𝒜$, $𝒜/M$ has either no non-zero proper left ideals or no non-zero proper right ideals, meaning that $\{0\}$ is either a maximal left or a maximal right ideal in $𝒜/M$ which is certainly complemented. From the preceding corollary, $𝒜/M$ is of finite dimension. Then any subspace containing $M$ is finite codimensional and hence complemented in $𝒜$. This is true for every maximal ideal $M$ and so, from Theorem 6.14, $𝒜$ is finite dimensional. \qed

**Corollary 6.17** (Johnson [44]). Suppose that $𝒜$ is a reflexive amenable Banach algebra. If every maximal left ideal of $𝒜$ is complemented in $𝒜$, then
\( \mathcal{A} \) is finite dimensional. In particular, an amenable Banach algebra with the underlying space a Hilbert space is finite dimensional (Ghahramani, Loy and Willis [26]).

**Proof.** Since \( \mathcal{A} \) has an identity, every maximal ideal of \( \mathcal{A} \) is contained in a maximal left ideal of \( \mathcal{A} \). Hence by Theorem 6.14, \( \mathcal{A} \) is finite dimensional. \( \square \)

**Corollary 6.18** (Runde [61]). Suppose that \( \mathcal{A} \) is a reflexive amenable Banach algebra. If for every maximal ideal \( M \), \( \mathcal{A}/M \) has a finite dimension, then \( \mathcal{A} \) has a finite dimension.

**Proof.** As pointed out in the proof of the preceding corollary, each maximal ideal \( M \) is contained in a maximal left ideal \( N \) of \( \mathcal{A} \). Since \( \mathcal{A}/M \) has a finite dimension, \( N \) is finite codimensional and hence complemented in \( \mathcal{A} \). From Theorem 6.14 we then have the result. \( \square \)
CHAPTER 7

Approximate Complementability and Nilpotent Ideals

An ideal \( \mathcal{N} \) in a Banach algebra is called \textit{nilpotent} if there is an integer \( m \) such that \( \mathcal{N}^m = \{0\} \), where \( \mathcal{N}^m = \{n_1n_2\cdots n_m; \ n_i \in \mathcal{N}, i = 1, 2, \cdots, m\} \).

It is known that non-zero nilpotent ideals in amenable Banach algebras, if any, must be infinite dimensional [16], and there do exist amenable Banach algebras which contain non-zero nilpotent ideals (see [54] and [68]). These properties lead to the following natural question: Can a contractible Banach algebra contain a non-zero nilpotent ideal? We see that this is impossible if \( \mathfrak{A} \) has the BCAP since for this case we have shown in Chapter 6 that \( \mathfrak{A} \) is a finite dimensional (amenable) algebra (Theorem 6.3). In [54] Loy and Willis have proved that if a biprojective Banach algebra has a central approximate identity, then no non-zero nilpotent ideal can have the AP. In Section 7.1 of this chapter we introduce the concept of approximately complemented subspaces and then discuss the relationship between this concept and the known ones such as weak complementability and the approximation property. In Section 7.2 we introduce a new class of Banach algebras called the approximately biprojective Banach algebras. Then in Section 7.3 we will improve Loy’s and Willis’ result above by showing that a non-zero nilpotent ideal in an approximately biprojective Banach algebra with both left and
right approximate identities can not be approximately complemented. We will use some of the techniques of [54].

7.1. Approximately Complemented Subspaces

We first introduce the following:

**Definition 7.1.** Suppose that $E$ is a subspace of a normed space $X$. Then $E$ is *approximately complemented* in $X$ if for each compact subset $K$ of $E$ and each $\epsilon > 0$, there is a continuous operator $P : X \to E$ such that $\|x - P(x)\| < \epsilon$, for all $x \in K$.

The following examples explain the use of the terminology “approximately complemented”. First, if a subspace $E$ of $X$ is such that for every compact subset $K$ of $E$ there is a subspace $N$, complemented in $X$, such that $K \subseteq N \subseteq E$, then $E$ is approximately complemented. In particular, every complemented subspace is approximately complemented. A subspace with the AP is also an approximately complemented subspace because every finite rank bounded operator on the subspace can be extended to the whole of the space. Another interesting example is that when $E$ is a right (left) ideal of a Banach algebra $\mathfrak{A}$ and has a left (resp. right) b.a.i., then $E$ is approximately complemented. In fact, suppose that $(e_\alpha)$ is a left b.a.i. of the right ideal $E$. Then for each compact $K \subseteq E$ and $\epsilon > 0$, there is an index $\alpha$ such that $\|x - e_\alpha x\| < \epsilon$ for all $x \in K$. So $P : X \to E$ defined by $P(x) = e_\alpha x$ satisfies the requirement in Definition 1. For $C^*$-algebras, it is well known
that every closed left ideal has a right b.a.i. (see [50, Proposition 4.2.12]).

This shows that closed left ideals of a $C^*$-algebra are approximately complemented. For every Banach space $X$, let $B(X)$ be the Banach algebra of all bounded operators on $X$ with the multiplication of the operator composition, and let $K(X)$ be the ideal of all compact operators on $X$. From [57, Theorem 5.1.12(d)], $K(X)$ has a left b.a.i. if $X$ has the BCAP. So $K(X)$ is approximately complemented in $B(X)$ if $X$ has this property. It is worth mentioning that $K(X)$ is not complemented in $B(X)$ when $X$ is an infinite dimensional Hilbert space (see [13]). For the theory concerning the existence of left (right) b.a.i. in ideals of $B(X)$ we refer to [21] and [62]. Also we refer to [34] for concepts related to the approximate complementability we just defined.

Suppose that $X$ and $Y$ are normed spaces. Denote by $X \hat{\otimes} Y$ (resp. $X \check{\otimes} Y$) the projective (resp. injective) tensor product of $X$ and $Y$. We will use $\| \cdot \|_{X,Y}$ to denote the projective tensor norm in case any confusion may occur.

Suppose that $X_1$, $X_2$, $Y_1$ and $Y_2$ are normed spaces, and $T_i$: $X_i \rightarrow Y_i$, $i = 1, 2$, are bounded operators. We denote by $T_1 \otimes T_2$: $X_1 \hat{\otimes} X_2 \rightarrow Y_1 \hat{\otimes} Y_2$ the linear operator specified by

$$(T_1 \otimes T_2)(a_1 \otimes a_2) = T_1(a_1) \otimes T_2(a_2), \quad (a_1 \in X_1, a_2 \in X_2).$$

Clearly, $T_1 \otimes T_2$ is continuous and $\| T_1 \otimes T_2 \| \leq \| T_1 \| \| T_2 \|$. If

$$A_1 \overset{T_1}{\longrightarrow} B_1 \overset{S_1}{\longrightarrow} C_1 \quad \text{and} \quad A_2 \overset{T_2}{\longrightarrow} B_2 \overset{S_2}{\longrightarrow} C_2$$
are bounded operator sequences of normed spaces, then we have

\[(S_1 \circ T_1) \otimes (S_2 \circ T_2) = (S_1 \otimes S_2) \circ (T_1 \otimes T_2).\]

**Lemma 7.2.** Suppose that \(X\) is a normed space and \(E\) is an approximately complemented subspace of \(X\). Let \(\iota : E \to X\) be the inclusion mapping. Then \(\iota \otimes I_Y : E \hat{\otimes} Y \to X \hat{\otimes} Y\) is injective for every normed space \(Y\), where \(I_Y\) is the identity mapping on \(Y\).

**Proof.** Let \(u \in E \hat{\otimes} Y\) satisfying \(\iota \otimes I_Y(u) = 0\) in \(X \hat{\otimes} Y\). We prove \(u = 0\) in \(E \hat{\otimes} Y\), or equivalently, \(\|u\|_{E,Y} = 0\). It is elementary that for every convergent series \(\sum_{n=1}^{\infty} x_n, x_n > 0\), there exists a sequence \((c_n)\) such that \(c_n \to \infty\) and \(\sum_{n=1}^{\infty} c_n x_n\) converges. Thus, without loss of generality, we can assume \(u = \sum_{n=1}^{\infty} a_n \otimes b_n\) with \(a_n \in E\) and \(b_n \in Y\), \(n = 1, 2, \ldots\), satisfying \(\lim_{n \to \infty} a_n = 0\), \(\sum_{n=1}^{\infty} \|b_n\| < 1\). Given \(\varepsilon > 0\), let \(P : X \to E\) be a continuous operator satisfying \(\|a_n - P(a_n)\| < \varepsilon\) for all \(n\). Then \(P \otimes I_Y : X \hat{\otimes} Y \to E \hat{\otimes} Y\) is a continuous operator and hence

\[\sum_{n=1}^{\infty} P(a_n) \otimes b_n = (P \otimes I_Y) \circ (\iota \otimes I_Y)(u) = 0.\]

Therefore

\[\|u\|_{E,Y} = \|\sum_{n=1}^{\infty} a_n \otimes b_n - \sum_{n=1}^{\infty} P(a_n) \otimes b_n\|_{E,Y}\]

\[\leq \sum_{n=1}^{\infty} \|a_n - P(a_n)\| \|b_n\| < \varepsilon.\]

Since \(\varepsilon\) was arbitrary, we have \(\|u\|_{E,Y} = 0\) and hence \(\iota \otimes I_Y\) is injective. \(\square\)
Remark 7.3. One can not expect that \( \iota \otimes I_Y \) will be injective for each closed subspace \( E \) of \( X \). In fact, if \( X \) is a Banach space with the approximation property and \( E \) is a closed subspace of \( X \) without this property, then there is a Banach space \( Y \) such that \( \iota \otimes I_Y : E \hat{\otimes} Y \to X \hat{\otimes} Y \) has a non-trivial kernel. Because, (see [35, p.156]) if \( E \) does not have the approximation property, then there is a Banach space \( Y \) such that a non-zero element \( u = \sum_{n=1}^{\infty} a_n \otimes b_n \in E \hat{\otimes} Y \) exists with \( \sum_{n=1}^{\infty} a_n \otimes b_n = 0 \) in \( E \hat{\otimes} Y \). Since \( E \hat{\otimes} Y \subset X \hat{\otimes} Y \), \( \sum_{n=1}^{\infty} a_n \otimes b_n = 0 \) in \( X \hat{\otimes} Y \). This implies \( \sum_{n=1}^{\infty} a_n \otimes b_n = 0 \) in \( X \hat{\otimes} Y \), since \( X \) has the approximation property. In other words \( \iota \otimes I_Y (u) = 0 \).

The problem of under what condition \( \iota \otimes I_Y \) is injective is still open.

Corollary 7.4. Suppose that \( T_i : A_i \to B_i, i = 1,2, \) are topologically isomorphic embeddings (injective and homeomorphic) of Banach spaces. If \( T_i(A_i) \) is approximately complemented in \( B_i, i = 1,2, \) then the operator \( T_1 \otimes T_2 \) is injective.

Proof. Let \( \hat{T}_i : A_i \to T_i(A_i), i = 1,2, \) be the canonical isomorphisms induced from \( T_i \), and \( \iota_i : T_i(A_i) \to B_i, i = 1,2, \) be the inclusion mappings. Then

\[
T_1 \otimes T_2 = (I_{B_1} \otimes \iota_2) \circ (I_{B_1} \otimes \hat{T}_2) \circ (\iota_1 \otimes I_{A_2}) \circ (\hat{T}_1 \otimes I_{A_2}).
\]

The operators \( I_{B_1} \otimes \hat{T}_2 \) and \( \hat{T}_1 \otimes I_{A_2} \) are invertible. From the condition and the preceding lemma, \( I_{B_1} \otimes \iota_2 \) and \( \iota_1 \otimes I_{A_2} \) are injective. Therefore, as a composition of injective operators \( T_1 \otimes T_2 \) is injective. \( \square \)
For Banach spaces with the AP we have the following result.

**Theorem 7.5.** Suppose that $X$ is a Banach space having the AP. Then a subspace $E$ is approximately complemented in $X$ if and only if $E$ has the AP.

**Proof.** We only need to prove the necessity. Suppose that $E$ is approximately complemented in $X$. Suppose that $K$ is a compact subset of $E$. Then for each $\varepsilon > 0$, there is a bounded operator $P_1: X \to E$, such that $\|P_1(k) - k\| < \varepsilon/2$, $k \in K$. Also there is a bounded finite-rank operator $P_2$ on $X$ such that $\|P_2(k) - k\| < \varepsilon/(2\|P_1\|)$. Let $P = (P_1 \circ P_2)|_E$. Then $P$ is a bounded finite-rank operator on $E$.

$$\|P(k) - k\| \leq \|P_1(P_2(k) - k)\| + \|P_1(k) - k\| < \varepsilon$$

for $k \in K$. This shows that $E$ has the AP. \qed

It is known that many Banach spaces enjoying the AP contain subspaces which lack the AP ([24], [18] and [66]). From the above theorem, such subspaces are not approximately complemented.

A subspace $E$ of a Banach space $X$ is said to be *weakly complemented* in $X$ if the dual operator $i^*$ of the inclusion mapping $i: E \to X$ has a continuous right inverse ([40, p.23]). In Banach spaces literature, this concept was first discussed in [51], where the terminology *local complementability* was used for it. An equivalent, but more simple, expression of this concept is that
$E^\perp = \{ f \in X^*; \ f(x) = 0 \text{ for } x \in E \}$ is complemented in $X^*$ (see [16] and [41]). In fact, one can see this equivalence easily by noting that $\ker(t^*) = E^\perp$.

In the following we are concerned with the relationship between approximate complementability and weak complementability. First the following example shows that the latter does not cover the former.

Recall that an infinite dimensional Banach space is prime if it is isomorphic to every infinite dimensional complemented subspace of it (see [53, Definition 2.a.6]).

**Example 7.6.** Consider the Banach space $L^1[0, 1]$. It has a subspace isomorphic to $l_2$ ([53, p.57]). This subspace is approximately complemented in $L^1[0, 1]$ since $l_2$ has the approximation property. We prove that it is not complemented and hence from Corollary 7.8 below it is not weakly complemented since it is a dual space.

**Proof.** Suppose, towards a contradiction, that this subspace is complemented. Then $L^\infty[0, 1]$, the conjugate of $L^1[0, 1]$, would have a complemented subspace isomorphic to $l_2$. Since $L^\infty[0, 1]$ is isomorphic to $l_\infty$ ([53, p.111]), $l_\infty$ would have a complemented subspace which is isomorphic to $l_2$. But $l_\infty$ is a prime space ([53, p.57]). This would imply that $l_\infty$ were isomorphic to $l_2$, which is impossible. \qed

It is not known whether every weakly complemented subspace is approximately complemented. In amenable Banach algebras, if the subspace is a left
(right) ideal, then this is true because it will have a right (resp. left) b.a.i. in this case (see Theorem 1.3). For the general case, to discuss this problem the following characterizing result for weak complementability seems to be useful.

**Proposition 7.7.** Suppose that $M$ is a closed subspace of a Banach space $X$. Then $M$ is weakly complemented in $X$ if and only if there is a bounded operator $P : X \rightarrow M^{**}$, such that, for each $m \in M$, $P \circ \iota(m) = \hat{m}$, where $\iota : M \rightarrow X$ is the inclusion mapping and $\hat{m}$ is the image of $m$ in $M^{**}$ under the canonical mapping.

**Proof.** To prove the necessity we assume that $M$ is weakly complemented and $p : M^* \rightarrow X^*$ is a continuous right inverse of $\iota^*$. Then $p^* : X^{**} \rightarrow M^{**}$ is a bounded operator satisfying $p^* \circ \iota^{**} = I_{M^{**}}$. Let $j : X \rightarrow X^{**}$ be the canonical mapping and set $P = p^* \circ j$. Then we have

$$P \circ \iota(m) = p^* \circ j(\iota(m)) = p^*(\iota^{**}(\hat{m})) = \hat{m}, \quad m \in M.$$ 

For proving the sufficiency, suppose that $P : X \rightarrow M^{**}$ is a continuous operator such that $P \circ \iota(m) = \hat{m}$ for $m \in M$. Then $p = P^*|_{M^*} : M^* \rightarrow X^*$ satisfies

$$\langle m, \iota^* \circ p(m^*) \rangle = \langle m^*, P \circ \iota(m) \rangle = \langle m^*, \hat{m} \rangle = \langle m, m^* \rangle,$$

for each $m^* \in M^*$ and $m \in M$. So $\iota^* \circ p(m^*) = m^*$ for all $m^* \in M^*$, i.e. $\iota^* \circ p = I|_{M^*}$. Therefore $M$ is weakly complemented. \qed
From the preceding proposition we have the following.

**Corollary 7.8.** Suppose that $M$ is a weakly complemented subspace of a Banach space $X$. If $M$ is also a dual space, then $M$ is complemented in $X$.

**Proof.** Since $M$ is a dual space, there is a bounded operator $\sigma : M^{**} \to M$, such that $\sigma(\hat{m}) = m$ for $m \in M$, where $\hat{m}$ is the image of $m$ in $M^{**}$ under the canonical mapping. Let $P : X \to M^{**}$ and $\iota : M \to X$ be the bounded operators described in Proposition 7.7, and let $P_M = \sigma \circ P : X \to M$. Then $P_M(\iota(m)) = m$ for $m \in M$. So $\iota \circ P_M$ is a bounded projection from $X$ onto $\iota(M)$. Therefore $\iota(M)$ (i.e. $M$) is complemented in $X$. \hfill \Box

**Corollary 7.9.** Suppose that $M$ is a weakly complemented subspace of a Banach space $X$. If $M$ is approximately complemented in $M^{**}$, then it is approximately complemented in $X$.

**Proof.** If $M$ is approximately complemented in $M^{**}$, then for every compact set $K \subset M$ and every $\varepsilon > 0$ there is a bounded operator $T : M^{**} \to M$, such that $\|T(\hat{k}) - k\| < \varepsilon$ for $k \in K$. Let $P$ and $\iota$ be the operators described in Proposition 7.7. Then $S = \iota \circ T \circ P : X \to \iota(M)$ will satisfy $\|S(\iota(k)) - \iota(k)\| = \|T(\hat{k}) - k\| < \varepsilon$ for $k \in K$. This shows that $\iota(M)$ (i.e. $M$) is approximately complemented in $X$. \hfill \Box
7.2. Approximately Biprojective Banach Algebras

In this section we introduce the following notion.

**Definition 7.10.** A Banach algebra $\mathcal{A}$ is *pointwise approximately biprojective* if for each $a \in \mathcal{A}$ there is a net $\{T_\alpha : \alpha \in \Lambda\}$ of continuous bimodule morphisms from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{A}$ such that $\pi \circ T_\alpha(a) \to a$ in norm. If in addition, the net $\{T_\alpha : \alpha \in \Lambda\}$ is independent of the element $a \in \mathcal{A}$, then $\mathcal{A}$ is *approximately biprojective*.

Of course every biprojective Banach algebra is approximately biprojective. Also, if $\mathcal{A}$ has a central approximate diagonal (i.e. a net $\{u_\alpha : \alpha \in \Lambda\}$ in $\mathcal{A} \otimes \mathcal{A}$ such that $au_\alpha = u_\alpha r$, for all $a \in \mathcal{A}$ and $r \in \Lambda$, and $\pi(au_\alpha) \to a$ in norm), then $\mathcal{A}$ is approximately biprojective. An example of such kind of Banach algebra is as follows.

**Example 7.11.** Let $S$ be any non-empty set and consider $\ell^2(S)$ with the pointwise multiplication. Let $\Lambda$ be the collection of all finite subsets of $S$ ordered by inclusion. Then $\{u_\alpha = \sum_{i \in \alpha} e_i \otimes e_i : \alpha \in \Lambda\}$ forms a central approximate diagonal of $\ell^2(S)$, where $e_i$ is the element of $\ell^2(S)$ equal to 1 at $i$ and 0 elsewhere. So $\ell^2(S)$ is approximately biprojective. We prove that $\ell^2(S)$ is not biprojective whenever $S$ is an infinite set.

**Proof.** Suppose conversely that $T : \ell^2(S) \to \ell^2(S) \otimes \ell^2(S)$ were a continuous bimodule morphism satisfying $\pi \circ T = I_{\ell^2(S)}$. Since $T(e_i) = e_i T(e_i) e_i$,
one sees easily that \( T(e_i) = e_i \otimes e_i \) for each \( i \in S \). So if \( x = \sum_{i \in S} \alpha_i e_i \in \ell^2(S) \),
then \( T(x) = \sum_{i \in S} \alpha_i e_i \otimes e_i \). Consider the identity operator \( \iota \in BL(\ell^2(S), \ell^2(S)) \),
which can be viewed as an element of \((\ell^2(S) \hat{\otimes} \ell^2(S))^*\), when the latter is iden-
tified with \( BL(\ell^2(S), \ell^2(S)) \) (cf. [4, §30]). We have
\[
|\langle T(x), \iota \rangle| \leq ||\iota||\|T\||\|x\| = \|T\||\|x\|.
\]
But it is clear that \( \langle e_i \otimes e_i, \iota \rangle = 1 \) for each \( i \in S \), and so
\[
\langle T(x), \iota \rangle = \sum_{i \in S} \alpha_i.
\]
Together with the preceding inequality, one is led to a contradiction that
\( \sum_{i \in S} \alpha_i \) converges for each \( x = \sum_{i \in S} \alpha_i e_i \in \ell^2(S) \).

**Remark 7.12.** The above example also shows that, for each commuta-
tive compact group \( G \), \( L^2(G) \) with convolution multiplication is approxi-
mately biprojective but is not biprojective unless \( G \) is finite. In fact, from
Plancherel's Theorem ([60, Theorem 1.6.1]) \( L^2(G) \) is isometrically isomor-
phic to \( \ell^2(\Gamma) \); \( \Gamma \) being the dual group of \( G \) and \( \ell^2(\Gamma) \) having pointwise
multiplication. It is worth mentioning that, although \( L^2(G) \) has a central
approximate diagonal as shown in the example, by Corollary 6.17 it is not
amenable unless \( G \) is a finite group.

### 7.3. Nilpotent ideals

In the following for two subsets \( M \) and \( N \) of an algebra \( \mathfrak{A} \), \( MN \) always
denotes the set \( \{mn \mid m \in M, n \in N\} \).
Lemma 7.13. Suppose that \( \mathfrak{A} \) is a pointwise approximately biprojective Banach algebra, \( \mathcal{N} \) is an approximately complemented closed ideal of \( \mathfrak{A} \), and \( E \) is a closed ideal of \( \mathfrak{A} \) satisfying \( E \subseteq \mathcal{N} \) and \( EN = \{0\} \) (\( EN = \{0\} \)). Then for every subset \( M \) of \( \mathfrak{A} \) satisfying \( ME \subseteq \overline{E\mathfrak{A}} \) (resp. \( EM \subseteq \overline{\mathfrak{A}E} \)), it is true that \( ME = \{0\} \) (resp. \( EM = \{0\} \)).

Proof. We prove the case \( EN = \{0\} \) and \( ME \subseteq \overline{E\mathfrak{A}} \). Let \( i : \mathcal{N} \to \mathfrak{A} \) be the inclusion operator, \( q : \mathfrak{A} \to \mathfrak{A}/E \) be the quotient operator, \( I_{\mathfrak{A}}, I_{\mathcal{N}} \) and \( I_{\mathfrak{A}/E} \) be the identity operators on \( \mathfrak{A}, \mathcal{N} \) and \( \mathfrak{A}/E \) respectively, and \( p : (\mathfrak{A}/E) \hat{\otimes} \mathcal{N} \to \mathcal{N} \) be the operator specified by \( p((a + E) \otimes c) = ac \) for each \( a + E \in \mathfrak{A}/E \) and \( c \in \mathcal{N} \). The operator \( p \) is well defined because \( EN = \{0\} \).

All the operators above are obviously continuous.

Suppose towards a contradiction that \( ME \neq \{0\} \), and assume \( mc \neq 0 \) for some \( m \in M \), \( c \in E \). From the assumption \( mc = \lim c_n a_n \) for some sequences \( (c_n) \subset E \) and \( (a_n) \subset \mathfrak{A} \). Since \( \mathfrak{A} \) is pointwise approximately biprojective, there is a continuous bimodule morphism \( T : \mathfrak{A} \to \mathfrak{A} \hat{\otimes} \mathfrak{A} \) such that \( \pi \circ T(m) \cdot c \neq 0 \). For \( b \in \mathcal{N} \), let \( R_b \) (\( L_b \)) : \( \mathfrak{A} \to \mathcal{N} \) be the map of right (resp. left) multiplication by \( b \). We now consider the element \( d \in (\mathfrak{A}/E) \hat{\otimes} \mathcal{N} \) defined by

\[
d = (q \otimes R_c) \circ T(m).
\]

Since

\[
p(d) = R_c \circ \pi \circ T(m) = \pi \circ T(m) \cdot c \neq 0,
\]
we have $d \neq 0$. Now consider its image under $I_{\mathfrak{A}/E} \otimes \imath$.

$$(I_{\mathfrak{A}/E} \otimes \imath)(d) = (I_{\mathfrak{A}/E} \otimes \imath) \circ (q \otimes I_{\mathcal{N}}) \circ (I_{\mathfrak{A}} \otimes R_{e}) \circ T(m)$$

$$= (q \otimes I_{\mathfrak{A}}) \circ (I_{\mathfrak{A}} \otimes \imath) \circ (I_{\mathfrak{A}} \otimes R_{e}) \circ T(m)$$

$$= (q \otimes I_{\mathfrak{A}})(T(m)c)$$

$$= (q \otimes I_{\mathfrak{A}})T(mc)$$

$$= \lim (q \otimes I_{\mathfrak{A}})T(c_{n}a_{n})$$

$$= \lim (q \otimes I_{\mathfrak{A}}) \circ (\imath \otimes I_{\mathfrak{A}}) \circ (L_{c_{n}} \otimes I_{\mathfrak{A}}) \circ T(a_{n})$$

$$= \lim ((q \circ \imath \circ L_{c_{n}}) \otimes I_{\mathfrak{A}}) \circ T(a_{n}) = 0,$$

since $q \circ \imath \circ L_{c_{n}} = 0$ as an operator from $\mathfrak{A}$ to $\mathfrak{A}/E$. But $I_{\mathfrak{A}/E} \otimes \imath$ is injective due to Lemma 7.2. Therefore the above equality implies that $d = 0$, a contradiction which shows that $ME = \{0\}$.

\[ \square \]

**Theorem 7.14.** Suppose that $\mathfrak{A}$ is a pointwise approximately biprojective Banach algebra. If $\mathfrak{A}$ has both left and right approximate identities, then $\mathfrak{A}$ possesses no non-zero nilpotent ideal whose closure is approximately complemented in $\mathfrak{A}$.

**Proof.** If there were a non-zero nilpotent ideal $\mathcal{N}$ whose closure is approximately complemented, then there would be an integer $n \geq 1$ such that $\mathcal{N}^{n} \neq \{0\}$ and $\mathcal{N}^{n+1} = \{0\}$. Without loss of generality, assume that $\mathcal{N}$ is closed. Let $E$ be the closure of the linear span of $\mathcal{N}^{n}$. Then $E$
would be a non-zero closed ideal of \( \mathfrak{A} \) satisfying \( E \subset N \) and \( EN = \{0\} \).

Since \( \mathfrak{A} \) has a left approximate identity, \( E = \overline{AE} \). Similarly \( E = \overline{EA} \). So \( \mathfrak{A}E \subseteq \overline{E\mathfrak{A}} \). Then \( \mathfrak{A}E = 0 \) from Lemma 2, which leads to a contradiction that \( E = \overline{AE} = \{0\} \).

**Remark 7.15.** The requirement of the existence of both left and right approximate identities in the above theorem can not be reduced to the existence of just one one-sided approximate identity, as seen from the following example of Helemskii. The Banach algebra of \( 2 \times 2 \) matrices with zero first column is biprojective. It has a right identity \( \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \) but no left (approximate) identity, and it does have a non-zero nilpotent ideal (which must be complemented since the algebra is finite dimensional) containing all the matrices of the form \( \left( \begin{array}{cc} 0 & a \\ 0 & 0 \end{array} \right) \).

**Corollary 7.16.** A pointwise approximately biprojective Banach algebra with both left and right approximate identities has no non-zero finite dimensional nilpotent ideals.

**Proof.** Any finite dimensional subspace is complemented. \( \Box \)

**Corollary 7.17.** Suppose that \( \mathfrak{A} \) is a pointwise approximately biprojective Banach algebra whose underlying space is a Hilbert space. If \( \mathfrak{A} \) has both left and right approximate identities, then \( \mathfrak{A} \) possesses no non-zero nilpotent ideals. If in addition, \( \mathfrak{A} \) is commutative, then it has no nontrivial nilpotent elements.
7.3. NILPOTENT IDEALS

**Proof.** Every closed subspace of a Hilbert space is complemented. If in addition, \( \mathfrak{A} \) is commutative, then the ideal generated by a nilpotent element is nilpotent. \( \square \)

Since contractible Banach algebras are biprojective and have identities, we can deduce immediately the following result.

**Corollary 7.18.** Suppose that \( \mathfrak{A} \) is a contractible Banach algebra. Then \( \mathfrak{A} \) possesses no non-zero nilpotent ideals which are approximately complemented.

Similar results can be obtained for Banach algebras with central approximate diagonals. For example we have:

**Corollary 7.19.** Suppose that \( \mathfrak{A} \) is a Banach algebra with a central approximate diagonal. Then \( \mathfrak{A} \) possesses no non-zero nilpotent ideal which is approximately complemented.
Bibliography


