Monadicity, Purity, and Descent Equivalence

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Abstract

This dissertation is intended to study monadicity, to introduce the notion of descent equivalence, and to initiate descent theory in locally presentable categories. At first, we give two monadicity results which do not involve explicitly checking preservation of certain coequalizer diagrams, as well as a type of "monadicity lifting" theorem. Then we summarize the fundamentals of descent theory in order to lay the groundwork for developing the new notion of descent equivalence and for investigating its properties. Finally, we study closedness of (effective) codescent morphisms under directed colimits and their connection with pure monomorphisms in locally presentable categories.

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0. Introduction.

Descent Theory plays an important role in the development of modern algebraic geometry by Grothendieck [13] [14], Giraud [12], and Demazure [10]. Generally speaking, it deals with the problem of which morphisms in a given "structured" category allow for *change of base* under minimal loss of information, and how to *compensate* for the occuring loss, such morphisms are called *effective descent morphisms.* In commutative algebra, the descent problem may roughly be described as follows: given a morphism $f : R \longrightarrow S$ of commutative unital rings, when does there exist, for each S-module M with *descent data* (see 2.1.3), an R-module N such that $M \cong N \otimes_R S$? Putting the problem into more precise terms, it is well known which morphisms f in the opposite of the category of commutative monoids in sup-lattices are *effective descent morphisms*, with respect to the fibration induced by the indexed category which assigns to each commutative monoid A in sup-lattices an A-module: precisely the pure monomorphisms (see [21]). In their monograph [21], Joyal and Tierney showed that Grothendieck's techniques of descent developed for commutative unital rings can be extended to *locale theory*: open surjections are *effective for descent*, not only in *locale theory* but also in *topos theory.* In *topology*, Reiterman and Tholen [27] completely characterized effective descent maps of topological spaces. Janelidze's *Galois theory* has a strong connection with *descent*

theory, as is shown in [16], [17], [9], and [20]. Janelidze and Tholen gave a very motivating introduction to descent theory in their paper [18], to which we refer the reader, as well as to its successor [19].

As is well known, the category $\mathrm{Des}_{\mathbf{E}}(p)$ of *descent data relative to a morphism* p can be defined in the general context of a *fibred category* $\mathbf{E}$ (see 2.1.3) or of a $\mathbf{C}$-indexed category $\mathbb{A}$ (see 2.2.4). However, in the case of a *bifibration* satisfying the *Beck-Chevalley condition* (see 2.1.4), descent problems can be converted into monadicity problems. Hence Beck's Monadicity Theorem turns out to be extremely useful. But, preservation of some kind of coequalizer diagrams, the crucial requirement of Beck's Theorem, remains difficult to check in some cases, including the module case. In Chapter 1, we give two monadicity results (Theorems 1.2.4 and 1.3.6) which do not involve checking preservation of some kind of coequalizer diagrams. In this chapter, we also give a "monadicity lifting" type result (Theorem 1.4.4).

Chapter 2 is devoted to a summary of some definitions and results which we shall use later on. They include: descent data with respect to a fibration, the Beck-Chevalley condition, Bénabou-Roubaud's Theorem and Beck's Theorem; indexed categories and their actions, descent data with respect to an indexed category; some theorems of Janelidze and Tholen.

As illustrated in [18] and [19], finding sufficient (and necessary) conditions for a

morphism in a given category to be effective descent can be a challenging problem. On the other hand, a natural question is:

How to compare morphisms (or bundles) in the descent sense?

More clearly, *which conditions can guarantee that two morphisms have the same descent structure?* In Chapter 3, we begin with a study of the induced equivalence relation functor Eq (Propositions 3.1.2 and 3.1.4). Then we give the notion of descent equivalence (Definition 3.2.2) and study its properties (Propositions 3.2.2 and 3.2.5, Theorems 3.2.2, 3.2.3, and 3.2.4).

The concept of *a locally presentable category* is due to P. Gabriel and F. Ulmer [11]. This notion has turned out to be extremely useful [2]. It is broad enough to include: varieties and quasivarieties of many-sorted ranked algebras. Horn classes of relational structures, and presheaf categories. It is strong enough to show that a locally presentable category is complete, cocomplete, wellpowered, and cowellpowered; and has a strong generator; moreover, pure morphisms have good properties [1]. We feel that locally presentable categories (and even the more general accessible categories with pushouts) provide a suitable environment for tackling descent problems. In Chapter 4, we give a first outline of descent theory in locally presentable categories and study closedness of (effective) codescent morphisms under directed colimits (Theorem 4.3.2, Proposition 4.3.4).

1. Monadicity

As mentioned in Chapter 0, the preservation of some kind of coequalizer diagrams, the crucial requirement of Beck's Theorem, remains difficult to check in some cases. Hence, in this chapter, we wish to *give some monadicity criteria which do not involve checking preservation of some kind of coequalizer diagrams.*

We also wish to revisit the "lifting problem" for monadicity. Hence, we consider a commutative (up to natural isomorphism) square of categories and functors:

$$\begin{tikzcd} \mathbf{B} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] \arrow[d, "V"'] & \mathbf{C} \arrow[l, shift right, "F"'] \arrow[d, "W"] \\ \mathbf{B}' \arrow[r, shift right, "G'"'] \arrow[r, phantom, "\perp"] & \mathbf{C}' \arrow[l, shift right, "F'"'] \end{tikzcd}$$

Our second problem is then to *find conditions which can guarantee that monadicity of G' implies monadicity of G.*

1.1. Review of Monads and Their Algebras

1.1.1. In algebra, a *monoid* M is a semigroup with an identity element. It may be viewed as a set with two operations

$$\eta : 1 \to M, \quad \mu : M \times M \to M$$

such that

$$\begin{tikzcd} M \times M \times M \arrow[r, "1 \times \mu"] \arrow[d, "\mu \times 1"'] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"] & M \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \times M \arrow[r, "\eta \times 1"] \arrow[dr, "\pi_2"'] & M \times M \arrow[d, "\mu"] & M \times 1 \arrow[l, "1 \times \eta"'] \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd}$$

are commutative, where the object 1 is the one-point set $\{0\}$, the morphism 1 is an identity map, and where π_1 and π_2 are projections.

Definition. *A* monad $T = < T, \eta, \mu >$ *in a category* $\mathbf{C}$ *consists of an endofunctor* $T : \mathbf{C} \to \mathbf{C}$ *and two natural transformations*

$$\eta : I \to T, \quad \mu : T^2 \to T$$

such that

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd}$$

and

$$\begin{tikzcd}
IT \arrow[r, "\eta T"] \arrow[dr, "1"'] & T^2 \arrow[d, "\mu"] & TI \arrow[l, "T\eta"'] \arrow[dl, "1"] \\
& T &
\end{tikzcd}$$

are commutative, where $I : \mathbf{C} \to \mathbf{C}$ is the identity functor.

Recall that an *adjunction* from $\mathbf{C}$ to $\mathbf{B}$ is a triple $< F, G, \varphi >: \mathbf{C} \to \mathbf{B}$, where F and G are functors:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, shift left, "F"] & \mathbf{B} \arrow[l, shift left, "G"]
\end{tikzcd}$$

φ is a function which assigns to each pair of objects $C \in \mathbf{C}, B \in \mathbf{B}$ a bijection of sets

$$\varphi = \varphi_{C,B} : \mathrm{B}(FC, B) \cong \mathrm{C}(C, GB)$$

which is natural in C and B.

By [24, p.83, Theorem 2], each adjunction $< F, G, \varphi >: \mathbf{C} \to \mathbf{B}$ is completely determined by

(v) *functors F, G and natural transformations $\eta : 1_{\mathbf{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathbf{B}}$ such that $G\varepsilon \cdot \eta G = 1_G$ and $\varepsilon F \cdot F\eta = 1_F$.*

Hence we often denote the adjunction $< F, G, \varphi >$ by $< F,\ G\colon \eta,\ \varepsilon >: \mathbf{C} \to \mathbf{B}$.

If $< F,\ G\colon \eta,\ \varepsilon >: \mathbf{C} \to \mathbf{B}$ is an adjunction, then $< GF,\ \eta,\ G\varepsilon F >$ is a monad on $\mathbf{C}$ (see [24], p.138). In fact, every monad arises this way (see 1.1.2).

1.1.2. Definition. *Let* $< T, \eta, \mu >$ *be a monad in* $\mathbf{C}$, *a* T-alegbra $< C, \xi >$ *is a pair consisting of an object* $C \in \mathbf{C}$ *and a morphism* $\xi : TC \to C$ *in* $\mathbf{C}$ *such that*

$$\begin{tikzcd} T^2C \arrow[r, "T\xi"] \arrow[d, "\mu_C"'] & TC \arrow[d, "\xi"] \\ TC \arrow[r, "\xi"] & C \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "1"'] & TC \arrow[d, "\xi"] \\ & C \end{tikzcd}$$

are commutative. A morphism $f : < C, \xi > \to < C', \xi' >$ *of* T*-algebras is a morphism* $f : C \to C'$ *of* $\mathbf{C}$ *such that*

$$\begin{tikzcd} TC \arrow[r, "Tf"] \arrow[d, "\xi"'] & TC' \arrow[d, "\xi'"] \\ C \arrow[r, "f"] & C' \end{tikzcd}$$

commutes.

Every monad is determined by its T-algebras, as specified by the following theorem:

Theorem. *If* $< T, \eta, \mu >$ *is a monad in* $\mathbf{C}$, *then all* T*-algebras and their morphisms form a category* $\mathbf{C}^T$. *There is an adjunction*

$$< F^T, G^T; \eta^T, \varepsilon^T >: \mathbf{C} \to \mathbf{C}^T,$$

where $G^T : \mathbf{C}^T \to \mathbf{C}$ *is the obvious forgetful functor and* F^T *is given by*

$$
F^T : \begin{array}{ccc} C & \longmapsto & < TC,\ \mu_C > \\ \Big\downarrow f & & \Big\downarrow Tf \\ C & \longmapsto & < TC',\ \mu_{C'} > \end{array}
$$

furthermore, $\eta^T = \eta$ *and* $\varepsilon^T_{<C,\xi>} = \xi$ *for each* T*-algebra* $< C,\ \xi >$*. The monad defined in* $\mathbf{C}$ *by this adjunction is* $< T,\ \eta,\ \mu >$ *.*

Proof. See [24], pp.140-141. □

Examples. a. *Modules.* Let R be a unital ring. Then

$$T_R A = A \otimes R,\ \ \eta_A : A \to A \otimes R : \ \ a \mapsto a \otimes 1$$

and

$\mu_A : (A \otimes R) \otimes R \to A \otimes R : \ \ (a \otimes r_1) \otimes r_2 \mapsto a \otimes (r_1 r_2)$ for every abelian group A

give a monad $(T_R,\ \eta,\ \mu)$ on the category $\mathbf{Ab}$ of all abelian groups. and $\mathbf{Ab}^{T_R}$ is the category $\mathbf{Mod}$-R of right R-modules.

b. *Group Actions.* Let G be a group. Then $\mathbf{Set}^{T_G}$ is the category $\mathbf{Set}^G$ of G-sets, where the monad $< T_G,\ \eta,\ \mu >$ on $\mathbf{Set}$ is defined by

$$T_G X = G \times X,\ \eta_X : X \to G \times X : \ \ x \mapsto (1_G,\ x),$$

and

$$\mu_X : G \times (G \times X) \to G \times X : \ \ (g_1,\ (g_2,\ x)) \mapsto (g_1 g_2,\ x).$$

More generally, every variety of universal algebra is the category of T-algebras over **Set**, where TX is the the underlying set of the free algebras over X.

1.1.3. Definition. *Let $T = (T, \eta, \mu)$ and $T' = (T', \eta', \mu')$ be two monads in a category* **C**. *A* monad morphism *from T to T' is a natural transformation $\alpha : T \to T'$ such that*

$$\begin{tikzcd} & I \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\ T \arrow[rr, "\alpha"] & & T' \end{tikzcd}$$

and

$$\begin{tikzcd} TT \arrow[r, "\alpha \cdot \alpha"] \arrow[d, "\mu"'] & T'T' \arrow[d, "\mu'"] \\ T \arrow[r, "\alpha"] & T' \end{tikzcd}$$

commute, where $\alpha \cdot \alpha$ is the morphism $T'\alpha \circ \alpha T$.

Remark. There is a bijection between monad morphisms $\alpha : T \to T'$ and functors $V : \mathbf{C}^{T'} \to \mathbf{C}^{T}$ for which

$$\begin{tikzcd} \mathbf{C}^{T'} \arrow[rr, "V"] \arrow[dr, "U^{T'}"'] & & \mathbf{C}^{T} \arrow[dl, "U^{T}"] \\ & \mathbf{C} & \end{tikzcd}$$

commutes.

For a functor $V : \mathbf{C}^{T'} \to \mathbf{C}^{T}$ with $U^{T}V = U^{T'}$, one defines $\alpha : T \to T'$ by $\alpha_C = \xi_C T\eta'_C$, where ξ_C is given by $V(T'C, \mu'_C) = (T'C, \xi_C)$.

Conversely, for a monad morphism $\alpha : T \to T'$, $V : \mathbf{C}^{T'} \to \mathbf{C}^{T}$ is given by $V(C, \xi) = (C, \xi\alpha_C)$ (see [3], p.128).

Moreover, one has a functor $\mathbb{T}$:

$$\begin{array}{ccc} (\mathbf{Monad(C)})^{\mathrm{op}} & \xrightarrow{\mathbb{T}} & \mathbf{CAT} \\ & & \\ T & \mapsto & \mathbf{C}^{T} \\ \alpha \Big\uparrow & \mapsto & \Big\uparrow V \\ T' & \mapsto & \mathbf{C}^{T'} \end{array}$$

Naturally, we have the following question:

Question. *When does the functor $\mathbb{T}$ reflect isomorphisms?*

Morita theory shows that one can study a ring R by investigating the category of all R-modules. A more general question is:

Can we characterize T by $\mathbf{C}$ and $\mathbf{C}^{T}$? In particular, what is the relationship between T_1 and T_2 if $\mathbf{C}^{T_1} \approx \mathbf{C}^{T_2}$ and $\mathbf{C}$ is "good"?

1.2. Note on Beck's Theorem

Throughout this section, $< F,\ G;\ \eta,\ \varepsilon >: \mathbf{C} \to \mathbf{B}$ is an adjunction, and $T = < GF,\ \eta,\ G\varepsilon F >$ is the induced monad.

1.2.1. Theorem *(Comparison of adjunctions with algebras). There is a unique*

functor $K : \mathbf{B} \to \mathbf{C}^T$ *given by*

$$KB = < GB,\ G\varepsilon_B >,\ \ Kf = Gf : < GB,\ G\varepsilon_B > \to < GB',\ G\varepsilon_{B'} >$$

such that $G^T K = G$ *and* $KF = F^T$:

$$\begin{tikzcd} \mathbf{B} \arrow[rr, "K"] \arrow[dr, shift left, "G"] & & \mathbf{C}^T \arrow[dl, shift left, "G^T"] \\ & \mathbf{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^T"] & \end{tikzcd}$$

Proof. See [24], pp.142-143. □

Definition. G *is* monadic (premonadic) *if the comparison functor* K *is an equivalence of categories (full and faithful).* F *is* (pre)comonadic *if* F^{op} *is (pre)monadic, where* F^{op} *belongs to the adjunction* $< G^{op},\ F^{op},\ \varepsilon,\ \eta >: \mathbf{B}^{op} \to \mathbf{C}^{op}$.

1.2.2. Recall that a *split fork* in a category is a diagram

$$\begin{tikzcd} E \arrow[r, shift left=2, "d"] \arrow[r, "e" description] & B \arrow[l, shift left=2, "s"] \arrow[r, shift left, "c"] & C \arrow[l, shift left, "t"] \end{tikzcd}$$

which satisfies the conditions

$$cd = ce,\ \ ct = 1_C,\ \ ds = 1_B,\ \ es = tc.$$

A pair of arrows $d,\ e : E \rightrightarrows B$ is a *G-split pair* if $Gd,\ Ge : GE \rightrightarrows GB$ is a part of a split fork; it is *reflexive* if d and e have a common right inverse.

Example. Given a monad $< T,\ \eta,\ \mu >$ in $\mathbf{C}$, if $(C,\ \xi)$ is a T-algebra, then

$$T^2C \overset{\mu_C}{\underset{T\xi}{\rightrightarrows}} TC \xrightarrow{\xi} C$$

is a fork in $\mathbf{C}$ split by

$$T^2C \xleftarrow{\eta_{TC}} TC \xleftarrow{\eta_C} C$$

Furthermore

$$(T^2C, \mu_{TC}) \overset{\mu_C}{\underset{T\xi}{\rightrightarrows}} (TC, \mu_C)$$

is a reflexive G^T-split pair since μ_C and $T\xi$ have a common right inverse $T\eta_C$.

In particular, for any $B \in \mathrm{ob}\mathbf{B}$,

$$GFGFGB \overset{G\varepsilon_{FGB}}{\underset{GFG\varepsilon_B}{\rightrightarrows}} GFGB \xrightarrow{G\varepsilon_B} GB$$

is a reflexive split fork and

$$(GFGFGB, G\varepsilon_{FGFGB}) \overset{G\varepsilon_{FGB}}{\underset{GFG\varepsilon_B}{\rightrightarrows}} (GFGB, G\varepsilon_{FGB})$$

is a reflexive G^T-split pair.

We now discuss the monadicity criteria.

Theorem (Beck). 1. *G is premonadic if and only if ε_B is a regular epimorphism for each $B \in \mathrm{ob}\mathbf{B}$.*

2. *G is monadic if and only if G reflects isomorphisms,* $\mathbf{B}$ *has coequalizers of reflexive G-split pairs, and G preserves them.*

Proof. See [3], pp.110-114. $\square$

From the proof of Beck's Theorem, we actually have:

G is monadic if and only if G reflects isomorphisms, and for each $(C, \xi) \in \mathbf{C}^T$, $F\xi, \varepsilon_{FC} : FGFC \rightrightarrows FC$ has a coequalizer, and G preserves it.

1.2.3. Let $f : R \to S$ be a morphism in $\mathbf{Rng}_1$. We define functors

$$\mathbf{Mod}\text{-}S \underset{f!}{\overset{f_\bullet}{\leftrightarrows}} \mathbf{Mod}\text{-}R$$

by

$$f!(M) = M \quad \text{(restriction of scalars)}$$

$$f_\bullet(N) = N \otimes_R S \quad \text{(extension of scalars)}$$

for each $M \in \mathbf{Mod}\text{-}S$ and $N \in \mathbf{Mod}\text{-}R$, with the obvious assignments for morphisms. Then $f_\bullet$ is a left adjoint of $f!$, with units and counits given by

$$\eta_N : N \to f!f_\bullet(N) = N \otimes_R S : \quad n \mapsto n \otimes 1$$

$$\varepsilon_M : f_\bullet f!(M) = M \otimes_R S \to M : \quad m \otimes_R s \mapsto ms$$

for all $N \in \mathbf{Mod}\text{-}R$, $M \in \mathbf{Mod}\text{-}S$.

Definition. *A morphism $n : N_1 \to N_2$ in R-*Mod *is* pure *if and only if for each $M \in$* Mod-R,

$$1_M \otimes n : M \otimes_R N_1 \to M \otimes_R N_2$$

is a monomorphism. Similarly, a morphism $n : N_1 \to N_2$ in Mod-R *is a* pure *morphism if $n \otimes 1_M$ is a monomorphism for each M in R-*Mod.

Since a ring homomorphism $f : R \to S$ can be viewed as a morphism in R-Mod, since

$$1 \otimes f : N \otimes_R R \to N \otimes_R S$$

is the unit η of $f_\bullet \dashv f^!$, for each $N \in$ Mod-R, and since every monomorphism in Mod-R is regular, we have:

Proposition. *f is pure in R-*Mod *if and only if $f_\bullet$ is precomonadic.*

1.2.4. Theorem. *The following are equivalent:*

(1) *G is monadic.*

(2) (i) *G reflects isomorphisms.*

(ii) *For any morphisms $f, g : B_1 \to B_2$ in* **B** *such that*

$$GB_1 \underset{Gg}{\overset{Gf}{\rightrightarrows}} GB_2 \xrightarrow{\ e\ } C$$

is a split fork, there is a morphism $h : B_2 \to B_3$ in **B** *with $e \cong Gh$, and f, g has a coequalizer*

$$B_1 \overset{f}{\underset{g}{\rightrightarrows}} B_2 \xrightarrow{b} B$$

in **B** *such that* Gb *is an epimorphism.*

(3) *G is premonadic, and for any morphisms $f, g : B_1 \to B_2$ in* **B** *such that*

$$GB_1 \overset{Gf}{\underset{Gg}{\rightrightarrows}} GB_2 \xrightarrow{e} C$$

is a split fork, there is a morphism $h : B_2 \to B_3$ in **B** *such that $e \cong Gh$ in $GB_2/$***C***.*

Proof. By [3, p.103, Proposition 1 and p.105, Proposition 3], clearly, (1)$\Rightarrow$ (2).

(2)$\Rightarrow$(3) Since $FG\varepsilon_B, \varepsilon_{FGB} : FGFGB \to FGB$ is a G-split pair for each $B \in$ ob**B**, by (ii), it has a coequalizer, say (B_1, b). Since $\varepsilon_B \cdot \varepsilon_{FGB} = \varepsilon_B \cdot FG\varepsilon_B$, there is the unique morphism $b_1 : B_1 \to B$ such that $b_1 b = \varepsilon_B$:

$$\begin{tikzcd} FGFGB \arrow[r, shift left, "FG\varepsilon_B"] \arrow[r, shift right, "\varepsilon_{FGB}"'] & FGB \arrow[r, "b"] \arrow[dr, "\varepsilon_B"'] & B_1 \arrow[d, "b_1"] \\ & & B \end{tikzcd}$$

Application of G gives the following commutative diagram

$$\begin{tikzcd} GFGFGB \arrow[r, shift left, "GFG\varepsilon_B"] \arrow[r, shift right, "G\varepsilon_{FGB_1}"'] & GFGB \arrow[r, "Gb"] \arrow[dr, "G\varepsilon_B"'] & GB_1 \arrow[d, "Gb_1"] \\ & & GB \end{tikzcd}$$

On the other hand, since $(G\varepsilon_B, GB)$ is the coequalizer of $(G\varepsilon_{FGB}, GFG\varepsilon_B)$, there is the unique morphism $c : GB \to GB_1$ in C such that

$$c \cdot G\varepsilon_B = Gb.$$

Since

$$c \cdot Gb_1 \cdot Gb = cG(b_1 b) = cG\varepsilon_B = Gb.$$

and Gb is an epimorphism (by (ii)), we have

$$cGb_1 = 1_{GB_1}.$$

On the other hand,

$$Gb_1 \cdot c \cdot G\varepsilon_B = Gb_1 \cdot Gb = G(b_1 b) = G\varepsilon_B,$$

so that $Gb_1 \cdot c = 1_{GB}$. Hence Gb_1 is an isomorphism, so that, by (i), b_1 is an isomorphism. Hence ε_B is a regular epimorphism. Therefore G is premonadic. Then (3) holds.

(3)⇒(1) We check the conditions of Beck's Theorem. Suppose that

$$\begin{tikzcd} GB_1 \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & GB_2 \arrow[r, "e"] & C \end{tikzcd}$$

is a split fork. By hypothesis, there is a morphism h such that

$$\begin{tikzcd} G^T K B_1 \arrow[r, shift left, "G^T K f"] \arrow[r, shift right, "G^T K g"'] & G^T K B_2 \arrow[r, "G^T K h"] & G^T K B_3 \end{tikzcd}$$

is also a split fork. Since G^T is monadic and $(Kf,\ Kg)$ is a G^T-split pair in $\mathbf{C}^T$, there is a coequalizer diagram

$$KB_1 \overset{Kf}{\underset{Kg}{\rightrightarrows}} KB_2 \xrightarrow{Kh} (GB_3,\ \theta)$$

in $\mathbf{C}^T$. Then we have the following commutative diagrams

$$\begin{array}{ccc} GFGB_2 & \xrightarrow{GFGh} & GFGB_3 \\ {\scriptstyle G\varepsilon_{B_2}}\downarrow & & \downarrow{\scriptstyle \theta} \\ GB_2 & \xrightarrow{Gh} & GB_3 \end{array}$$

and

$$\begin{array}{ccc} GFGB_2 & \xrightarrow{GFGh} & GFGB_3 \\ {\scriptstyle G\varepsilon_{B_2}}\downarrow & & \downarrow{\scriptstyle G\varepsilon_{B_3}} \\ GB_2 & \xrightarrow{Gh} & GB_3 \end{array}$$

Hence

$$G\varepsilon_{B_3} \cdot GFGh = Gh \cdot G\varepsilon_{B_2} = \theta \cdot GFGh.$$

Since $Gh \cong e$ is a split epimorphism, $GFGh$ is an epimorphism. Hence $\theta = G\varepsilon_{B_3}$ and therefore

$$KB_1 \overset{Kf}{\underset{Kg}{\rightrightarrows}} KB_2 \xrightarrow{Kh} KB_3$$

is a coequalizer diagram in $\mathbf{C}^T$. Since K is full and faithful,

$$B_1 \overset{f}{\underset{g}{\rightrightarrows}} B_2 \xrightarrow{h} B_3$$

is a coequalizer diagram in $\mathbf{B}$ and G preserves it since

$$GB_1 \overset{Gf}{\underset{Gg}{\rightrightarrows}} GB_2 \xrightarrow{e} C$$

is a split fork. By Beck's Theorem, G is monadic. $\square$

1.3. A Monadicity Result in Abelian Categories

1.3.1. We call $k,\ k' : K \rightrightarrows E$ a *joint kernel pair* of $f,\ g : E \rightrightarrows B$ if (i) $fk = fk'$ and $gk = gk'$, and (ii) for any pair $l,\ l' : L \rightrightarrows E$ of morphisms with $fl = fl'$ and $gl = gl'$, there is a unique morphism $v : L \to K$ such that $l = kv$ and $l' = k'v$:

$$\begin{tikzcd}
K \arrow[r, shift left, "k"] \arrow[r, shift right, "k'"'] & E \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
L \arrow[u, "v"] \arrow[ur, shift left, "l"] \arrow[ur, shift right, "l'"'] & &
\end{tikzcd}$$

A pair $f, g : E \rightrightarrows B$ is *a G-split equivalence relation* if $Gf, Gg : GE \rightrightarrows GB$ is an equivalence relation which is a part of a split fork.

Duskin proved:

Theorem. *Suppose* $\mathbf{B}$ *has joint kernel pairs and* $\mathbf{C}$ *has kernel pairs of split epimorphisms. Then G is monadic if and only if G reflects isomorphisms. G-split equivalence relations have coequalizers, and G preserves them.*

Proof. See [3], pp.304-308. $\square$

1.3.2. Recall that a *zero object* 0 in a category is an object which is both *initial* and *terminal*. If a category $\mathbf{A}$ has a zero object 0, then to any $A, B \in \mathrm{ob}\mathbf{A}$ the unique morphisms $A \to 0$ and $0 \to B$ have a composite $0 = 0_B^A : A \to B$ called the *zero morphism* from A to B. For a morphism $f : A \to B$ of $\mathbf{A}$, A *kernel* of f is defined to be an equalizer of morphisms $f, 0 : A \to B$. The dual notion of *kernel* is *cokernel*.

Definition. *An abelian category* $\mathbf{A}$ *is a category satisfying the following conditions:*

(1) $\mathbf{A}$ *has a zero object,*

(2) $\mathbf{A}$ *has binary products and coproducts,*

(3) *every morphism in* $\mathbf{A}$ *has a kernel and a cokernel,*

(4) *every monomorphism is a kernel and every epimorphism is a cokernel.*

For example, **Ab**, **Mod**-R, and R-**Mod** are *abelian categories*.

Every abelian category is finitely complete, finitely cocomplete, additive, and exact.

Now we want to see what happens if both $\mathbf{B}$ and $\mathbf{C}$ are abelian categories in Duskin's Theorem.

In the rest of this section, we assume that $\mathbf{A}_1$ and $\mathbf{A}_2$ are two abelian categories and $< F, G; \eta, \varepsilon >: \mathbf{A}_1 \to \mathbf{A}_2$ is an adjunction.

1.3.3. Proposition. 1. *F is right exact and G is left exact.*

2. *F and G preserve biproducts.*

Proof. By [6, vol.2, Propositions 1.3.4 and 1.11.2]. □

1.3.4. In a category,

$$P \overset{u}{\underset{v}{\rightrightarrows}} A \xrightarrow{f} B$$

is said to be *an exact sequence* if (u, v) is the kernel pair of f and f is the coequalizer of (u, v).

Proposition. *In an abelian category, the following are equivalent :*

(1)

$$P \overset{u}{\underset{v}{\rightrightarrows}} A \xrightarrow{f} B$$

is an exact sequence.

(2)

$$0 \longrightarrow P \xrightarrow{\binom{u}{v}} A \oplus A \xrightarrow{(f, -f)} B \longrightarrow 0$$

is a short exact sequence.

Proof. See [6], vol.2, p.95. □

1.3.5. Basically, the following Proposition is [3, p.305, Lemma 3]. Here we give a complete proof.

Proposition. *Let* $< F, G; \eta, \varepsilon >: \mathbf{C} \to \mathbf{B}$ *be an adjunction. If* G *is premonadic, then* G *reflects equivalence relations.*

Proof. Let $d, e : E \rightrightarrows B$ be a pair of arrows such that

$$Gd, Ge : GE \rightrightarrows GB$$

is an equivalence relation. If $x, y : X \rightrightarrows E$ is a pair of arrows with

$$dx = dy \text{ and } ex = ey,$$

then

$$Gd \cdot Gx = Gd \cdot Gy \text{ and } Ge \cdot Gx = Ge \cdot Gy,$$

and so

$$Gx = Gy,$$

since $Gd, Ge : GE \rightrightarrows GB$ is a jointly monic pair. Since G is faithful, $x = y$ follows. Hence, $d, e : E \rightrightarrows B$ is a relation.

For any object X, we want to prove that

$$R_X = \{(dx, ex) | x \in \hom(X, E)\}$$

is an equivalence relation on the set $\hom(X, B)$.

Claim 1. *R_X is reflexive.*

For any object C, the induced maps

$$\hom(C, GE) \rightrightarrows \hom(C, GB)$$

is an equivalence relation in **Set**, and by adjointness, so is

$$\hom(FC,\ E) \rightrightarrows \hom(FC,\ B).$$

Since G is premonadic,

$$\begin{tikzcd} FGFGX \arrow[r, shift left, "FG\varepsilon_X"] \arrow[r, shift right, "\varepsilon_{FGX}"'] & FGX \arrow[r, "\varepsilon_X"] & X \end{tikzcd}$$

is a coequalizer diagram. Now we consider the following commutative diagram:

$$\begin{tikzcd} \hom(X, E) \arrow[r, "\hom(\varepsilon_X, E)"] \arrow[d, shift right, "\hom(X,d)"'] \arrow[d, shift left, "\hom(X,e)"] & \hom(FGX, E) \arrow[d, shift right, "\hom(FGX,e)"'] \arrow[d, shift left, "\hom(FGX,d)"] \\ \hom(X, B) \arrow[r, "\hom(\varepsilon_X, B)"'] & \hom(FGX, B) \end{tikzcd}$$

For any $a \in \hom(X, B)$, $a\varepsilon_X \in \hom\ (FGX, B)$. Since

$$\hom(FGX,\ E) \rightrightarrows \hom(FGX,\ E)$$

is an equivalence relation in **Set**, there is a morphism $b : FGX \to E$ such that

$$db = a\varepsilon_X \ \text{ and } \ eb = a\varepsilon_X.$$

Since

$$dbFG\varepsilon_X = a\varepsilon_X FG\varepsilon_X = a\varepsilon_X \varepsilon_{FGX} = db\varepsilon_{FGX}$$

and

$$ebFG\varepsilon_X = a\varepsilon_X FG\varepsilon_X = a\varepsilon_X \varepsilon_{FGX} = eb\varepsilon_{FGX},$$

we have

$$bFG\varepsilon_X = b\varepsilon_{FGX}.$$

Then there is a unique morphism $c : X \to E$ such that

$$c\varepsilon_X = b.$$

$$\begin{tikzcd}
FGFGX \arrow[r, shift left, "FG\varepsilon_X"] \arrow[r, shift right, "\varepsilon_{FGX}"'] & FGX \arrow[r, "\varepsilon_X"] \arrow[d, dashed, "b"'] & X \arrow[d, "a"] \arrow[dl, dashed, "c"'] \\
& E \arrow[r, shift left, "e"] \arrow[r, shift right, "d"'] & B
\end{tikzcd}$$

Hence

$$dc\varepsilon_X = db = a\varepsilon_X, \quad ec\varepsilon_X = eb = a\varepsilon_X,$$

and therefore

$$dc = a, \quad ec = a.$$

Thus, $(a, a) \in R_X$. Then R_X is reflexive.

Claim 2. *R_X is symmetric.*

If $(a, b) \in R_X$, then there is a morphism $x_0 : X \to E$ such that

$$a = dx_0 \text{ and } b = ex_0,$$

and so

$$a\varepsilon_X = dx_0\varepsilon_X, \quad b\varepsilon_X = ex_0\varepsilon_X.$$

$$\begin{tikzcd}
 & X \arrow[r, "x_0"] \arrow[d, equal] & E \arrow[d, shift right, "d"'] \arrow[d, shift left, "e"] \\
FGX \arrow[ur, "\varepsilon_X"] \arrow[r, "\varepsilon_X"'] & X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & B
\end{tikzcd}$$

Hence

$$(a\varepsilon_X,\ b\varepsilon_X) \in R_{FGX}.$$

Since

$$\mathrm{hom}(FGX,\ E) \rightrightarrows \mathrm{hom}(FGX,\ B)$$

is an equivalence relation, we get

$$(b\varepsilon_X,\ a\varepsilon_X) \in R_{FGX}.$$

Then there is a morphism $y_0 : FGX \to E$ such that

$$b\varepsilon_X = dy_0,\quad a\varepsilon_X = ey_0.$$

$$\begin{tikzcd}
FGFGX \arrow[r, shift left, "FG\varepsilon_X"] \arrow[r, shift right, "\varepsilon_{FGX}"'] & FGX \arrow[r, "\varepsilon_X"] \arrow[d, dashed, "y_0"'] & X \arrow[dl, dashed, "x_1"'] \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] \\
 & E \arrow[r, shift left, "e"] \arrow[r, shift right, "d"'] & B
\end{tikzcd}$$

Note that

$$dy_0FG\varepsilon_X = b\varepsilon_XFG\varepsilon_X = b\varepsilon_X\varepsilon_{FGX} = dy_0\varepsilon_{FGX}$$

and

$$ey_0FG\varepsilon_X = a\varepsilon_XFG\varepsilon_X = a\varepsilon_X\varepsilon_{FGX} = ey_0\varepsilon_{FGX}.$$

Hence

$$y_0 FG\varepsilon_X = y_0 \varepsilon_{FGX},$$

and therefore there is a unique morphism $x_1 : X \to E$ such that

$$x_1 \varepsilon_X = y_0.$$

It follows that

$$dx_1\varepsilon_X = dy_0 = b\varepsilon_X, \quad ex_1\varepsilon_X = ey_0 = a\varepsilon_X.$$

Then

$$dx_1 = b, \quad ex_1 = a,$$

and so

$$(b, \quad a) \in R_X.$$

Hence R_X is symmetric.

CLAIM 3. *R_X is transitive.*

If $(a, \ b), \ (b, \ c) \in R_X$, then

$$(a\varepsilon_X, \ b\varepsilon_X), \ (b\varepsilon_X, \ c\varepsilon_X) \in R_{FGX},$$

and so

$$(a\varepsilon_X, \ c\varepsilon_X) \in R_{FGX}.$$

Hence there is a morphism $z_0 : FGX \to E$ such that

$$dz_0 = a\varepsilon_X, \quad ez_0 = c\varepsilon_X.$$

Then

$$dz_0FG\varepsilon_X = a\varepsilon_X FG\varepsilon_X = a\varepsilon_X\varepsilon_{FGX} = dz_0\varepsilon_{FGX}$$

and

$$ez_0FG\varepsilon_X = c\varepsilon_X FG\varepsilon_X = c\varepsilon_X\varepsilon_{FGX} = ez_0\varepsilon_{FGX},$$

and so

$$z_0FG\varepsilon_X = z_0\varepsilon_{FGX}.$$

Hence there is a unique morphism z_1 such that

$$z_1\varepsilon_X = z_0.$$

$$\begin{tikzcd}
FGFGX \arrow[r, shift left, "FG\varepsilon_X"] \arrow[r, shift right, "\varepsilon_{FGX}"'] & FGX \arrow[rr, "\varepsilon_X"] \arrow[d, dashed, "z_0"'] & & X \arrow[d, shift right, "a"'] \arrow[d, shift left, "c"] \arrow[dll, dashed, "z_1"] \\
& E \arrow[rr, shift left, "e"] \arrow[rr, shift right, "d"'] & & B
\end{tikzcd}$$

Therefore

$$dz_1\varepsilon_X = dz_0 = a\varepsilon_X, \quad ez_1\varepsilon_X = ez_0 = c\varepsilon_X.$$

It follows that $dz_1 = a$, $ez_1 = c$. Then

$$(a,\ c) \in R_X.$$

Hence R_X is transitive, as desired. □

1.3.6. Recall that an equivalence relation $d,\ e : E \rightrightarrows B$ is *effective* if $(d,\ e)$ is the kernel pair of some morphism q. If an equivalence relation is effective and has a

coequalizer then it is the kernel pair of its coequalizer (see [3], p.49). Since abelian categories are exact in the sense of Barr, every equivalence relation is effective in abelian categories.

Theorem. *If G is premonadic, then the following are equivalent :*

(1) *G is monadic.*

(2) *For each reflexive G-split pair $d,\ e : E \rightrightarrows B$, if*

$$E \underset{d}{\overset{e}{\rightrightarrows}} B \xrightarrow{c} C$$

is a coequalizer diagram, then Gc is an epimorphism.

(3) *For each G-split equivalence relation $d,\ e : E \rightrightarrows B$, if*

$$E \underset{e}{\overset{d}{\rightrightarrows}} B \xrightarrow{c} C$$

is a coequalizer diagram, then Gc is an epimorphism.

Proof. (1) $\Rightarrow$ (2). By [3, p.103, Proposition 1 and p.105, Proposition 3].

(2) $\Rightarrow$ (3). Suppose that $d,\ e : E \rightrightarrows B$ is a G-split equivalence relation. Then $Gd, Ge : GE \rightrightarrows GB$ is an equivalence relation, and so is $d,\ e : E \rightrightarrows B$ by Proposition 1.3.5. Since $\mathbf{A}_2$ is abelian, $d, e : E \rightrightarrows B$ is effective. If

$$E \underset{e}{\overset{d}{\rightrightarrows}} B \xrightarrow{c} C$$

is a coequalizer diagram, then

$$\begin{tikzcd} E \arrow[r, "d"] \arrow[d, "e"'] & B \arrow[d, "c"] \\ B \arrow[r, "c"] & C \end{tikzcd}$$

is a pullback diagram, and therefore $d,\ e : E \rightrightarrows B$ is a reflexive G-split pair as one sees by considering the following diagram

$$\begin{tikzcd} B \arrow[dr, dashed] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\ & E \arrow[r, "d"] \arrow[d, "e"] & B \arrow[d, "c"] \\ & B \arrow[r, "c"'] & C \end{tikzcd}$$

By (2), Gc is an epimorphism.

(3) $\Rightarrow$ (1). Clearly, $G = G^T K$ reflects isomorphisms. Let $d,\ e : E \rightrightarrows B$ be a G-split equivalence relation. Then $Gd,\ Ge : GE \rightrightarrows GB$ is a split equivalence relation. By Proposition 1.3.5, $d,\ e : E \rightrightarrows B$ is an equivalence relation. Then it is effective, and so there is a morphism $g : B \to Y$ such that

$$\begin{tikzcd} E \arrow[r, shift left, "e"] \arrow[r, shift right, "d"'] & B \arrow[r, "g"] & Y \end{tikzcd}$$

is an exact sequence. Hence, by Proposition 1.3.4,

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "\binom{d}{e}"] & B \bigoplus B \arrow[r, "{(g, -g)}"] & Y \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence. By Proposition 1.3.3,

$$0 \longrightarrow GE \xrightarrow{\binom{Gd}{Ge}} GB \bigoplus GB \xrightarrow{(Gg,-Gg)} GY$$

is exact. By condition (3),

$$0 \longrightarrow GE \xrightarrow{\binom{Gd}{Ge}} GB \bigoplus GB \xrightarrow{(Gg,-Gg)} GY \longrightarrow 0$$

is exact and therefore, by Proposition 1.3.4 again,

$$\begin{tikzcd} GE \arrow[r, shift left, "Ge"] \arrow[r, shift right, "Gd"'] & GB \arrow[r, "Gg"] & GY \end{tikzcd}$$

is a coequalizer diagram. Thus, by Duskin's Theorem, G is monadic. □

From Theorem 1.3.6, we immediately have:

Corollary. *If G preserves epimorphisms, then G is monadic if and only if G is premonadic.*

1.3.7. If $f : R \to S$ is a morphism of $\mathbf{Rng}_1$, then we have an adjoint pair:

$$\begin{tikzcd} \textbf{Mod-}S \arrow[r, shift right, "f!"'] & \textbf{Mod-}R \arrow[l, shift right, "f_*"'] \end{tikzcd}$$

Recall that the category of all right modules over a ring is abelian, and in such a category the epimorphisms are precisely those morphisms which are surjective on the underlying sets, and epimorphisms are all regular. Hence, $f!$ preserves epimorphisms. Clearly, $f!$ is premonadic since each ε_M is an epimorphism (see 1.2.3). Then, by Corollary 1.3.6, we have:

Proposition. *$f!$ is monadic.*

Surprisingly, **Mod**-S is an Eilenberg-Moore category over **Mod**-R whenever there is a ring homomorphism $f : R \to S$.

1.4. Transformations of Monads

In [31], Tholen studied adjoint morphisms and their properties. In this section, we consider a special case of adjoint morphisms. Throughout this section, $< F, G: \eta, \varepsilon >: \mathbf{C} \to \mathbf{B}$ and $< F', G': \eta', \varepsilon' >: \mathbf{C}' \to \mathbf{B}'$ are two adjunctions. $T = < GF, \eta, G\varepsilon F >$ and $T' = < G'F', \eta', G'\varepsilon' F >$ are the two induced monads.

1.4.1. Definition. *Let $V : \mathbf{B} \to \mathbf{B}'$ and $W : \mathbf{C} \to \mathbf{C}'$ be two functors:*

$$\begin{tikzcd} \mathbf{B} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] \arrow[d, "V"'] & \mathbf{C} \arrow[l, shift right, "F"'] \arrow[d, "W"] \\ \mathbf{B}' \arrow[r, shift right, "G'"'] \arrow[r, phantom, "\perp"] & \mathbf{C}' \arrow[l, shift right, "F'"'] \end{tikzcd}$$

$< F, G; \eta, \varepsilon >: \mathbf{C} \to \mathbf{B}$ *is* transformable into $< F', G'; \eta', \varepsilon' >: \mathbf{C}' \to \mathbf{B}'$ *under V and W if there are natural isomorphisms $\alpha : WG \to G'V$ and $\beta : F'W \to VF$ such that*

$$\begin{tikzcd} W \arrow[r, "\eta' W"] \arrow[d, "W\eta"'] & G'F'W \arrow[d, "G'\beta"] \\ WGF \arrow[r, "\alpha F"] & G'VF \end{tikzcd}$$

and

$$\begin{tikzcd} F'WG \arrow[r, "\beta G"] \arrow[d, "F'\alpha"'] & VFG \arrow[d, "V\varepsilon"] \\ F'G'V \arrow[r, "\varepsilon' V"] & V \end{tikzcd}$$

commute.

1.4.2. Let $\mathbf{C}$ be a category and $B \in \mathrm{ob}\,\mathbf{C}$. Recall that the *comma category of* $\mathbf{C}$ *over* B is the category $\mathbf{C}/B$ whose objects are pairs (E, p) with $\mathbf{C}$-morphisms $p : E \to B$, and whose morphisms $g : (E, p) \to (E', p')$ are $\mathbf{C}$-morphisms $g : E \to E'$ such that

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

commutes.

Let $\mathbf{C}$ be a category with pullbacks, and let $p : E \to B$ be a morphism in $\mathbf{C}$. Then we have the following adjoint pair:

$$\begin{tikzcd} \mathbf{C}/B \arrow[r, shift right, "p^*"'] & \mathbf{C}/E \arrow[l, shift right, "p!"'] \end{tikzcd}$$

where $p!(D,\ s) = ps$, $p^*(C,\ r) = \pi_1$ which is given by the following pullback diagram:

$$\begin{tikzcd} E \times_B C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "r"] \\ E \arrow[r, "p"] & B \end{tikzcd}$$

The unit and counit of $p! \dashv p^*$ is given by $\eta_{(s:C \to E)} =< s, 1_C >: C \to E \times_B C$ and $\varepsilon_{(r:C \to B)} = \pi_2$, respectively.

Examples. Let $\mathbf{C}$ be a category with pullbacks. If

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "s"'] & B \arrow[d, "t"] \\ E' \arrow[r, "p'"] & B' \end{tikzcd}$$

is a pullback diagram in $\mathbf{C}$, then:

1. $p! \dashv p^* : \mathbf{C}/B \to \mathbf{C}/E$ is *transformable into* $p'! \dashv p'^* : \mathbf{C}/B' \to \mathbf{C}/E'$ under $t!$ and $s!$:

$$\begin{tikzcd} \mathbf{C}/B \arrow[d, "t!"'] \arrow[r, shift right, "p^*"'] \arrow[r, phantom, "\perp"] & \mathbf{C}/E \arrow[l, shift right, "p!"'] \arrow[d, "s!"] \\ \mathbf{C}/B' \arrow[r, shift right, "p'^*"'] \arrow[r, phantom, "\perp"] & \mathbf{C}/E' \arrow[l, shift right, "p'!"'] \end{tikzcd}$$

2. $p'! \dashv p'^* : \mathbf{C}/B' \to \mathbf{C}/E'$ is *transformable into* $p! \dashv p^* : \mathbf{C}/B \to \mathbf{C}/E$ under t^* and s^*:

$$
\begin{tikzcd}
\mathbf{C}/B' \arrow[d, "t^*"'] \arrow[r, phantom, "\perp"] & \mathbf{C}/E' \arrow[l, shift right=1.5ex, "p'!"'] \arrow[d, "s^*"] \\
\mathbf{C}/B \arrow[r, phantom, "\perp"] & \mathbf{C}/E \arrow[l, shift right=1.5ex, "p!"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}/B' \arrow[r, "p'^*"'] & \mathbf{C}/E' \\
\mathbf{C}/B \arrow[r, "p^*"'] & \mathbf{C}/E
\end{tikzcd}
$$

1.4.3. Proposition. *Suppose that* $< F, G; \eta, \varepsilon >: \mathbf{C} \to \mathbf{B}$ *is transformable into* $< F', G'; \eta', \varepsilon' >: \mathbf{C}' \to \mathbf{B}'$ *under* V *and* W. *Then there is a functor*

$$L : \mathbf{C}^{\mathcal{T}} \to \mathbf{C}'^{\mathcal{T}'}$$

such that

$$
\begin{tikzcd}
\mathbf{B} \arrow[rr, "K"] \arrow[dd, "V"'] \arrow[dr, shift right, "G"'] & & \mathbf{C}^{\mathcal{T}} \arrow[dl, shift left, "G^{\mathcal{T}}"] \arrow[dd, "L"] \\
& \mathbf{C} \arrow[ul, shift right, "F"'] \arrow[ur, shift left, "F^{\mathcal{T}}"] \arrow[dd, "W"] & \\
\mathbf{B}' \arrow[rr, "K'"] \arrow[dr, shift right, "G'"'] & & \mathbf{C}'^{\mathcal{T}'} \arrow[dl, shift left, "G'^{\mathcal{T}'}"] \\
& \mathbf{C}' \arrow[ul, shift right, "F'"'] \arrow[ur, shift left, "F'^{\mathcal{T}'}"] &
\end{tikzcd}
$$

commutes (up to isomorphism).

Proof. Define $L : \mathbf{C}^{\mathcal{T}} \to \mathbf{C}'^{\mathcal{T}'}$ by

$$L(C, \xi) = (WC, W\xi \cdot \alpha_{FC}^{-1} G' \beta_C) \text{ for each } (C, \xi) \in \mathbf{C}^{\mathcal{T}}.$$

If $(C, \xi) \in \mathbf{C}^T$, then

$$\xi\eta_C = 1_C \text{ and } \xi GF\xi = \xi G\varepsilon_{FC},$$

and so, by applying W,

$$W\xi W\eta_C = 1_{WC} \text{ and } W\xi WGF\xi = W\xi WG\varepsilon_{FC}.$$

But

$$(W\xi \cdot \alpha_{FC}^{-1}G'\beta_C)\eta'_{WC} = W\xi\alpha_{FC}^{-1} \cdot \alpha_{FC}W\eta_C = W\xi W\eta_C = 1_{WC}.$$

Then

$$\begin{tikzcd}
WC \arrow[r, "\eta'_{WC}"] \arrow[dr, "1_{WC}"'] & G'F'WC \arrow[d, "W\xi \cdot \alpha_{FC}^{-1}G'\beta_C"] \\
 & WC
\end{tikzcd}$$

commutes.

By the naturality of α, β, and ε, for any object C of $\mathbf{C}$,

$$\begin{tikzcd}
WGFGFC \arrow[r, "\alpha_{FGFC}"] \arrow[d, "WGF\xi"'] & G'VFGFC \arrow[d, "G'VF\xi"] \\
WGFC \arrow[r, "\alpha_{FC}"] & G'VFC
\end{tikzcd}$$

$$\begin{tikzcd}
WGFGFC \arrow[r, "\alpha_{FGFC}"] \arrow[d, "WG\varepsilon_{FC}"'] & G'VFGFC \arrow[d, "G'V\varepsilon_{FC}"] \\
WGFC \arrow[r, "\alpha_{FC}"] & G'VFC
\end{tikzcd}$$

$$
\begin{tikzcd}
F'WGFC \arrow[r, "\beta_{GFC}"] \arrow[d, "F'W\xi"'] & VFGFC \arrow[d, "VF\xi"] \\
F'WC \arrow[r, "\beta_C"] & VFC
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
F'G'F'WC \arrow[r, "\varepsilon'_{F'WC}"] \arrow[d, "F'G'\beta_C"'] & F'WC \arrow[d, "\beta_C"] \\
F'G'VFC \arrow[r, "\varepsilon'_{VFC}"] & VFC
\end{tikzcd}
$$

commute. Then

$$
\begin{aligned}
& W\xi \cdot \alpha_{FC}^{-1} G'\beta_C G'F'W\xi G'F'\alpha_{FC}^{-1} G'F'G'\beta_C \\
&= W\xi\alpha_{FC}^{-1} \cdot G'VF\xi G'\beta_{GFC} \cdot G'F'\alpha_{FC}^{-1} G'F'G'\beta_C \\
&= W\xi \cdot WGF\xi\alpha_{FGFC}^{-1} \cdot G'\beta_{GFC} G'F'\alpha_{FC}^{-1} G'F'G'\beta_C \\
&= W\xi WG\varepsilon_{FC} \cdot \alpha_{FGFC}^{-1} G'\beta_{GFC} G'F'\alpha_{FC}^{-1} G'F'G'\beta_C \\
&= W\xi \cdot \alpha_{FC}^{-1} G'V\varepsilon_{FC} \cdot G'\beta_{GFC} G'F'\alpha_{FC}^{-1} G'F'G'\beta_C \\
&= W\xi\alpha_{FC}^{-1} \cdot G'\varepsilon'_{VFC} G'F'\alpha_{FC} \cdot G'F'\alpha_{FC}^{-1} G'F'G'\beta_C \\
&= W\xi\alpha_{FC}^{-1} \cdot G'\varepsilon'_{VFC} G'F'G'\beta_C \\
&= W\xi\alpha_{FC}^{-1} \cdot G'\beta_C G'\varepsilon'_{F'WC},
\end{aligned}
$$

and so

$$\begin{tikzcd}
G'F'G'F'WC \arrow[r, "G'F'(W\xi\alpha_{FC}^{-1}G'\beta_C)"] \arrow[d, "G'\varepsilon'_{F'WC}"'] & G'F'WC \arrow[d, "W\xi\alpha_{FC}^{-1}G'\beta_C"] \\
G'F'WC \arrow[r, "W\xi\alpha_{FC}^{-1}G'\beta_C"] & WC
\end{tikzcd}$$

commutes. Hence $(WC,\ W\xi \cdot \alpha_{FC}^{-1}G'\beta_C) \in \mathbf{C}'^{T'}$ and therefore L is well-defined.

For any $C \in \mathrm{ob}\mathbf{C}$,

$$LF^T C = L(GFC,\ G\varepsilon_{FC}) = (WGFC,\ WG\varepsilon_{FC}\alpha_{FGFC}^{-1}G'\beta_{GFC}),$$

and

$$F'^{T'} WC = (G'F'WC,\ G'\varepsilon'_{F'WC}).$$

Hence

$$F'^{T'} W \cong LF^T,$$

since

$$\begin{tikzcd}
G'F'G'F'WC \arrow[r, "G'F'(\alpha_{FC}^{-1}G'\beta_C)"] \arrow[d, "G'\varepsilon'_{F'WC}"'] & G'F'WGFC \arrow[d, "WG\varepsilon_{FC}\alpha_{FGFC}^{-1}G'\beta_{GFC}"] \\
G'F'WC \arrow[r, "\alpha_{FC}^{-1}G'\beta_C"] & WGFC
\end{tikzcd}$$

commutes.

For any $(C,\ \xi) \in \mathbf{C}^T$,

$$G'^{T'} L(C,\ \xi) = WC = WG^T(C,\ \xi).$$

For any $B \in \mathrm{ob}\mathbf{B}$,

$$LKB = L(GB,\ G\varepsilon_B) = (WGB,\ WG\varepsilon_B\alpha^{-1}_{FGC}G'\beta_{GB}),$$

and

$$K'VB = (G'VB,\ G'\varepsilon'_{VB}).$$

Hence

$$LK \cong K'V,$$

since

$$\begin{tikzcd}
G'F'WGB \arrow[r, "G'F'\alpha_B"] \arrow[d, "WG\varepsilon_B\alpha^{-1}_{FGB}G'\beta_{GB}"'] & G'F'G'VB \arrow[d, "G'\varepsilon'_{VB}"] \\
WGB \arrow[r, "\alpha_B"] & G'VB
\end{tikzcd}$$

commutes. □

1.4.4. Theorem. *Suppose that* $< F,\ G;\ \eta,\ \varepsilon >: \mathbf{C} \to \mathbf{B}$ *is transformable into* $< F',\ G';\ \eta',\ \varepsilon' >: \mathbf{C}' \to \mathbf{B}'$ *under V and W. If*

(1) $\mathbf{B}$ *has the coequalizers of G-split pairs and V preserves them.*

(2) G (or V) reflects isomorphism.

(3) W reflects split forks into coequalizer diagrams.

(4) G' is monadic.

then G is monadic.

Proof. For any $(C, \xi) \in \mathbf{C}^T$, by Example 1.2.2,

$$\begin{tikzcd} GFGFC \arrow[r, shift left, "GF\xi"] \arrow[r, shift right, "G\varepsilon_{FC}"'] & GFC \arrow[r, "\xi"] & C \end{tikzcd}$$

is a split fork. By (1), $F\xi$, $\varepsilon_{FC} : FGFC \rightrightarrows FC$ has the coequalizer, say (D, c), and

$$\begin{tikzcd} VFGFC \arrow[r, shift left, "VF\xi"] \arrow[r, shift right, "V\varepsilon_{FC}"'] & VFC \arrow[r, "Vc"] & VD \end{tikzcd}$$

is a coequalizer diagram, which is equivalent to saying that

$$\begin{tikzcd} F'G'F'WC \arrow[r, shift left, "F'(W\xi\alpha_{FC}^{-1}G'\beta_C)"] \arrow[r, shift right, "\varepsilon'_{F'WC}"'] & F'WC \arrow[r, "Vc \cdot \beta_C"] & VD \end{tikzcd}$$

is a coequalizer diagram since

$$\begin{tikzcd} F'G'F'WC \arrow[r, shift left, "F'(W\xi\alpha_{FC}^{-1}G'\beta_C)"] \arrow[r, shift right, "\varepsilon'_{F'WC}"'] \arrow[d, "t"'] & F'WC \arrow[d, "\beta_C"] \\ VFGFC \arrow[r, shift left, "VF\xi"] \arrow[r, shift right, "V\varepsilon_{FC}"'] & VFC \end{tikzcd}$$

commutes, where $t = \beta_{GFC}F'\alpha_{FC}^{-1}F'G'\beta_C$. But $(WC, W\xi\alpha_{FC}^{-1}G'\beta_C) \in \mathbf{C}'^{T'}$, by Beck's Theorem and (4), $F'(W\xi\alpha_{FC}^{-1}G'\beta_C)$, $\varepsilon'_{F'WC} : F'G'F'WC \rightrightarrows F'WC$ has the coequalizer, say (D', c') $(\cong (VD, Vc \cdot \beta_C))$, and

$$\begin{tikzcd} G'F'G'F'WC \arrow[r, shift left, "G'F'(W\xi\alpha_{FC}^{-1}G'\beta_C)"] \arrow[r, shift right, "G'\varepsilon'_{F'WC}"'] & G'F'WC \arrow[r, "G'(Vc\beta_C)"] & G'VD \end{tikzcd}$$

is a split fork, which amounts to saying

$$\begin{tikzcd} WGFGFC \arrow[r, shift left, "WGF\xi"] \arrow[r, shift right, "WG\varepsilon_{FC}"'] & WGFC \arrow[r, "WGc"] & WGD \end{tikzcd}$$

is a split fork since

$$\begin{tikzcd} WGFGFC \arrow[r, shift left, "WGF\xi"] \arrow[r, shift right, "WG\varepsilon_{FC}"'] \arrow[d, "m"'] & WGFC \arrow[r, "WGc"] \arrow[d, "n"] & WGD \arrow[d, "\alpha_D"] \\ G'F'G'F'WC \arrow[r, shift left, "G'F'(W\xi\alpha_{FC}^{-1}G'\beta_C)"] \arrow[r, shift right, "G'\varepsilon'_{F'WC}"'] & G'F'WC \arrow[r, "G'(Vc\beta_C)"] & G'VD \end{tikzcd}$$

commutes, where $m = G'F'G'\beta_C^{-1}G'F'\alpha_{FC}G'\beta_{GFC}^{-1}\alpha_{FGFC}$, $n = G'\beta_C^{-1}\alpha_{FC}$.

By (3),

$$\begin{tikzcd} GFGFC \arrow[r, shift left, "GF\xi"] \arrow[r, shift right, "G\varepsilon_{FC}"'] & GFC \arrow[r, "Gc"] & GD \end{tikzcd}$$

is a coequalizer diagram. Then, by Beck's Theorem, G is monadic. □

Combining Theorem 1.4.4 and Example 1.4.2.1 yields the following corollary which was shown in [29] for the first time.

Corollary. *Let* **C** *be a category with pullbacks, and let*

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "s"'] & B \arrow[d, "t"] \\ E' \arrow[r, "p'"] & B' \end{tikzcd}$$

be a pullback diagram in $\mathbf{C}$. *If* $(p')^*$ *is monadic, so is* p^*.

Proof. If $b_1, b_2 : (C_1, r_1) \to (C_2, r_2)$ is a p^*-split pair in $\mathbf{C}/B$, then we have a $(p')^*$-split pair $b_1, b_2 : (C_1, tr_1) \to (C_2, tr_2)$ in $\mathbf{C}/B'$. Since $(p')^*$ is monadic, by Beck's Theorem, b_1, b_2 has a coequalizer, say $(c, (C, r))$, in $\mathbf{C}/B'$. Clearly there is a unique morphism $b : (C, r) \to (B, t)$ in $\mathbf{C}/B'$ such that

$$bc = r_2.$$

Now it is easy to check $(c, (C, b))$ is a coequalizer of

$$b_1, b_2 : (C_1, r_1) \to (C_2, r_2)$$

in $\mathbf{C}/B$, which is preserved by $t!$. Clearly, $s!$ reflects split forks into coequalizer diagrams and p^* reflects isomorphisms. Hence, by Theorem 1.4.4, p^* is monadic. □

2. Preliminaries for Descent Theory

In this chapter, we recall the basic notions and results of descent theory which will be used in Chapters 3 and 4. They are: fibrations and descent theory, the Beck-Chevalley condition; internal categories and their actions, indexed categories and descent theory; some theorems of Janelidze and Tholen.

2.1. Fibrations and Descent Theory

This section is devoted to the presentations of the fundamentals of fibrational descent theory.

2.1.1. Let $P : \mathbf{E} \to \mathbf{C}$ be a functor and $p : E \to B$ a morphism of $\mathbf{C}$. *The fibre* of P at B is the non-full subcategory $\mathbf{E}(B)$ of $\mathbf{E}$ whose objects are in $P^{-1}B$ (i.e., those objects A of $\mathbf{E}$ with $PA = B$) and whose morphisms $f : A \to A'$ are $\mathbf{E}$-morphisms such that $Pf = 1_B$. Let $X \in \mathbf{E}(B)$, a morphism $\vartheta_p X : p^* X \to X$ of $\mathbf{C}$ is *a cartesian lifting over p at X* if

C1. $P(\vartheta_p X) = p$,

C2. for any morphism $v : Y \to X$ of $\mathbf{E}$ and any morphism $h : PY \to E$ in $\mathbf{C}$ satisfying $ph = Pv$, there is a unique $w : Y \to p^* X$ in $\mathbf{E}$ such that

$$\vartheta_p X \cdot w = v \ \text{ and } \ Pw = h.$$

$$\begin{array}{cc}
\mathbf{E} & \begin{tikzcd} & Y \arrow[dl, "w"'] \arrow[dr, "v"] & \\ p^*X \arrow[rr, "\vartheta_p X"'] & & X \end{tikzcd} \\
P \downarrow & \\
\mathbf{C} & \begin{tikzcd} & PY \arrow[dl, "h"'] \arrow[dr, "Pv"] & \\ E \arrow[rr, "p"'] & & B \end{tikzcd}
\end{array}$$

A functor $P : \mathbf{E} \to \mathbf{C}$ is called *a fibration* if for any morphism $p : E \to B$ in $\mathbf{C}$ and every object X in $\mathbf{E}(B)$ there is a cartesian lifting $(p^*X,\ \vartheta_p X)$ over p at X.

Examples. 1. For every category $\mathbf{C}$, the identity functor $1_{\mathbf{C}}$ is a fibration.

2. If $\mathbf{C}$ is a category with pullbacks, and if $\mathbf{C}^2$ is the morphism category of $\mathbf{C}$, so that

- objects are morphisms $f : E \to B$ in $\mathbf{C}$,

- morphisms from $f : E \to B$ to $f' : E' \to B'$ are pairs $(u,\ v)$ of morphisms in $\mathbf{C}$ such that $f'u = vf$, where $u : E \to E'$ and $v : B \to B'$ are $\mathbf{C}$-morphisms:

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "f"'] & E' \arrow[d, "f'"] \\ B \arrow[r, "v"] & B' \end{tikzcd}$$

then the codomain functor

$$\partial : \mathbf{C}^2 \to \mathbf{C} : \ (f : A \to B) \mapsto B, \ (u, v) \mapsto v$$

is a fibration, called *the basic fibration of* $\mathbf{C}$: for any morphism $p : E \to B$ in $\mathbf{C}$ and an object $(x : X \to B)$ in $\partial^{-1}(B)$, the following pullback diagram:

$$\begin{tikzcd} \partial^{\bullet} X \arrow[r, "\vartheta_p X"] \arrow[d, "f"'] & X \arrow[d, "x"] \\ E \arrow[r, "p"] & B \end{tikzcd}$$

yields a morphism $(\vartheta_p X, \ p) : (f : \partial^{\bullet} X \to E) \to (x : X \to B)$ in $\mathbf{C}^2$, which is a cartesian lifting over $(p : E \to B)$ at $(x : X \to B)$.

On the other hand, ∂ is an *opfibration* (i.e., ∂^{op} is a fibration). In fact, for any morphism $p : E \to B$ in $\mathbf{C}$ and any object $(x : X \to E)$ in $\partial^{-1}(E)$, an *opcartesian lifting* over $(p : E \to B)$ at $(x : X \to E)$ is given by

$$(1_X, \ p) : (x : X \to E) \to (px : X \to B) :$$

$\mathbf{C}^2$

$$
\begin{tikzcd}
& Z \arrow[dd, "z"] & \\
X \arrow[ur, "m"] \arrow[rr, "1_X"'] \arrow[dd, "x"'] & & X \arrow[ul, "m"'] \arrow[dd, "px"] \\
& D & \\
E \arrow[ur, "m"] \arrow[rr, "p"] & & B \arrow[ul, "h"']
\end{tikzcd}
$$

$\partial \downarrow$

$\mathbf{C}$

$$
\begin{tikzcd}
& D & \\
E \arrow[ur, "m"] \arrow[rr, "p"'] & & B \arrow[ul, "h"']
\end{tikzcd}
$$

Hence ∂ *is a bifibration.*

3. Let MOD be the category defined as follows:

- An object of MOD is a pair (R, M), where $R \in \mathbf{CRng}_1$, $M \in \mathbf{Mod}\text{-}R$.

- A morphism $(f, u) : (R, M) \to (R', M')$ has $f : R' \to R$ a morphism of $\mathbf{CRng}_1$ and $u : M \to M' \otimes_{R'} R$ a morphism of $\mathbf{Mod}\text{-}R$.

- If $(f, u) : (R, M) \to (R', M')$, $(g, v) : (R', M') \to (R'', M'')$ are morphisms of MOD, then

$$(g, v) \circ (f, u) : (R, M) \to (R'', M'')$$

is given by

$$(g, v) \circ (f, u) = (f \circ g,\ (v \otimes_{R'} 1_R) \circ u).$$

Then the functor mod: MOD $\to (\mathbf{CRng}_1)^{\text{op}}$ given by

$$\begin{array}{ccc} (R, M) & \mapsto & R \\ \big\downarrow (f, u) & \mapsto & \big\downarrow f \\ (R', M') & \mapsto & R' \end{array}$$

is a fibration. In fact, for any unital commutative ring homomorphism $f : R \to S$ and any $(R, M) \in \text{mod}^{-1}(R)$, $(f, 1_{M \otimes_R S}) : (S, M \otimes_R S) \to (R, M)$ is a cartesian lifting over f at (R, M).

4. Let $F : \mathbf{Top} \to \mathbf{Set}$ be the forgetful functor. Then F is a fibration: for any morphism $p : E \to B$ in **Set** and for $B \in \mathbf{Top}$, if E is equipped with the coarsest topology which makes $p : E \to B$ continuous, then $p : E \to B$ is a cartesian lifting over p at B. F is also an opfibration: for any morphism $p : E \to B$ in **Set** and for $E \in \mathbf{Top}$, $p : E \to B$ is a cocartesian lifting over p at E if B is equipped with the finest topology which makes $p : E \to B$ continuous. Hence F is *a bifibration*.

2.1.2. If $P : \mathbf{E} \to \mathbf{C}$ is a fibration and $p : E \to B$ is a morphism in $\mathbf{C}$, then we have *the inverse-image functor*

$$p^* : \mathbf{E}(B) \to \mathbf{E}(E)$$

given by $A \mapsto p^*A$ and obvious assignments on morphisms, and *a cleavage*

$$\vartheta_p : J_E p^* \to J_B,$$

where $J_B : \mathbf{E}(B) \to \mathbf{E}$ is the inclusion functor and $P\vartheta_p = \triangle p : \triangle E \to \triangle B$ is the constant natural transformation. Hence one gets a *pseudo-functor*

$$(\ )^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$$

which is given by

$$\begin{array}{ccc} B & & \mathbf{E}(B) \\ p \downarrow & \mapsto & \downarrow p^* \\ E & & \mathbf{E}(E) \end{array}$$

since there are the uniquely determined natural equivalences

$$i_B : 1_{\mathbf{E}(B)} \to (1_B)^* \ \text{ and } \ j_{p,q} : q^* p^* \to (pq)^*$$

such that

$$\vartheta_{1_B} \cdot J_B i_B = 1_{J_B},\ PJ_B i_B = \triangle 1_B,$$

and

$$\vartheta_{pq} \cdot J_X j_{p,q} = \vartheta_p \cdot \vartheta_q p^*,\ PJ_X j_{p,q} = \triangle 1_X,$$

for any $p : E \to B$ and $q : X \to E$ in $\mathbf{C}$ by the definition of the cartesian lifting. This means that $(\ )^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ is a $\mathbf{C}$-indexed category (see 2.2.3).

For example, the basic fibration yields the basic $\mathbf{C}$-indexed category given by the sliced categories and pullback functors (see 2.2.3). On the other hand, given a $\mathbf{C}$-indexed category $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$, by the Grothendieck construction, one may

construct a fibration (see [4] or 2.2.3 below). Any C-indexed category essentially arises in this way (see [19]).

2.1.3. Let C be a category with pullbacks, $p : E \to B$ a morphism in C and $P : \mathbf{E} \to \mathbf{C}$ a fibration. *Descent data* (C, ξ) *for* $C \in \mathbf{E}(E)$ *(relative to* p*)* are given by certain morphisms $\xi : p_1^*C \to p_2^*C$ in $\mathbf{E}(E \times_B E)$ such that

$$\begin{tikzcd}
p_1^*C \arrow[rr, "\xi"] & & p_2^*C \arrow[dl, "\vartheta_{p_2}C"] \\
& C \arrow[ul, "\delta_1"] &
\end{tikzcd}$$

commutes at C, and

$$\begin{tikzcd}
& (\pi_1)^*p_2^*C \arrow[r, "j"] & (\pi_2)^*p_1^*C \arrow[dr, "(\pi_2)^*\xi"] & \\
(\pi_1)^*p_1^*C \arrow[ur, "(\pi_1)^*\xi"] \arrow[dr, "j_1^{-1}"'] & & & (\pi_2)^*p_2^*C \\
& \pi^*p_1^*C \arrow[r, "\pi^*\xi"] & \pi^*p_2^*C \arrow[ur, "j_2"'] &
\end{tikzcd}$$

commutes, where (p_1, p_2) is the kernel pair of p and

$$\pi = < p_1\pi_1, p_2\pi_2 >: (E \times_B E) \times_E (E \times_B E) \to E \times_B E$$

is given by the following pullback diagram:

$$\begin{tikzcd}
(E \times_B E) \times_E (E \times_B E) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \times_B E \arrow[d, "p_1"] \\
E \times_B E \arrow[r, "p_2"] & E
\end{tikzcd}$$

$\delta_i : C \to p_i^* C$ is such that $P\delta_i = \delta$ and $\vartheta_{p_i} C \cdot \delta_i = 1_C$ ($\delta : E \to E \times_B E$ is the "diagonal"), the canonical isomorphisms j and j_i arise from the identities

$$p_1\pi_2 = p_2\pi_1 \text{ and } p_i\pi = p_i\pi_i \quad (i = 1, 2)$$

as in 2.1.2.

These descent data, with morphisms $h : (C, \xi) \to (C', \xi')$ given by morphisms $h : C \to C'$ of $\mathbf{E}(E)$ such that

$$\begin{tikzcd}
p_1^* C \arrow[r, "p_1^* h"] \arrow[d, "\xi"'] & p_1^* C' \arrow[d, "\xi'"] \\
p_2^* C \arrow[r, "p_2^* h"] & p_2^* C'
\end{tikzcd}$$

commutes, form *the category* $\mathrm{Des}_{\mathbf{E}}(p)$.

For any $A \in \mathbf{E}(B)$, $p^* A$ is provided with canonical descent data

$$(j^{-1}_{p,p_2} A)(j_{p,p_1} A) : p_1^* p^* A \to p_2^* p^* A.$$

Hence p^* can be lifted to the comparison functor Φ^p such that

$$\begin{tikzcd}
\mathbf{E}(B) \arrow[rr, "\Phi^p"] \arrow[dr, "p^*"'] & & \mathrm{Des}_{\mathbf{E}}(p) \arrow[dl, "U^p"] \\
& \mathbf{E}(E) &
\end{tikzcd}$$

commutes.

p is called *an* $\mathbf{E}$*-descent (effective* $\mathbf{E}$*-descent) morphism* if Φ^p is full and faithful (an equivalence of categories).

2.1.4. Let $P : \mathbf{E} \to \mathbf{C}$ be a *bifibration* (i.e., both P and P^{op} are fibrations). Then, dually to *the inverse image functor* p^* and *the cleavage* ϑ_p, we have a *direct image functor*

$$p! : \mathbf{E}(E) \to \mathbf{E}(B)$$

and a *cocleavage*

$$\delta_p : J_E \to J_B p!.$$

Hence there exist uniquely determined natural transformations

$$\rho_p : 1_{\mathbf{E}(E)} \to p^* p! \text{ with } (\vartheta_p p!)(J_E \rho_p) = \delta_p,$$

$$\sigma_p : p! p^* \to 1_{\mathbf{E}(B)} \text{ with } (J_B \sigma_p)(\delta_p p^*) = \vartheta_p,$$

which serve as unit and counit of the adjunction $p! \dashv p^*$. Furthermore, we have *the Beck transformation*

$$\mathfrak{z}_p : (p_2)! p_1^* \to p^* p! \text{ with } (J_E \mathfrak{z}_p)(\delta_{p_2} p_1^*) = (J_E \rho_p) \vartheta_{p_1}.$$

One says that P *satisfies the Beck-Chevalley condition for* p if $\mathfrak{z}_p$ is a natural equivalence.

Example. *The basic fibration* $\partial : \mathbf{C}^2 \to \mathbf{C}$ *(see Example 2.1.1 (2)) satisfies the Beck-Chevalley condition for any* $p : E \to B$.

In fact, for any $(x : X \to E) \in \partial^{-1}(E)$,

$$(p_2)!p_1^*(x : X \to E) = (p_2\sigma_1 : E \times_B X \to E)$$

is given by the following pullback diagrams:

$$\begin{tikzcd}
E \times_B X \arrow[r, "\sigma_2"] \arrow[d, "\sigma_1"'] & X \arrow[d, "x"] \\
E \times_B E \arrow[r, "p_1"] \arrow[d, "p_2"'] & E \arrow[d, "p"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

On the other hand,

$$p^*p!(x : X \to E) = (\bar{\pi}_1 : E \times_B X \to E)$$

is given by the following pullback diagram:

$$\begin{tikzcd}
E \times_B X \arrow[r, "\bar{\pi}_2"] \arrow[dd, "\bar{\pi}_1"'] & X \arrow[d, "x"] \\
 & E \arrow[d, "p"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

Clearly, there is an isomorphism $(p_2\sigma_1 : E \times_B X \to E) \cong (\bar{\pi}_1 : E \times_B X \to E)$.

Given **E**-descent data $\xi : p_1^*C \to p_2^*C$ for $C \in \mathbf{E}(E)$, one has a bijective correspondence $(\xi \leftrightarrow (\xi)^{\#})$ by the universal property:

$$\begin{tikzcd}
(p_2)!p_1^*C \arrow[d, "(p_2)!\xi"'] \arrow[dr, "(\xi)^{\#}"] & & p_1^*C \arrow[d, "\xi"] \\
(p_2)!p_2^*C \arrow[r, "\sigma_{p_2}C"'] & C & p_2^*C
\end{tikzcd}$$

If $P : \mathbf{E} \to \mathbf{C}$ is a bifibration satisfying *the Beck-Chevalley condition for* p, then β_p is a natural equivalence, and so it yields a bijective correspondence between $(\xi)^{\#} : (p_2)!p_1^*C \to p^*p!C$ and the algebra structures $\theta : p^*p!C \to C$ with respect to the monad induced by $p! \dashv p^*$:

$$\begin{tikzcd}
(p_2)!p_1^*C \arrow[rr, "\beta_p C"] \arrow[dr, "(\xi)^{\#}"'] & & p^*p!C \arrow[dl, "\theta"] \\
& C &
\end{tikzcd}$$

This essentially establishes the bijective correspondence $(\xi \leftrightarrow \theta)$. Hence:

Theorem. (Bénabou-Roubaud [5], Beck) *For* $\mathbf{E}$ *bifibred over a category* $\mathbf{C}$ *with pullbacks and for* $p : E \to B$ *in* $\mathbf{C}$ *such that the Beck-Chevalley condition holds for* p, $\mathrm{Des}_{\mathbf{E}}(p)$ *is isomorphic to the Eilenberg-Moore category of the monad given by* $p! \dashv p^*$. *Hence* p *is an (effective)* $\mathbf{E}$*-descent morphism if and only if* p^* *is premonadic (monadic).*

2.2. Internal Categories, Actions, Indexed Categories, and Descent Theory

As mentioned in Chapter 0, the category of *descent data relative to a morphism* p can also be defined in the context of a $\mathbf{C}$-indexed category $\mathbb{A}$, namely the category of actions of the equivalence relation $\mathrm{Eq}(p)$ induced by p on $\mathbb{A}$. It is convenient to work with indexed categories instead of fibrations in some cases (see [19]). In this section, we summarize the notions of internal categories, actions, and indexed categories and some related results. Throughout this section, $\mathbf{C}$ denotes a category with pullbacks.

2.2.1. *An internal category* D in $\mathbf{C}$ is given by a diagram

$$\begin{tikzcd} D_2 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_1"'] & D_1 \arrow[r, shift left=2, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=2, "c"'] & D_0 \end{tikzcd}$$

which satisfies

I1. $de = 1_{D_0} = ce$,

I2. $dm = d\pi_2$, $cm = c\pi_1$,

I3. $m(1_{D_1} \times m) = m(m \times 1_{D_1})$,

I4. $m < 1_{D_1}, ed > = 1_{D_1} = m < ec, 1_{D_1} >$,

where $D_0, D_1 \in \mathrm{ob}\mathbf{C}$, $d,\ c,\ e,\ m \in \mathrm{Mor}\mathbf{C}$, and $D_2,\ \pi_1,\ \pi_2$ are given by the following pullback diagram in $\mathbf{C}$:

$$\begin{tikzcd} D_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D_1 \arrow[d, "c"] \\ D_1 \arrow[r, "d"] & D_0 \end{tikzcd}$$

An internal functor $f : D \to D'$ between two internal categories D, D' in $\mathbf{C}$ is given by two morphisms $f_0 : D_0 \to D'_0, f_1 : D_1 \to D'_1$ of $\mathbf{C}$ such that

F1. $f_0 d = d' f_1,\ f_0 c = c' f_1,$

F2. $f_1 e = e' f_0,\ f_1 m = m' f_2,$

where $f_2 = f_1 \times f_1 : D_1 \times_{D_0} D_1 \to D'_1 \times_{D'_0} D'_1$.

Composition of internal functors is given by the obvious way. Hence we obtain $\mathbf{cat}(\mathbf{C})$-the category of all internal categories and internal functors in $\mathbf{C}$. $\mathbf{cat}(\mathbf{C})$ is actually a 2-category (see [19]) since one can define *an internal natural transformation* $\alpha : f \to g$ of internal functors $f,\ g : D \to D'$, which is given by a morphism $\alpha : D_0 \to D'_1$ in $\mathbf{C}$ such that

T1. $d'\alpha = f_0,\ c'\alpha = g_0;$

T2. $m' < \alpha c,\ f_1 > = m' < g_1,\ \alpha d >$.

Let $f,\ g,\ h : D \to D'$ be internal functors and let $\alpha : f \to g$ and $\beta : g \to h$ be *internal natural transformation.* one defines the *composition* $\beta\alpha : f \to h$ to be the morphism

$$m' < \beta,\ \alpha >: D_0 \to D'_1,$$

and the *identity internal natural transformation* $1_f : f \to f$ to be the morphism

$$e' f_0 : D_0 \to D'_1.$$

An internal functor $f : D \to D'$ of $\mathbf{C}$ is an *internal category equivalence* if there is an internal functor $g : D' \to D$ such that

$$gf \cong 1_D \ \text{ and } \ fg \cong 1_{D'}.$$

For example, if $p : E \to B$ is a morphism in $\mathbf{C}$, then

$$\begin{tikzcd}[column sep=large] (E \times_B E) \times_E (E \times_B E) \cong E \times_B E \times_B E \arrow[r, shift left=2, "\pi_{23}"] \arrow[r, "\pi_{13}" description] \arrow[r, shift right=2, "\pi_{12}"'] & E \times_B E \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\pi_1"'] & E \arrow[l, "e" description] \end{tikzcd}$$

is an internal category in $\mathbf{C}$, where $e =< 1_E,\ 1_E >$, $(\pi_1,\ \pi_2)$ is the kernel pair of p, π_{12} and π_{23} are such that $\pi_1\pi_{23} = \pi_2\pi_{12}$ (pullback square) and $\pi_{13} =< \pi_1\pi_{12},\ \pi_2\pi_{23} >$. This internal category is denoted by $\mathrm{Eq}(p)$. If $B \in \mathrm{ob}(\mathbf{C})$, then B can be viewed as a discrete internal category B of $\mathbf{C}$:

$$\begin{tikzcd}[column sep=large] B \arrow[r, shift left=2, "1_B"] \arrow[r, "1_B" description] \arrow[r, shift right=2, "1_B"'] & B \arrow[r, shift left=2, "1_B"] \arrow[r, shift right=2, "1_B"'] & B \arrow[l, "1_B" description] \end{tikzcd}$$

Clearly $\mathrm{Eq}(1_B)$ is isomorphic to the above discrete internal category B.

2.2.2. Let D be an internal category in **C**. *An action* of D in **C** is given by a triple (C, γ, ξ) such that

A1. $\gamma\xi = c\cdot \mathrm{proj}_1$,

A2. $\xi(m \times 1_C) = \xi(1_{D_1} \times \xi)$,

A3. $\xi < e\gamma, 1_C > = 1_C$,

where $C \in \mathrm{ob}\mathbf{C}$, $\gamma : C \to D_0$ and $\xi : D_1 \times_{D_0} C \to C$ are morphisms in **C**, and $D_1 \times_{D_0} C$ is given by the pullback diagram

$$\begin{tikzcd} D_1 \times_{D_0} C \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"'] & C \arrow[d, "\gamma"] \\ D_1 \arrow[r, "d"] & D_0 \end{tikzcd}$$

Morphisms $h : (C, \gamma, \xi) \to (C', \gamma', \xi')$ between two actions of D are given by **C**-morphisms $h : C \to C'$ over D_0 ($\gamma' h = \gamma$) such that

M1. $h\xi = \xi'(1_{D_1} \times h)$.

All actions and morphisms between actions of D form the category $\mathbf{C}^D$- actions of D in **C**.

Example. Let D be an internal category in **C**. One defines functors

$$\begin{tikzcd} \mathbf{C}^D \arrow[r, shift left, leftarrow, "F^D"] \arrow[r, shift right, "U^D"'] & \mathbf{C}/D_0 \end{tikzcd}$$

by

$$U^D(C,\gamma,\xi) = (C,\gamma) \text{ for each } (C,\gamma,\xi) \in \mathrm{ob}\mathbf{C}^D,$$

$$F^D(C,\gamma) = (D_1 \times_{D_0} C, c \cdot \mathrm{proj}_1, m \times_{D_0} 1) \text{ for each } (C,\gamma) \in \mathrm{ob}\mathbf{C}/D_0,$$

and the obvious assignments for morphisms. Then F^D is a left adjoint of U^D. By Theorem 1.2.4 (3), U^D is monadic.

2.2.3. *A* **C**-*indexed category* $\mathbb{A}$ is a pseudo-functor $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ such that for every $f : E \to D$, $g : D \to C$ in **C** there are natural isomorphisms:

$$i^D : 1_{\mathbb{A}^D} \to (1_D)^*, \quad j^{f,g} : f^*g^* \to (gf)^*$$

which satisfy that

$$\begin{tikzcd}
f^* \arrow[r, "f^*i^D"] \arrow[d, "i^E f^*"'] \arrow[dr, "1_{f^*}"] & f^*(1_D)^* \arrow[d, "j^{f,1_D}"] \\
(1_E)^*f^* \arrow[r, "j^{1_E,f}"'] & f^*
\end{tikzcd}$$

and

$$\begin{tikzcd}
f^*g^*h^* \arrow[r, "f^*j^{g,h}"] \arrow[d, "j^{f,g}h^*"'] & f^*(hg)^* \arrow[d, "j^{f,hg}"] \\
(gf)^*h^* \arrow[r, "j^{gf,h}"'] & (hgf)^*
\end{tikzcd}$$

commute, where $\mathbb{A}^D = \mathbb{A}(D)$ and x^* denotes $\mathbb{A}(x) : \mathbb{A}^D \to \mathbb{A}^E$ for any morphism $x : E \to D$ in $\mathbf{C}$.

For example,

$$\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$$

given by $B \longmapsto \mathbf{C}/B$ and $(f : E \to B) \longmapsto f^* : \mathbf{C}/B \to \mathbf{C}/E$, the pullback functor along f, is a $\mathbf{C}$-indexed category, and we call it the *basic* $\mathbf{C}$-indexed category.

Grothendieck Construction. Given a $\mathbf{C}$-indexed category $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$, one defines a category $G(\mathbf{C}, \mathbb{A})$ as follows:

- An object of $G(\mathbf{C}, \mathbb{A})$ is a pair (C, x), where C is an object of $\mathbf{C}$ and x is an object of $\mathbb{A}(C)$.

- A morphism $(f, u) : (C, x) \to (C', x')$ consists of a $\mathbf{C}$-morphism $f : C \to C'$ and a $\mathbb{A}(C)$-morphism $u : x \to \mathbb{A}(f)(x')$.

- If $(f, u) : (C, x) \to (C', x')$, $(g, v) : (C', x') \to (C'', x'')$ are morphisms of $G(\mathbf{C}, \mathbb{A})$, then

  $$(g, v) \circ (f, u) : (C, x) \to (C'', x'')$$

  is given by

  $$(g, v) \circ (f, u) = (g \circ f, \mathbb{A}(f)(v) \circ u).$$

The projection functor $P : G(\mathbf{C}, \mathbb{A}) \to \mathbf{C}$ given by

$$\begin{tikzcd}
(C, x) \arrow[d, "{(f,u)}"] & \mapsto & C \arrow[d, "f"] \\
(C', x') & \mapsto & C'
\end{tikzcd}$$

is a fibration. This process is called *the Grothendieck construction* (see [4]).

Clearly, mod: MOD $\to (\mathbf{CRng}_1)^{\mathrm{op}}$ is given by the Grothendieck construction applied to the $(\mathbf{CRng}_1)^{\mathrm{op}}$-indexed category $R \mapsto$**Mod**-R, in Example 2.1.1(3). Hence it is a fibration.

2.2.4. Let $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ be a $\mathbf{C}$-indexed category and D an internal category in $\mathbf{C}$ as in 2.2.1. One defines $\mathbb{A}^D$ to be the category with

- objects: pairs of (C, ξ), where $C \in \mathrm{ob}\mathbb{A}^{D_0}$ and $\xi : d^*C \to c^*C$ is a morphism in $\mathbb{A}^{D_1}$ such that

$$\begin{tikzcd}
e^*d^*C \arrow[rr, "e^*\xi"] \arrow[dr, "\cong"'] & & e^*c^*C \arrow[dl, "\cong"] \\
& C &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& (\pi_2)^*c^*C \arrow[r, "\cong"] & (\pi_1)^*d^*C \arrow[dr, "(\pi_1)^*\xi"] & \\
(\pi_2)^*d^*C \arrow[ur, "(\pi_2)^*\xi"] \arrow[dr, "\cong"'] & & & (\pi_1)^*c^*C \\
& m^*d^*C \arrow[r, "m^*\xi"] & m^*c^*C \arrow[ur, "\cong"'] &
\end{tikzcd}$$

commute, in $\mathbb{A}^{D_0}$ and $\mathbb{A}^{D_2}$, respectively. The above natural "$\cong$"'s arise from I1 and I2 in 2.2.1 and Definition 2.2.3.

- morphisms: $h : (C, \xi) \to (C', \xi')$ of $\mathbb{A}^D$ are given by morphisms of $h : C \to C'$ of $\mathbb{A}^{D_0}$ such that

$$\begin{tikzcd} d^*C \arrow[r, "d^*h"] \arrow[d, "\xi"'] & d^*C' \arrow[d, "\xi'"] \\ c^*C \arrow[r, "c^*h"] & c^*C' \end{tikzcd}$$

commutes in $\mathbb{A}^{D_1}$.

2.2.5. Janelidze and Tholen [19] proved:

Theorem. *For every* $\mathbf{C}$*-indexed category* $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$*, the extension*

$$\mathbb{A} : \mathbf{cat}(\mathbf{C})^{\mathrm{op}} \to \mathbf{CAT}$$

given by the assignment $D \to \mathbb{A}^D$*, is a pseudo-functor of* 2*-categories.*

They also proved:

Lemma. *For every* $\mathbf{C}$*-indexed category* $\mathbb{A}$*, and every internal category equivalence* $f : D \to D'$ *of* $\mathbf{C}$*, the functor* $f^* : \mathbb{A}^{D'} \to \mathbb{A}^D$ *is an equivalence of categories.*

2.2.6. Let $\mathbb{A}$ be a $\mathbf{C}$-indexed category and $p : E \to B$ a morphism of $\mathbf{C}$. The category

$$\mathrm{Des}_{\mathbb{A}}(p) = \mathbb{A}^{\mathrm{Eq}(p)}$$

is called *the category of* $\mathbb{A}$*-descent data relative to* p.

Since the discrete functor $p : E \to B$ can be factored as

$$\begin{tikzcd} B & & \mathrm{Eq}(p) \arrow[ll, "\overline{p}"'] \\ & E \arrow[ul, "p"] \arrow[ur, "\bar{\delta}"'] & \end{tikzcd}$$

where $\overline{p}_0 = p$, $\overline{p}_1 = p\pi_1 = p\pi_2$, $\bar{\delta}_0 = 1_E$, $\bar{\delta}_1 = e$, and (π_1, π_2) is the kernel pair of p, we have a commutative diagram (up to natural isomorphisms) in **CAT**:

$$\begin{tikzcd} \mathbb{A}^B \arrow[rr, "\Phi^p = \overline{p}^*"] \arrow[dr, "p^*"'] & & \mathrm{Des}_{\mathbb{A}}(p) \arrow[dl, "\bar{\delta}^*"] \\ & \mathbb{A}^E & \end{tikzcd}$$

p is called *an* $\mathbb{A}$*-descent (effective* $\mathbb{A}$*-descent) morphism* if the comparison functor Φ^p is full and faithful (an equivalence of categories). p is called *an absolutely (effective) descent morphism* if it is an (effective) $\mathbb{A}$-descent morphism for every $\mathbf{C}$-indexed category $\mathbb{A}$.

For $\mathbb{A}$ defined by a fibration $P : \mathbf{E} \to \mathbf{C}$, the category $\mathrm{Des}_{\mathbb{A}}(p)$ is the category $\mathrm{Des}_{\mathbf{E}}(p)$ in 2.1.3, and the notion of (effective) $\mathbb{A}$-descent is equivalent to the notion of (effective) $\mathbf{E}$-descent in 2.1.3 (see [19], Section 3.3, for reasons).

Janelidze and Tholen [19] also showed:

Theorem. *A morphism in a category with pullbacks is an absolutely effective morphism if and only if it is a split epimorphism.*

3. Descent Equivalence

Throughout this chapter, $\mathbf{C}$ is a category with pullbacks, and $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ is a $\mathbf{C}$-indexed category. Let $B \in \mathrm{ob}\mathbf{C}$, for a given morphism $q : (E, p) \to (X, \varphi)$ in $\mathbf{C}/B$, one may ask:

Which conditions can guarantee that bundles p and φ have the same $\mathbb{A}$-descent structure?

In this chapter, we give the notion of descent equivalence and study its properties.

3.1. The Functor Eq

3.1.1. For any morphism $q : (E, p) \to (X, \varphi)$ in $\mathbf{C}/B$, Janelidze and Tholen [19] constructed the following commutative diagram in $\mathbf{cat}(\mathbf{C})$:

$$\begin{tikzcd}
\mathrm{Eq}(q) \arrow[r, "\overline{q}"] \arrow[d, "i_\varphi"'] & X \arrow[d, "\delta_X"] \\
\mathrm{Eq}(p) \arrow[r, "\tilde{q}"] & \mathrm{Eq}(\varphi)
\end{tikzcd} \tag{1}$$

where $(i_\varphi)_0 = 1_E$, $(i_\varphi)_1 = 1_E \times_\varphi 1_E$, $(\tilde{q})_0 = q$, $(\tilde{q})_1 = q \times_B q$, $(\delta_X)_0 = 1_X$, $(\delta_X)_1 = \Delta_X$, $(\overline{q})_0 = q$, $(\overline{q})_1 = q\overline{\pi}_1 = q\overline{\pi}_2$, and $(\overline{\pi}_1, \overline{\pi}_2)$ is the kernel pair of q.

They called it the *basic equivalence diagram (BED)* induced by the morphism $q : (E, p) \to (X, \varphi)$. They proved:

Proposition. *Each BED is a pullback diagram in the ordinary category* $\mathbf{cat}(\mathbf{C})$, *and it is also a pushout diagram in* $\mathbf{cat}(\mathbf{C})$ *if q is a pullback-stable regular epimorphism of* $\mathbf{C}$.

Recall that for each morphism $f : D \to C$ in a category with binary products and for each object X in the category,

$$\begin{tikzcd} D \times X \arrow[r, "f \times 1"] \arrow[d] & C \times X \arrow[d] \\ D \arrow[r, "f"] & C \end{tikzcd}$$

and

$$\begin{tikzcd} X \times D \arrow[r, "1 \times f"] \arrow[d] & X \times C \arrow[d] \\ D \arrow[r, "f"] & C \end{tikzcd}$$

are pullback diagrams.

Since products in $\mathbf{C}/B$ are given by pullbacks in $\mathbf{C}$, we have:

3.1.2. Lemma. *If $q : (E, p) \to (X, \varphi)$ is a morphism of $\mathbf{C}/B$, then, for any* $(Y, y) \in \mathrm{ob}(\mathbf{C}/B)$,

$$\begin{tikzcd}
E \times_B Y \arrow[r, "q \times_B 1_Y"] \arrow[d, "\pi_1"'] & X \times_B Y \arrow[d, "\bar{\pi}_1"] \\
E \arrow[r, "q"] & X
\end{tikzcd}$$

is a pullback diagram, where π_1 *and* $\bar{\pi}_1$ *are given by the following pullback diagrams:*

$$\begin{tikzcd}
E \times_B Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "y"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \times_B Y \arrow[r, "\bar{\pi}_2"] \arrow[d, "\bar{\pi}_1"'] & Y \arrow[d, "y"] \\
X \arrow[r, "\varphi"] & B
\end{tikzcd}$$

Similarly,

$$\begin{tikzcd}
Y \times_B E \arrow[r, "1_Y \times_B q"] \arrow[d] & Y \times_B X \arrow[d] \\
E \arrow[r, "q"] & X
\end{tikzcd}$$

is also a pullback diagram.

If

$$
\begin{tikzcd}
Y \arrow[r, "q"] \arrow[d, "y"'] & X \arrow[d, "x"] \\
E \arrow[r, "p"] & B
\end{tikzcd}
\tag{2}
$$

is a commutative diagram in $\mathbf{C}$, then we have the following commutative diagram in $\mathbf{cat}(\mathbf{C})$:

$$
\begin{tikzcd}
\mathrm{Eq}(q) \arrow[r, "\overline{q}"] \arrow[d, "c_y"'] & X \arrow[d, "x"] \\
\mathrm{Eq}(p) \arrow[r, "\overline{p}"] & B
\end{tikzcd}
\tag{3}
$$

where $(c_y)_0 = y$, $(c_y)_1 = y \times_x y$ which is given by

$$
\begin{tikzcd}
& E \times_B E \arrow[rr, "\pi_2"] \arrow[dd, "\pi_1" near start] & & E \arrow[dd, "p"] \\
Y \times_X Y \arrow[ur, "y \times_x y"] \arrow[rr, "\overline{\pi}_2" near start] \arrow[dd, "\overline{\pi}_1"'] & & Y \arrow[ur, "y"'] \arrow[dd, "q" near start] & \\
& E \arrow[rr, "p"' near end] & & B \\
Y \arrow[ur, "y"'] \arrow[rr, "q"'] & & X \arrow[ur, "x"'] &
\end{tikzcd}
\tag{4}
$$

here the front face and the back face are pullback diagrams in $\mathbf{C}$, and the other faces are commutative.

Note that the morphism $y \times_x y$ can be decomposed as

$$Y \times_X Y \xrightarrow{1 \times_x 1} Y \times_B Y \xrightarrow{1 \times_B y} Y \times_B E \xrightarrow{y \times_B 1} E \times_B E$$

Proposition. 1. *If* (2) *is a pullback diagram, then*

(i) *the left face and top face of* (4) *are pullback diagrams in* **C**.

(ii) (3) *is a pullback diagram in* **cat**(**C**).

2. *If y is a regular epimorphism and $y \times_x y$ is an epimorphism, then the left face of* (4) *is a pushout diagram in* **C**.

Moreover, if (2) *is a pushout diagram in* **C**, *then* (3) *is a pushout diagram in* **cat**(**C**).

Proof. 1. (i) If (2) is a pullback diagram, then, by the pullback diagram composition law, in diagram (4), the left face + the back face = the front face + the right face is a pullback diagram. But the back face is a pullback diagram, so, by the pullback diagram cancellation law, the left face of is a pullback diagram. Similarly, the top face of (4) is a pullback diagram, as desired.

(ii) Let $f : W \to \mathrm{Eq}(p)$, $g : W \to X$ be morphisms in **cat**(**C**) such that

$$\overline{p}f = xg.$$

Then

$$pf_0 = xg_0,$$

$$p\pi_1 f_1 = p\pi_2 f_1 = xg_1.$$

Hence there are a unique morphism $h_0 : W_0 \to Y$ such that

$$yh_0 = f_0,\ qh_0 = g_0,$$

and a unique morphism $h_1 : W_1 \to Y \times_X Y$ such that

$$(y \times_x y)h_1 = f_1,\ q\overline{\pi}_1 h_1 = q\overline{\pi}_2 h_1 = g_1.$$

It is easy to check that $(h_0,\ h_1) : D \to \mathrm{Eq}(q)$ is a morphism of $\mathbf{cat}(\mathbf{C})$. Hence (3) is a pullback diagram.

2. Let $w_1 : Y \to W$ and $w_2 : E \times_B E \to W$ be morphisms in $\mathbf{C}$ such that

$$w_1\overline{\pi}_1 = w_2(y \times_x y).$$

We claim that for all $u_1,\ u_2 : U \to Y$,

$$yu_1 = yu_2 \Rightarrow w_1u_1 = w_1u_2.$$

There are $u,\ u' : U \to Y \times_X Y$ such that

$$\overline{\pi}_1 u = u_1,\ \overline{\pi}_2 u = u_1,\ \text{ and }\ \overline{\pi}_1 u' = u_2,\ \overline{\pi}_2 u' = u_2.$$

But

$$\pi_1(y \times_x y)u = y\overline{\pi}_1 u = yu_1 = yu_2 = y\overline{\pi}_1 u'$$

$$= \pi_1(y \times_x y)u',$$

$$\pi_2(y \times_x y)u = y\overline{\pi}_2 u = yu_1 = yu_2 = y\overline{\pi}_2 u'$$

$$= \pi_2(y \times_x y)u'.$$

So

$$(y \times_x y)u = (y \times_x y)u'.$$

Hence

$$w_1 u_1 = w_1\overline{\pi}_1 u = w_2(y \times_x y)u = w_2(y \times_x y)u' = w_1\overline{\pi}_1 u'$$

$$= w_1 u_2.$$

Since y is a regular epimorphism, there is a unique $w : E \to W$ such that

$$wy = w_1.$$

On the other hand,

$$w\pi_1(y \times_x y) = wy\overline{\pi}_1 = w_1\overline{\pi}_1 = w_2(y \times_x y).$$

Hence $w\pi_1 = w_2$, as desired.

Let $f : \mathrm{Eq}(p) \to D$, $g : X \to D$ be morphisms in $\mathbf{cat}(\mathbf{C})$ such that

$$fc_y = g\overline{q}.$$

Then

$$f_0 y = g_0 q,$$

$$f_1(y \times_x y) = g_1 q\overline{\pi}_1 = g_1 q\overline{\pi}_2.$$

If (2) is a pushout diagram, then there is a unique $h_0 : B \to D_0$ and a unique $h_1 : B \to D_1$ such that

$$h_0 x = g_0, \ h_0 p = f_0$$

and

$$h_1 x = g_1, \ h_1 p \pi_1 = h_1 p \pi_2 = f_1.$$

Since y is an epimorphism, x is an epimorphism by the pushout stability of epimorphisms. Now it is routine to check that $(h_0, \ h_1) : B \to D$ is an internal functor. Hence (3) is a pushout diagram in $\mathbf{cat}(\mathbf{C})$. □

For a morphism $q : (E, \ p) \to (X, \ \varphi)$ in $\mathbf{C}/B$, by considering the commutative diagram:

$$\begin{tikzcd} E \arrow[rr, "q"] \arrow[dr, "p"'] & & X \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

with (3) we get the commutative diagram:

$$\begin{tikzcd} \mathrm{Eq}(p) \arrow[rr, "\bar{q} = c_q"] \arrow[dr, "\overline{p}"'] & & \mathrm{Eq}(\varphi) \arrow[dl, "\overline{\varphi}"] \\ & B & \end{tikzcd}$$

Moreover, one can also look at

$$
\begin{tikzcd}
E \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "\varphi"] \\
B \arrow[r, "1_B"] & B
\end{tikzcd}
$$

If

$$
\begin{tikzcd}
W \arrow[rr, "q'"] \arrow[d, "x'"'] \arrow[ddr, "px'" near start] & & Y \arrow[d, "x"] \arrow[ddl, "\varphi x"' near start] \\
E \arrow[rr, "q"'] \arrow[dr, "p"'] & & X \arrow[dl, "\varphi"] \\
& B &
\end{tikzcd}
\tag{5}
$$

is commutative in $\mathbf{C}/B$, then one has the following commutative diagram in $\mathbf{cat}(\mathbf{C})$:

$$
\begin{tikzcd}
& \mathrm{Eq}(\varphi) & & \mathrm{Eq}(p) \arrow[ll, "\tilde{q}"'] \\
X \arrow[ur, "\delta_X"] & & \mathrm{Eq}(q) \arrow[ll, "\overline{q}"] \arrow[ur, "i_\varphi"] & \\
& \mathrm{Eq}(\varphi x) \arrow[uu, "\tilde{x}"] & & \mathrm{Eq}(px') \arrow[ll, "\tilde{q}'"'] \arrow[uu, "\tilde{x}'"'] \\
Y \arrow[uu, "x"] \arrow[ur, "\delta_Y"] & & \mathrm{Eq}(q') \arrow[ll, "\overline{q'}"] \arrow[uu, "c_{x'}"'] \arrow[ur, "i_{\varphi x}"'] &
\end{tikzcd}
\tag{6}
$$

3.1.3. Let $\mathbf{C}$ be a category with pullbacks. Recall that for a morphism $p : E \to B$ in $\mathbf{C}$ we have the internal category $\mathrm{Eq}(p)$ (see 2.2.1). Then, for a fixed object B of $\mathbf{C}$, the assignments:

$$(E,\ p) \mapsto \mathrm{Eq}(p) \ \text{ and } \ (q : (E,\ p) \to (X,\ \varphi)) \mapsto \tilde{q},$$

define a functor:

$$\mathrm{Eq} : \ \mathbf{C}/B \to \mathbf{cat}(\mathbf{C}).$$

3.1.4. Proposition. *Let* $\mathbf{C}$ *be a category with pullbacks and* B *a fixed object of* $\mathbf{C}$. *Then the functor*

$$\mathrm{Eq} : \ \mathbf{C}/B \to \mathbf{cat}(\mathbf{C})$$

preserves those limits which are preserved by the forgetful functor $\mathbf{C}/B \to \mathbf{C}$.

Proof. Let $p : E \to B$ be the limit of $p_i : E_i \to B$ in $\mathbf{C}/B$:

$$\begin{tikzcd} E \arrow[rr, "t_i"] \arrow[dr, "p"'] & & E_i \arrow[dl, "p_i"] \\ & B & \end{tikzcd}$$

such that $\lim E_i = E$. We claim that $\mathrm{Eq}(p) = \lim \mathrm{Eq}(p_i)$ in $\mathbf{cat}(\mathbf{C})$.

For any I-cone $(u_i,\ v_i) : D \to \mathrm{Eq}(p_i)$ in $\mathbf{cat}(\mathbf{C})$, since $\lim E_i = E$, there is a unique $x : D_0 \to E$ such that

$$t_i x = u_i \ \text{ for all } i$$

Similarly, since $\lim E_i \times_B E_i = E \times_B E$, and $\lim E_i \times_B E_i \times_B E_i = E \times_B E \times_B E$, there are a unique morphism $y : D_1 \to E \times_B E$ with

$$(t_i \times t_i)y = v_i \text{ for all } i$$

and a unique morphism $z : D_2 \to E \times_B E \times_B E$ $(z \cong y \times y)$ with

$$(t_i \times t_i \times t_i)z = v_i \times v_i \text{ for all } i$$

$$\begin{tikzcd}
E \times_B E \times_B E \arrow[r, shift left=2, "\pi_{2,3}"] \arrow[r, "\pi_{1,3}" description] \arrow[r, shift right=2, "\pi_{1,2}"'] \arrow[d, "t_i \times t_i \times t_i"] & E \times_B E \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\pi_1"'] \arrow[d, "t_i \times t_i"] & E \arrow[l, "e" description] \arrow[d, "t_i"] \\
E_i \times_B E_i \times_B E_i \arrow[r, shift left=2, "\pi_{i,2,3}"] \arrow[r, "\pi_{i,1,3}" description] \arrow[r, shift right=2, "\pi_{i,1,2}"'] & E_i \times_B E_i \arrow[r, shift left=2, "\pi_{i2}"] \arrow[r, shift right=2, "\pi_{i1}"'] & E_i \arrow[l, "e_i" description] \\
D_2 \arrow[r, shift left=2, "\sigma_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "\sigma_1"'] \arrow[u, "v_i \times v_i"'] \arrow[uu, bend left, "z"] & D_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[u, "v_i"'] \arrow[uu, bend left, "y"] & D_0 \arrow[l, "f" description] \arrow[u, "u_i"'] \arrow[uu, bend left, "x"]
\end{tikzcd}$$

Note that, for all i,

$$t_i\pi_1 y = \pi_{1i}(t_i \times t_i)y = \pi_{1i}v_i = u_i c = t_i x c,$$

$$t_i\pi_2 y = \pi_{2i}(t_i \times t_i)y = \pi_{2i}v_i = u_i d = t_i x d,$$

and

$$(t_i \times t_i)ex = e_i t_i x = e_i u_i = v_i f = (t_i \times t_i)yf.$$

Then

$$\pi_1 y = xc, \quad ex = yf, \text{ and } \pi_2 y = xd.$$

Similarly, we can check that

$$
\begin{tikzcd}
E \times_B E \times_B E \arrow[rr, shift left=2, "\pi_{2,3}"] \arrow[rr, "\pi_{1,3}"] \arrow[rr, shift right=2, "\pi_{1,2}"'] & & E \times_B E \\
D_2 \arrow[u, "z"] \arrow[rr, shift left=2, "\sigma_2"] \arrow[rr, "m"] \arrow[rr, shift right=2, "\sigma_1"'] & & D_1 \arrow[u, "y"']
\end{tikzcd}
$$

commutes serially. Therefore $\lim \mathrm{Eq}(p_i) = \mathrm{Eq}(p)$ in $\mathbf{cat}(\mathbf{C})$. □

3.1.5. Corollary. *Let* C *be a category with pullbacks and* B *a fixed object of* C. *Then*

$$\mathrm{Eq}: \ \mathrm{C}/B \to \mathbf{cat}(\mathrm{C})$$

preserves pullback diagrams and inverse limits.

Proof. It is easy to check that the forgetful functor $\mathrm{C}/B \to \mathrm{C}$ preserves pullback diagrams and inverse limits. By Theorem 3.1.4, the Corollary is clear. □

3.2. Descent Equivalence

3.2.1. Let C be a category with pullbacks and $\mathbb{A}$: $\mathrm{C}^{\mathrm{op}} \to \mathbf{CAT}$ a C-indexed category. one has a pseudo-functor

$$\mathbb{A} : \mathbf{cat}(\mathrm{C})^{\mathrm{op}} \to \mathbf{CAT}$$

of 2-categories, which extends the C-indexed category $\mathbb{A} : \mathrm{C}^{\mathrm{op}} \to \mathbf{CAT}$ (see 2.2.5

or [19]). Recall that for a morphism $p : E \to B$ in $\mathbf{C}$, one defines

$$\mathrm{Des}_{\mathbb{A}}(E,\ p) = \mathbb{A}^{\mathrm{Eq}(p)}.$$

Hence, for any fixed object B of $\mathbf{C}$, $\mathrm{Des}_{\mathbb{A}}(\ ) = \mathbb{A} \circ \mathrm{Eq}$ becomes a pseudo-functor:

$$\begin{tikzcd}
(\mathbf{C}/B)^{\mathrm{op}} \arrow[rr, "\mathrm{Des}_{\mathbb{A}}(\ )"] \arrow[dr, "\mathrm{Eq}"'] & & (\mathbf{C}/B)\backslash\mathbf{CAT} \\
& \mathbf{cat}(\mathbf{C})^{\mathrm{op}} \arrow[ur, "\mathbb{A}"'] &
\end{tikzcd}$$

Proposition. *Let* $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ *be a* $\mathbf{C}$*-indexed category such that*

$\mathbb{A} : \mathbf{cat}(\mathbf{C})^{\mathrm{op}} \to \mathbf{CAT}$ *preserves pushout diagrams (directed colimits).*

Then so does $\mathrm{Des}_{\mathbb{A}}(\ )$.

Proof. By Corollary 3.1.5, Eq: $(\mathbf{C}/B)^{\mathrm{op}} \to (\mathbf{cat}(\mathbf{C}))^{\mathrm{op}}$ preserves pushout diagrams (directed colimits), so does

$$\mathrm{Des}_{\mathbb{A}}(\ ) = \mathbb{A} \circ \mathrm{Eq}.$$

□

For a morphisms $q : (E,\ p) \to (X,\ \varphi)$ in $\mathbf{C}/B$, Reiterman, Sobral, and Tholen [26] considered the following diagram in $\mathbf{cat}(\mathbf{C})$:

$$\begin{tikzcd}
 & \mathrm{Eq}(1_B) & \\
\mathrm{Eq}(\varphi) \arrow[ur, "\overline{\varphi}"] & & \mathrm{Eq}(p) \arrow[ul, "\overline{p}"'] \arrow[ll, "\tilde{q}"] \\
\mathrm{Eq}(1_X) \arrow[u, "\delta_X"] & & \mathrm{Eq}(q) \arrow[u, "i_\varphi"'] \arrow[ll, "\overline{q}"']
\end{tikzcd} \tag{7}$$

where $(i_\varphi)_0 = 1_E$, $(i_\varphi)_1 = 1_E \times_\varphi 1_E$, $(\tilde{q})_0 = q$, $(\tilde{q})_1 = q \times_B q$, $(\delta_X)_0 = 1_X$, $(\delta_X)_1 = \Delta_X$, $(\overline{p})_0 = p$, $(\overline{p})_1 = p\pi_1 = p\pi_2$, and (π_1, π_2) is the kernel pair of p.

Applying $\mathbb{A}$ to the above diagram (7), one obtains the following commutative diagram (up to natural isomorphisms) in **CAT**:

$$\begin{tikzcd}
 & \mathbb{A}^B \arrow[dl, "\Phi^\varphi"'] \arrow[dr, "\Phi^p"] & \\
\mathrm{Des}_{\mathbb{A}}(X, \varphi) \arrow[rr, "\mathrm{Des}_{\mathbb{A}}(q)"'] \arrow[d, "U^\varphi"'] & & \mathrm{Des}_{\mathbb{A}}(E, p) \arrow[d, "V^\varphi"] \\
\mathbb{A}^X \arrow[rr, "\Phi^q"'] & & \mathrm{Des}_{\mathbb{A}}(E, q)
\end{tikzcd} \tag{8}$$

where $U^\varphi = (\delta_X)^*$, $V^\varphi = (i_\varphi)^*$, $\Phi^p = (\overline{p})^*$, and $\mathrm{Des}_{\mathbb{A}}(q) = (\tilde{q})^*$.

3.2.2. Definition. *Let $q : (E, p) \to (X, \varphi)$ be a morphism in $\mathbf{C}/B$. We call q*

an $\mathbb{A}$-descent equivalence ($\mathbb{A}$-descent pre-equivalence) *if* $\mathrm{Des}_{\mathbb{A}}(q)$ *is an equivalence of categories (full and faithful). We call* q an absolute descent equivalence (absolute descent pre-equivalence) *if* $\mathrm{Des}_{\mathbb{A}}(q)$ *is an equivalence of categories (full and faithful) for every* $\mathbf{C}$*-indexed category* $\mathbb{A}$.

By the functoriality of $\mathrm{Des}_{\mathbb{A}}(\ )$, immediately we have:

Proposition. *Let* $q : (E,\ p) \to (X,\ \varphi)$, $s : (X,\ \varphi) \to (Y,\ \xi)$ *be morphisms in* $\mathbf{C}/B$.

(1) *If two of* q, s, *and* sq *are* $\mathbb{A}$*-descent equivalences, so is the third one.*

(2) *If* s *is an* $\mathbb{A}$*-descent pre-equivalence, then* q *is an* $\mathbb{A}$*-descent pre-equivalence if and only if* sq *is an* $\mathbb{A}$*-descent pre-equivalence.*

The following result shows us why $\mathbb{A}$-descent (pre-)equivalence is a useful notion.

Invariance Theorem. *Let* $q : (E,\ p) \to (X,\ \varphi)$ *be an* $\mathbb{A}$*-descent pre-equivalence (*$\mathbb{A}$*-descent equivalence) in* $\mathbf{C}/B$. *Then* p *is an* $\mathbb{A}$*-descent (effective* $\mathbb{A}$*-descent) morphism if and only if* φ *is an* $\mathbb{A}$*-descent (effective* $\mathbb{A}$*-descent) morphism.*

Proof. By Diagram 3.2.1(7), $\mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi} = \Phi^{p}$ (up to natural isomorphism). If q is an $\mathbb{A}$-descent equivalence, then $\mathrm{Des}_{\mathbb{A}}(q)$ is an equivalence of categories and therefore Φ^{φ} is an equivalence of categories if and only if Φ^{p} is an equivalence of categories. Hence p is an effective $\mathbb{A}$-descent morphism if and only if φ is an effective $\mathbb{A}$-descent morphism.

Suppose now that q is an $\mathbb{A}$-descent pre-equivalence. Then $\mathrm{Des}_{\mathbb{A}}(q)$ is full and

faithful. If φ is $\mathbb{A}$-descent pre-quivalence morphism, then $\Phi^p = \mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi}$ (up to isomorphism) is full and faithful. Hence p is an $\mathbb{A}$-descent morphism.

On the other hand, if p is $\mathbb{A}$-descent morphism, then $\mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi} = \Phi^p$ (up to isomorphism) is faithful and so is Φ^{φ}.

For any $B_1, B_2 \in \mathbb{A}^B$ and any morphism $h : \Phi^{\varphi}(B_1) \to \Phi^{\varphi}(B_2)$ in $\mathrm{Des}_{\mathbb{A}}(X, \varphi)$, $\mathrm{Des}_{\mathbb{A}}(q)(h) : \mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi}(B_1) \to \mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi}(B_2)$ is a morphism of $\mathrm{Des}_{\mathbb{A}}(p)$. Since the following diagram

$$\begin{tikzcd} & \mathbb{A}^B \arrow[dl, "\Phi^{\varphi}"'] \arrow[dr, "\Phi^p"] & \\ \mathrm{Des}_{\mathbb{A}}(X, \varphi) \arrow[rr, "\mathrm{Des}_{\mathbb{A}}(q)"'] & & \mathrm{Des}_{\mathbb{A}}(E, p) \end{tikzcd}$$

commutes (up to natural isomorphism), without loss of generality we may assume $\mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi}(B_i) = \Phi^p(B_i)$, $i = 1, 2$. Since Φ^p is full, there is a morphism $f : B_1 \to B_2$ of $\mathbb{A}^B$ such that

$$\Phi^p(f) = \mathrm{Des}_{\mathbb{A}}(q)(h).$$

That is

$$\mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi}(f) = \mathrm{Des}_{\mathbb{A}}(q)(h).$$

Since $\mathrm{Des}_{\mathbb{A}}(q)$ is faithful, $\Phi^{\varphi}(f) = h$. Hence Φ^{φ} is full. Then φ is an $\mathbb{A}$-descent pre-equivalence. $\square$

3.2.3. Absolutely effective descent morphisms are precisely the split epimorphisms

(see 2.2.6 or [19]). However, for absolute descent equivalences, we have:

Theorem. *Let $q : (E, p) \to (X, \varphi)$ be a morphism in $\mathbf{C}/B$.*

1. *If q is a split epimorphism in $\mathbf{C}$, then q is an absolute descent equivalence.*

2. *If q is a split monomorphism in $\mathbf{C}/B$, then q is an absolute descent equivalence.*

Proof. 1. Suppose $qs = 1_X$ for some morphism $s : X \to E$ in $\mathbf{C}$. Then, clearly, $\bar{q}\bar{s} = 1_{\mathrm{Eq}(\varphi)}$. On the other hand, we claim $\bar{s}\bar{q} \cong 1_{\mathrm{Eq}(\varphi)}$ and the internal natural transformations

$$\bar{s}\bar{q} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 1_{\mathrm{Eq}(p)}$$

are given by

$$\alpha = < 1_E, sq >, \ \beta = < sq, 1_E >: E \to E \times_B E \ \text{ in } \mathbf{C}.$$

By Definition 2.2.1,

$$\beta\alpha = \pi_{13} << sq, 1_E >, < 1_E, sq >>$$

and

$$\alpha\beta = \pi_{13} << 1_E, sq >, < sq, 1_E >> .$$

Since

$$\pi_1\pi_{13} << sq, 1_E >, < 1_E, sq >>= \pi_1 < sq, 1_E >= sq = \pi_1 esq,$$

$$\pi_2\pi_{13} << sq,\ 1_E >,\ < 1_E,\ sq >>= \pi_2 < 1_E,\ sq >= sq = \pi_2 esq,$$

and

$$\pi_1\pi_{13} << 1_E,\ sq >,\ < sq,\ 1_E >>= \pi_1 < 1_E,\ sq >= 1_E = \pi_1 e,$$

$$\pi_2\pi_{13} << 1_E,\ sq >,\ < sq,\ 1_E >>= \pi_2 < sq,\ 1_E >= 1_E = \pi_2 e,$$

we obtain

$$\pi_{13} << sq,\ 1_E >,\ < 1_E,\ sq >>= esq$$

$$\pi_{13} << 1_E,\ sq >,\ < sq,\ 1_E >>= e.$$

Hence $\beta\alpha = 1_{\tilde{s}\tilde{q}} : \tilde{s}\tilde{q} \to \tilde{s}\tilde{q}$ and $\alpha\beta = 1_{1_{\mathrm{Eq}(p)}} : 1_{\mathrm{Eq}(p)} \to 1_{\mathrm{Eq}(p)}$. Then the internal functor $\tilde{q} : \mathrm{Eq}(p) \to \mathrm{Eq}(\varphi)$ has an inverse $\tilde{s} : \mathrm{Eq}(\varphi) \to \mathrm{Eq}(p)$. Therefore, by Lemma 2.2.5, $\mathrm{Des}_{\mathbb{A}}(q)$ is an equivalence of categories.

2. Since q is a split monomorphism in C/B, there is a morphism s in C/B such that $sq = 1$. By 1, s and 1 are absolute descent equivalences. Hence, q is an absolute descent equivalence by Proposition 3.2.2(1). ◻

Recall that a morphism in **Set** is a split epimorphism if and only if it is surjective. Hence, in **Set**, epimorphisms, extremal epimorphisms, regular epimorphisms, universal epimorphisms, and split epimorphisms are all the same. Recall also that **Set** has the (epi, mono)-factorization system. Then, in **Set**, a given morphism $p : E \to B$ can be decomposed into $m \circ e$ with a monomorphism m and an epimorphism e:

$$\begin{tikzcd} E \arrow[rr, "e"] \arrow[dr, "p"'] & & X \arrow[dl, "m"] \\ & B & \end{tikzcd}$$

Since $e : (E, p) \to (X, m)$ is an absolute descent equivalence, we have

$$\mathrm{Des}_{\mathbf{A}}(E, p) \approx \mathrm{Des}_{\mathbf{A}}(X, m).$$

Hence, in **Set**, we need only consider the computability of $\mathrm{Des}_{\mathbf{A}}(p)$ with monomorphisms p.

However, an absolute descent equivalence does not need to be a split epimorphism or a split monomorphism. See the following:

Example. Let $B = \{b_1, b_2\}$, $Y = \{y\}$, and $X = \{x_1, x_2\}$. We define

$$p : Y \to B, \quad q : Y \to X, \quad h : X \to Y, \quad \varphi : X \to B$$

by

$$p(y) = b_1, \quad q(y) = x_1,$$

$$h(x_1) = h(x_2) = y,$$

and

$$\varphi(x_1) = \varphi(x_2) = b_1.$$

Then

$$\varphi q = p, \quad ph = \varphi, \quad hq = 1_Y :$$

$$\begin{tikzcd} Y \arrow[rr, shift right, "q"'] \arrow[dr, "p"'] & & X \arrow[ll, shift right, "h"'] \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

Hence $q : (Y, p) \to (X, \varphi)$ is a split monomorphism in $\mathbf{Set}/B$.

For any morphism $f : E \to Y$ with $|E| \geq 2$ in $\mathbf{Set}$, f is an epimorphism, so $qf : (E, pf) \to (X, \varphi)$ is an absolute equivalence. But qf is neither an epimorphism nor a monomorphism in $\mathbf{Set}$.

3.2.4. Recall that a functor $F : \mathbf{B} \to \mathbf{D}$ is called *essentially surjective* if for each object D of $\mathbf{D}$ there is an object B of $\mathbf{B}$ such that $FB \cong D$.

Theorem. *Let $q : (E, p) \to (X, \varphi)$ be a morphism in $\mathbf{C}/B$. If $\mathrm{Des}_{\mathbb{A}}(q)$ is essentially surjective for every $\mathbf{C}$-indexed category $\mathbb{A}$, then there is a morphism $s : X \to E$ in $\mathbf{C}$ such that*

$$psq = p.$$

In particular, if q is an absolute descent equivalence, then there is a morphism $s : X \to E$ in $\mathbf{C}$ such that

$$psq = p.$$

Moreover,

1. *If p is a monomorphism and if q is an absolute descent equivalence, then q is a split monomorphism.*

2. *If φ is a monomorphism, then there is a morphism $s : X \to E$ in* $\mathbf{C}$ *such that*

$$qsq = q.$$

Proof. We define a C-indexed category $\mathbb{A}_p$ as follows:

$$\begin{array}{ccc} \mathbf{C}^{\mathrm{op}} & \xrightarrow{\mathbb{A}_p} & \mathbf{CAT} \\ X & \longmapsto & \mathbf{C}(X,\ E) \\ \uparrow t & & \uparrow t^{*} \\ Y & \longmapsto & \mathbf{C}(Y,\ E) \end{array}$$

where $\mathbb{A}_p^X = \mathbf{C}(X,\ E)$ carries an equivalence relation given by

$$u \sim v \Leftrightarrow pu = pv.$$

and $t^{*} : \mathbf{C}(Y,\ E) \to \mathbf{C}(X,\ E)$ is the composing functor with t. Since

$$p\pi_1 = p\pi_2,\ \pi_1^{*}(1_E) = \pi_1 \sim \pi_2 = \pi_2^{*}(1_E),$$

the object 1_E of $\mathbb{A}_p^E$ has a descent structure $\xi : \pi_2^{*}(1_E) \to \pi_1^{*}(1_E)$, where $(\pi_1,\ \pi_2)$ is the kernel pair of p. Hence,

$$V^{\varphi}(1_E,\ \xi) = (i_\varphi)^{*}(1_E,\ \xi) = ((1_E)^{*}(1_E),\ \xi_{i_\varphi}) = (1_E,\ \xi') \in \mathrm{Des}_{\mathbb{A}_p}(E,\ q).$$

But $\mathrm{Des}_{\mathbb{A}_p}(q)$ is essentially surjective, so there is $(g,\ \mu) \in \mathrm{Des}_{\mathbb{A}_p}(X,\ \varphi)$ such that

$$\mathrm{Des}_{\mathbb{A}_p}(q)(g,\ \mu) \cong (1_E,\ \xi),$$

and therefore

$$V^{\varphi}\mathrm{Des}_{\mathbb{A}_p}(q)(g,\ \mu) \cong V^{\varphi}(1_E,\ \xi) = (1_E,\ \xi').$$

That is

$$\Phi^q U^{\varphi}(g,\ \mu) \cong (1_E,\ \xi').$$

But Φ^q is just a lifting of $q^{\bullet}$ (see 2.2.6),

$$q^{\bullet}U^{\varphi}(g,\ \mu) \cong \delta^{\bullet}\Phi^q U^{\varphi}(g,\ \mu) \cong \delta^{\bullet}(1_E,\ \xi').$$

Hence there is a $s \in \mathbb{A}_p^X = \mathbf{C}(X,\ E)$ such that

$$q^{\bullet}s \sim 1_E \ \text{ in } \ \mathbb{A}_p^E,$$

and therefore

$$psq = p,$$

as desired.

1. If q is an absolute descent equivalence, then $\mathrm{Des}_{\mathbb{A}}(q)$ are always essentially surjective for all $\mathbb{A}$'s. Hence there is a morphism $s : X \to E$ in $\mathbf{C}$ such that $psq = p$ and therefore $sq = 1$ since p is a monomorphism.

2. Since $p = \varphi q$, $\varphi qsq = psq = \varphi q$. □

3.2.5. In what follows we investigate the relation between $\mathbb{A}$-descent pre-equivalence ($\mathbb{A}$-descent equivalence) and $\mathbb{A}$-descent (effective $\mathbb{A}$-descent).

Proposition. *The morphism $p : (E, p) \to (B, 1_B)$ in $\mathbf{C}/B$ is an $\mathbb{A}$-descent pre-equivalence ($\mathbb{A}$-descent equivalence) if and only if p is an $\mathbb{A}$-descent (effective $\mathbb{A}$-descent) morphism.*

Proof. Applying Eq to the following commutative diagram:

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "p"'] & & B \arrow[dl, "1_B"] \\ & B & \end{tikzcd}$$

we obtain the commutative diagram:

$$\begin{tikzcd} & \mathrm{Eq}(1_B) & \\ \mathrm{Eq}(p) \arrow[ur, "\overline{p}"] \arrow[rr, "\tilde{p}"'] & & \mathrm{Eq}(1_B) \arrow[ul, "\overline{1}_B"'] \end{tikzcd}$$

Clearly, $\overline{1}_B = 1_{\mathrm{Eq}(1_B)}$, $\tilde{p} = \overline{p}$. Hence

$$\mathbb{A}^{\tilde{p}} \text{ is an equivalence of categories} \Leftrightarrow \mathbb{A}^{\overline{p}} \text{ is an equivalence of categories.}$$

as desired. □

Remark. From a theorem of Reiterman, Sobral, and Tholen [26], we have the following:

Let $\mathbb{E} : B \mapsto \mathbb{E}(B)$ be the subfibration of the basic fibration of a category $\mathbf{C}$ given by a pullback-stable class $\mathbb{E}$ of morphisms in $\mathbf{C}$, and $q : (E, p) \to (X, \varphi)$ a morphism

in $\mathbf{C}/B$. *If* q *is a pullback stable regular epimorphism of* $\mathbf{C}$, *then* $q : E \to X$ *is an* $\mathbb{E}$*-descent (effective* $\mathbb{E}$*-descent) morphism in* $\mathbf{C}$ *implies* $q : (E, p) \to (X, \varphi)$ *is an* $\mathbb{E}$*-descent pre-equivalence (*$\mathbb{E}$*-descent equivalence).*

By the above Remark and Theorem 3.2.2, immediately we have the following:

Proposition. *Let* $q : (E, p) \to (X, \varphi)$ *be a morphism in a category* $\mathbf{C}$ *with pullbacks, and let* $\mathbb{E} : B \mapsto \mathbb{E}(B)$ *be the subfibration of the basic fibration of a category* $\mathbf{C}$ *given by a pullback-stable class* $\mathbb{E}$ *of morphisms in* $\mathbf{C}$. *Then the following are equivalent:*

1. q*:* $E \to X$ *is an* $\mathbb{E}$*-descent (effective* $\mathbb{E}$*-descent) morphism in* $\mathbf{C}$;

2. *For each morphism* $x : X \to B$ *in* $\mathbf{C}$, $q : (E, xq) \to (X, x)$ *is an* $\mathbb{E}$*-descent pre-equivalence (*$\mathbb{E}$*-descent equivalence) morphism;*

3. *For each morphism* $x : X \to B$ *in* $\mathbf{C}$, xq *is an* $\mathbb{E}$*-descent (effective* $\mathbb{E}$*-descent) morphism in* $\mathbf{C}$ *if and only if* x *is an* $\mathbb{E}$*-descent (effective* $\mathbb{E}$*-descent) morphism in* $\mathbf{C}$.

Naturally, the following question may be interesting:

Question. *For which* $\mathbf{C}$*-indexed categories* $\mathbb{A}$, *is an* $\mathbb{A}$*-descent (effective* $\mathbb{A}$*-descent) morphism* $q : (E, p) \to (X, \varphi)$ *also an* $\mathbb{A}$*-descent pre-equivalence (*$\mathbb{A}$*-descent equivalence)?*

Now we come back to the commutative diagram (5):

$$
\begin{tikzcd}
W \arrow[rr, "q'"] \arrow[d, "x'"'] \arrow[ddr, "px'"] & & Y \arrow[d, "x"] \arrow[ddl, "\varphi x"'] \\
E \arrow[rr, "q"'] \arrow[dr, "p"'] & & X \arrow[dl, "\varphi"] \\
& B &
\end{tikzcd}
$$

in $\mathbf{C}/B$. which, as we already saw, induces the following commutative diagram (6) in $\mathrm{cat}(\mathbf{C})$:

$$
\begin{tikzcd}
& \mathrm{Eq}(\varphi) & & \mathrm{Eq}(p) \arrow[ll, "\bar{q}"'] \\
X \arrow[ur, "\delta_X"] & & \mathrm{Eq}(q) \arrow[ll, "\overline{q}"] \arrow[ur, "i_\varphi"'] & \\
& \mathrm{Eq}(\varphi x) \arrow[uu, "\bar{x}"] & & \mathrm{Eq}(px') \arrow[uu, "\bar{x}'"'] \arrow[ll, "\bar{q}'"'] \\
Y \arrow[uu, "x"] \arrow[ur, "\delta_Y"'] & & \mathrm{Eq}(q') \arrow[ll, "\overline{q'}"] \arrow[uu, "c_{x'}"] \arrow[ur, "i_{\varphi x}"'] &
\end{tikzcd}
$$

Applying $\mathbb{A}$ to the last diagram, we get the commutative diagram (up to isomorphism) in **CAT**:

$$
\begin{tikzcd}
& \mathrm{Des}_{\mathbb{A}}(X,\varphi) \arrow[rr, "\mathrm{Des}_{\mathbb{A}}(q)"] \arrow[dl, "U^{\varphi}"'] \arrow[dd, "\mathrm{Des}_{\mathbb{A}}(x)"] & & \mathrm{Des}_{\mathbb{A}}(E,p) \arrow[dl, "V^{\varphi}"] \arrow[dd, "\mathrm{Des}_{\mathbb{A}}(x')"] \\
\mathbb{A}^X \arrow[rr, "\Phi^q"'] \arrow[dd, "x^*"'] & & \mathrm{Des}_{\mathbb{A}}(E,q) \arrow[dd, "W^{x'}"] & \\
& \mathrm{Des}_{\mathbb{A}}(Y,\varphi x) \arrow[rr, "\mathrm{Des}_{\mathbb{A}}(q')"'] \arrow[dl, "U^{\varphi x}"] & & \mathrm{Des}_{\mathbb{A}}(W,px') \arrow[dl, "V^{\varphi x}"] \\
\mathbb{A}^Y \arrow[rr, "\Phi^{q'}"'] & & \mathrm{Des}_{\mathbb{A}}(W,q') &
\end{tikzcd}
\tag{9}
$$

Clearly, an answer to the above question as well as pullback stability of (effective) $\mathbb{A}$-descent morphisms largely depend on the top face and the front face of diagram (9), respectively.

3.2.6. Reiterman, Sobral, and Tholen [26] proved the following composition and cancellation theorem:

In a category with pullbacks, both the class $\mathcal{D}$ of descent morphisms and the class $\mathcal{E}$ of effective descent morphisms with respect to the basic fibration satisfy the following composition-cancellation rule: given a composite $p = \varphi \cdot q$, then

$$\textit{if } q \in \mathcal{D}, \textit{ then } p \in \mathcal{D} \Leftrightarrow \varphi \in \mathcal{D}$$

and

$$\textit{if } q \in \mathcal{E}, \textit{ then } p \in \mathcal{E} \Leftrightarrow \varphi \in \mathcal{E}.$$

Let φ and q be two morphisms of $\mathbf{C}$ such that $\varphi \cdot q$ is defined. If $p = \varphi \cdot q$ is a

descent (an effective descent) morphism with respect to the basic fibration, then we form the following pullback diagram:

$$\begin{tikzcd}
 & E \times_B X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
E \arrow[rr, "q"] \arrow[dr, "p"'] & & X \arrow[dl, "\varphi"] \\
 & B &
\end{tikzcd}$$

Since π_1 is a split epimorphism, π_1 is an effective descent morphism. On the other hand, by the pullback stability of descent (effective descent) morphisms [29], π_2 is a descent (an effective descent) morphism. Then, by the the above composition-cancellation rule, q and φ are descent (effective descent) morphisms, and so we have the following proposition.

Proposition. *Let φ and q be two morphisms of $\mathbf{C}$ such that $\varphi \cdot q$ is defined. Then $\varphi \cdot q$ is a descent (an effective descent) morphism with respect to the basic fibration if and only if φ and q are descent (effective descent) morphisms.*

4. Purity and Effective Descent

A theorem of Joyal and Tierney [21] states that pure morphisms are effective descent in the opposite category of commutative monoids in sup-lattices with respect to the fibration induced by the indexed category which assigns to each commutative monoid A in sup-lattices an A-module. Mesablishvili [25] gave a proof that pure morphisms are effective descent in the opposite category of commutative unital rings with respect to the fibration induced by the indexed category $R \mapsto \mathbf{Mod}\text{-}R$ for each commutative unital ring R. In this chapter, we want to study the relationship between purity and effective descent.

4.1. Pure morphisms are effective descent for modules

Recall that for a unital ring homomorphism $f : R \to S$, we have the adjoint pair $f_* \dashv f!$ (see 1.2.3):

$$\text{Mod-}S \underset{f!}{\overset{f_*}{\rightleftarrows}} \text{Mod-}R$$

In this section, our objective is to answer the following question:

For which morphisms $f : R \to S$ of unital rings, are f_* comonadic?

4.1.1. Recall that $\mathbf{Q}/\mathbf{Z}$ is *an injective cogenerator* for $\mathbf{Ab}$. If M is a right R-module,

then it is a bimodule ${}_ZM_R$. Hence the left R-module structure on $\hom_Z(M, \mathbf{Q}/\mathbf{Z})$ is given by $rf : m \mapsto f(mr)$. One calls this left R-module *the character module* of M, and denotes it by $C_R(M)$. Hence, one has the representable functor

$$C_R : (\mathbf{Mod}\text{-}R)^{\mathrm{op}} \to R\text{-}\mathbf{Mod}$$

Similarly, one has the functor:

$$C_R : (R\text{-}\mathbf{Mod})^{\mathrm{op}} \to \mathbf{Mod}\text{-}R.$$

4.1.2. We need the following properties of the above functor C_R:

Proposition. 1. *C_R is faithful.*

2. *A sequence of right R-modules*

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow C_R(M_3) \xrightarrow{C_R(\beta)} C_R(M_2) \xrightarrow{C_R(\alpha)} C_R(M_1) \longrightarrow 0$$

is exact.

3. *C_R is exact.*

4. *C_R preserves and reflects coequalizer diagrams.*

Proof. 1. Since $\mathbf{Q}/Z$ is a cogenerator of **Ab**.

2. See [28], p.87, Lemma 3.51.

3. Clearly C_R is additive. By (2) and [6, vol.2, p.50, Proposition 1.11.3], it is clear.

4. Since C_R is exact, C_R preserves coequalizer diagrams. Clearly, C_R reflects isomorphisms. Hence, by [15, Theorem 24.7], it also reflects coequalizer diagrams.

$\square$

In commutative unital rings, B. Mesablishvili [25] showed that $C_R(f)$ is a split epimorphism in R-**Mod** for each pure morphism f of R-**Mod**. As Mesablishvili did in [25], we have the following:

Lemma. *Let M_1, M_2 be two R-bimodules. Suppose that $f : M_1 \to M_2$ is a pure morphism of R-**Mod**. Then*

1. *$C_R(f)$ is a split epimorphism in R-**Mod**.*

2. *The natural transformation*

$$\alpha : C_R \circ ((\ ) \otimes_R M_1) \to C_R \circ ((\ ) \otimes_R M_2) : (\mathbf{Mod}\text{-}R)^{\mathrm{op}} \to R\text{-}\mathbf{Mod}$$

given by $\alpha_M = C_R(1_M \otimes f) : C_R(M \otimes_R M_2) \to C_R(M \otimes_R M_1)$ for each right R-module M, is a split epic natural transformation.

Proof. 1. By adjointness of $\otimes$ and hom, one has the following commutative diagram:

$$\begin{tikzcd}
\hom_R(C_R(M_1), C_R(M_2)) \arrow[r, "\hom(C_R(M_1), C_R(f))"] \arrow[d, "\cong"'] & \hom_R(C_R(M_1), C_R(M_1)) \arrow[d, "\cong"] \\
C_R(C_R(M_1) \otimes_R M_2) \arrow[r, "C_R(1 \otimes f)"'] & C_R(C_R(M_1) \otimes_R M_1)
\end{tikzcd}$$

in which both vertical morphisms are isomorphisms. Since $f : M_1 \to M_2$ is pure in R-**Mod**, $1 \otimes f$ is a monomorphism. Since $\mathbf{Q/Z}$ is injective in **Ab**, $C_R(f \otimes 1)$ is an epimorphism, so is $\hom_R(C_R(M_2), C_R(f))$. Therefore $1_{C_R(M_1)}$ produces a retraction for $C_R(f)$.

2. Clearly α is a natural transformation. Again, by adjointness of $\otimes$ and hom, one has the following commutative diagram:

$$\begin{tikzcd}
C_R(M \otimes_R M_2) \arrow[r, "C_R(1 \otimes_R f) = \alpha_M"] \arrow[d, "\cong"'] & C_R(M \otimes_R M_1) \arrow[d, "\cong"] \\
\hom_R(M, C_R(M_2)) \arrow[r, "\hom_R(M, C_R(f))"'] & \hom_R(M, C_R(M_1))
\end{tikzcd}$$

for each $M \in$ **Mod**-R.

By 1, $C_R(f)$ has a right inverse, say h, which gives a right inverse of α_M and α is natural in M. $\square$

4.1.3. Theorem. *Let $f : R \to S$ be a morphism of rings. Then f_* is comonadic if and only if f is a pure morphism in R-**Mod**.*

Proof. $\Rightarrow$: By Proposition 1.2.3.

$\Leftarrow$: If $f : R \to S$ is a pure morphism in R-**Mod**, the unit $\eta_N : N \to N \otimes_R S$ of $f_* \dashv f^! : \mathbf{Mod}\text{-}S \to \mathbf{Mod}\text{-}R$ is a monomorphism for each $N \in \mathbf{Mod}\text{-}S$, then f_* is precomonadic. For any $(M, \theta) \in (\mathbf{Mod}\text{-}S)^{f_* f^!}$, consider the following equalizer diagram:

$$\begin{tikzcd} N \arrow[r, "i"] & M \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\eta_M"'] & M \otimes_R S \end{tikzcd}$$

in **Mod**-R, where $i : N = \{m \in M,\ \theta(m) = \eta_M(m)\} \to M$ is an inclusion map, and a split fork:

$$\begin{tikzcd} M \arrow[r, "\theta"] & M \otimes_R S \arrow[r, shift left, "\theta \otimes 1"] \arrow[r, shift right, "\eta_M \otimes 1"'] & M \otimes_R S \otimes_R S \end{tikzcd}$$

in **Mod**-R. Then we have the following commutative diagram

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[d, "l"'] & M \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\eta_M"'] \arrow[d, "\eta_M"] & M \otimes_R S \arrow[d, "\eta_{M \otimes S}"] \\ M \arrow[r, "\theta"] & M \otimes_R S \arrow[r, shift left, "\theta \otimes 1"] \arrow[r, shift right, "\eta_M \otimes 1"'] & M \otimes_R S \otimes_R S \end{tikzcd}$$

for some morphism $l : N \to M$ in **Mod**-R. Applying C_R to the above, we get the following commutative diagram in R-**Mod**:

$$\begin{tikzcd} C_R(M \otimes_R S \otimes_R S) \arrow[r, shift left, "C_R(\theta \otimes 1)"] \arrow[r, shift right, "C_R(\eta_M \otimes 1)"'] \arrow[d, "C_R(\eta_{M \otimes_R S})"'] & C_R(M \otimes_R S) \arrow[r, "C_R(\theta)"] \arrow[d, "C_R(\eta_M)"] & C_R(M) \arrow[d, "C_R(l)"] \\ C_R(M \otimes_R S) \arrow[r, shift left, "C_R(\theta)"] \arrow[r, shift right, "C_R(\eta)"'] & C_R(M) \arrow[r, "C_R(i)"] & C_R(N) \end{tikzcd} \tag{10}$$

in which the bottom row is a coequalizer diagram and the top row is a split fork. If $f : R \to S$ is pure in R-**Mod**, then, by Lemma 4.1.2(2), $C_R(\eta_{M\otimes S})$ ($\cong C_R(1_{M\otimes S}\otimes f)$) and $C_R(\eta_M)$ ($\cong C_R(1_M \otimes f)$) have right inverses s, t respectively, such that

$$\begin{tikzcd}
C_R(M \otimes_R S \otimes_R S) \arrow[r, shift left, "C_R(\theta\otimes 1)"] \arrow[r, shift right, "C_R(\eta_M \otimes 1)"'] & C_R(M \otimes_R S) \\
C_R(M \otimes_R S) \arrow[u, "s"] \arrow[r, shift left, "C_R(\theta)"] \arrow[r, shift right, "C_R(\eta)"'] & C_R(M) \arrow[u, "t"']
\end{tikzcd}$$

commutes serially. Then there is a morphism $k : C_R(N) \to C_R(M)$ in R-**Mod** such that

$$C_R(\theta)t = kC_R(i).$$

Since the top row in diagram (10) is a split fork, by an easy diagram chasing, the bottom row:

$$\begin{tikzcd}
C_R(M \otimes_R S) \arrow[r, shift left, "C_R(\theta)"] \arrow[r, shift right, "C_R(\eta)"'] & C_R(M) \arrow[r, "C_R(i)"] & C_R(N)
\end{tikzcd} \qquad (11)$$

is also a split fork. Applying the functor $\hom_R(S, -)$ to (11) and by adjointness of $\otimes$ and hom, we get the following split fork:

$$\begin{tikzcd}
C_R(M \otimes_R S \otimes_R S) \arrow[r, shift left, "C_R(\theta\otimes 1)"] \arrow[r, shift right, "C_R(\eta \otimes 1)"'] & C_R(M \otimes_R S) \arrow[r, "C_R(i\otimes 1)"] & C_R(N \otimes_R S)
\end{tikzcd}$$

By Proposition 4.1.2 (4),

$$\begin{tikzcd}
N \otimes_R S \arrow[r, "i\otimes 1"] & M \otimes_R S \arrow[r, shift left, "\theta\otimes 1"] \arrow[r, shift right, "\eta \otimes 1"'] & M \otimes_R S \otimes_R S
\end{tikzcd}$$

is an equalizer diagram. Hence, by Beck's Theorem, f_* is comonadic. □

4.1.4. Let B be a subcategory of $\mathbf{Rng}_1$ such that

(*) $\mathbf{CRng}_1 \subseteq \mathbf{B}$ *and for any morphism* $f : R \to S$ *of* B, *f is descent (effective descent) with respect to the fibration induced by* B^{op}*-indexed category* $R \mapsto$ Mod-R *if and only if* f_* *is precomonadic (comonadic).*

By the same process as one applied in the proof of Theorem 4.1.3, we have the following:

Let $f : R \to S$ *be a morphism in* B. *Then f is effective for descent with respect to the fibration induced by* B^{op}*-indexed category* $R \mapsto$ Mod-R *if and only if f is pure morphism in* R-Mod.

Clearly, letting $\mathrm{B} = \mathbf{CRng}_1$ in the above, by [6, vol.2, Theorem 4.7.4], we obtain the theorem of Mesablishvili [25].

4.2. Locally Presentable Categories, Accessible Categories, and Pure Morphisms

This section collects some basic notions and results of locally presentable categories, accessible categories, and pure morphisms. It is the preparation for the next section.

4.2.1. Recall that a non-empty poset is *directed* if each pair of its elements has an upper bound, and *directed colimits* are colimits of diagrams whose schemes are directed posets.

Definition. *An object K of a category* $\mathbf{K}$ *is called* finitely presentable *if*

$$\hom(K, -): \ \mathbf{K} \to \mathbf{Set}$$

preserves directed colimits.

For example, an object K in **Set** is *finitely presentable* if and only if $|K| < \aleph_0$. An object G in **Grp** is *finitely presentable* if and only if G is isomorphic to the quotient group of the free group $F(x_1, \cdots, x_n)$ on n generators modulo a congruence generated by finitely many equations on $F(x_1, \cdots, x_n)$. The group $(Z, +)$ of integers is finitely presentable but the group $(R, +)$ of the real numbers is not (see [2], p.10).

Definition. *A category* $\mathbf{K}$ *is called* locally finitely presentable *if it is cocomplete and has a set S of finitely presentable objects such that every object is a directed colimits of objects from S.*

For example, **Set**, **Pos**, and **Grp** are locally finitely presentable categories.

Theorem. *Every variety of finitary algebras is a locally finitely presentable category.*

Proof. See [2], p.141. $\square$

4.2.2. Throughout, λ is a regular cardinal. Recall that a poset is λ*-directed* if every

subset of cardinality smaller than λ has an upper bound and *λ-directed colimits* are colimits of diagrams whose schemes are λ-directed posets. An object K of a category is called *λ-presentable* if $\hom(K, -)$ preserves λ-directed colimits. An object is called *presentable* if it is λ-presentable for some λ.

For example, an object in **Set** is *λ-presentable* if and only if it has cardinality smaller than λ (see [2], p.19).

Definition. *A category is called* locally λ-presentable *if it is cocomplete and has a set S of λ-presentable objects such that every object is a λ-directed colimits of objects from S. A category is called* locally presentable *if it is locally λ-presentable for some λ.*

For example, locally $\aleph_0$-presentable categories are precisely the locally finitely presentable ones. The category **Ban** of complex Banach spaces and linear contractions is locally $\aleph_1$-presentable (see [2], p.40, Example 1.48). But the self-dual category **Hil** of Hilbert spaces and linear contractions is *not* locally presentable (see [2], p.51, Example 1.65 (2)).

4.2.3. Definition. *A category* **K** *is called* λ-accessible category *if* **K** *has λ-directed colimits and has a set S of λ-presentable objects such that every object in* **K** *is a λ-directed colimits of objects from S. A category is called* accessible *if it is λ-accessible category for some λ.*

For example, every locally λ-presentable category is λ-accessible category. The category **Hil** of Hilbert spaces and linear contractions is $\aleph_1$-accessible (see [2], p.70, Example 2.3 (9)).

Recall that a morphism $f : A \to B$ in a category $\mathbf{K}$ is *λ-pure* if for every commutative square

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "u"'] & B' \arrow[d, "v"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

in which A' and B' are λ-presentable objects, the morphism u factors through f'. If $\mathbf{K}$ is a λ-accessible category, then *λ-pure* morphisms are monomorphisms. If $\mathbf{K}$ is *locally λ-presentable*, then *λ-pure* morphisms are regular monomorphisms.

Adámek, Hu, and Tholen [1] showed:

Theorem 1. *In a λ-accessible category $\mathbf{K}$ with pushouts, every λ-pure morphism f is a λ-directed colimit (in $\mathbf{K}^2$) of split monomorphisms with the same domain as f.*

They also proved:

Theorem 2. *In a λ-accessible category $\mathbf{K}$ with pushouts, regular monomorphisms are closed under λ-directed colimits in $\mathbf{K}^2$.*

4.3. Descent Theory in Accessible Categories

Let $\mathbf{K}$ be a λ-accessible category with pushouts. In this section, we want to study descent theory in $\mathbf{K}^{\mathrm{op}}$ with respect to the basic fibration. By Theorem 2.1.4 and Example 2.1.4 (1), we have:

a morphism $p : E \to B$ in $\mathbf{K}^{\mathrm{op}}$ is descent (effective descent) with respect to the basic fibration if and only if p_* is precomonadic (comonadic) in the situation $p_* \dashv p! :$

$$\begin{tikzcd} E/\mathbf{K} \arrow[r, shift right, "p!"'] & B/\mathbf{K} \arrow[l, shift right, "p_*"'] \end{tikzcd}$$

Hence we need to study comonadicity of p_*.

4.3.1. For a morphism $p : B \to E$ of $\mathbf{K}$, *codescent data* for $(K, r) \in \mathrm{ob}(E/\mathbf{K})$ are given by morphisms $\theta : K \to K +_B E$ in $\mathbf{K}$ which make

$$\begin{tikzcd}
& B \arrow[r, "p"] \arrow[dl, "p"'] \arrow[dd, "rp"] \arrow[ddr, phantom, "(1)"] & E \arrow[r, "r"] \arrow[dd, "\pi_2"] & K \arrow[dd, "1_K"] \arrow[ddl, "\theta"] \\
E \arrow[dr, "r"'] & & & \\
& K \arrow[r, "\pi_1"'] & K +_B E \arrow[r, "\langle 1, r \rangle"'] & K
\end{tikzcd}$$

and

$$\begin{tikzcd}
K \arrow[r, "\theta"] \arrow[d, "\theta"'] & K +_B E \arrow[d, "\theta + 1"] \\
K +_B E \arrow[r, "\pi_1 + 1"'] & K +_B E +_B E
\end{tikzcd}$$

commute, where (1) is a pushout diagram. These are objects of the category $\mathrm{Des}^{\bullet}(p)$; a *morphism* $h : (K,\ r;\ \theta) \to (K',\ r';\ \theta')$ of $\mathrm{Des}^{\bullet}(p)$ is a $\mathbf{K}$-morphism which makes

$$\begin{tikzcd} & E \arrow[dl, "r"'] \arrow[dr, "r'"] & \\ K \arrow[rr, "h"'] \arrow[d, "\theta"'] & & K' \arrow[d, "\theta'"] \\ K +_B E \arrow[rr, "h +_B 1"'] & & K' +_B E \end{tikzcd}$$

commute.

The comparison functor

$$\Phi_p : \ B/\mathbf{K} \to \mathrm{Des}^{\bullet}(p)$$

given by $\Phi_p(L,\ \alpha) = (L +_B E,\ \sigma_2;\ \sigma_1 + 1)$ and the following pushout diagram:

$$\begin{tikzcd} B \arrow[r, "p"] \arrow[d, "\alpha"'] & E \arrow[d, "\sigma_2"] \\ L \arrow[r, "\sigma_1"'] & L +_B E \end{tikzcd}$$

A morphism $p : B \to E$ of $\mathbf{K}$ is called *a codescent* (*an effective codescent*) *morphism* if Φ_p is full and faithful (an equivalence of categories).

Clearly, the category $\mathrm{Des}^{\bullet}(p)$ we defined above is the *category of Eilenberg-Moore coalgebras* given by $p_{\bullet} \dashv p_!$. Hence, by the dual of Beck's Theorem,

p is a codescent morphism in $\mathbf{K}$

$\iff p_*$ is precomonadic

$\iff$ units of the adjunction $p_* \dashv p!$ are regular monomorphisms

$\iff p$ is a pushout stable regular monomorphism.

Now, let $B \in \text{ob}\mathbf{K}$,

$$\mathcal{D}_B = \{\text{all codescent morphisms in } \mathbf{K} \text{ with the domain } B\},$$

and

$$\mathcal{E}_B = \{\text{all effective codescent morphisms in } \mathbf{K} \text{ with the domain } B\}.$$

4.3.2. Theorem. *If $\mathbf{K}$ is a λ-accessible category with pushouts, then $\mathcal{D}_B$ is closed under λ-directed colimits in $B/\mathbf{K}$.*

Proof. Let $p_i : B \to E_i$ $(i \in I)$ be a λ-directed diagram in $B/\mathbf{K}$, $p_i \in \mathcal{D}_B$, and $p : B \to E$ λ-directed colimit of $\{p_i\}_{i\in I}$:

$$\begin{tikzcd} & B \arrow[dl, "p"'] \arrow[dr, "p_i"] & \\ E & & E_i \arrow[ll, "t_i"] \end{tikzcd}$$

Consider the following pushout diagrams:

$$\begin{tikzcd} B \arrow[r, "p"] \arrow[d, "x"'] & E \arrow[d, "\pi_2"] \\ X \arrow[r, "\pi_1"'] & X +_B E \end{tikzcd}$$

and

$$\begin{tikzcd} B \arrow[r, "p_i"] \arrow[d, "x"'] & E \arrow[d, "\pi_{2i}"] \\ X \arrow[r, "\pi_{1i}"'] & X +_B E \end{tikzcd}$$

for any morphism $x : B \to X$ in $\mathbf{K}$. Taking λ-directed colimit in the last pushout diagram gives

$$\operatorname{colim} \pi_{1i} = \pi_1.$$

By Theorem 4.2.3.2, π_1 is a regular monomorphism. Hence p is a pushout stable regular monomorphism and therefore $p \in \mathcal{D}_B$. □

By Theorem 4.2.3.1 and Theorem 4.3.2, one has the following:

Corollary [2, Proposition 2.31]. *In a λ-accessible category with pushouts pure morphisms are codescent morphisms and so they are pushout stable regular monomorphisms.*

4.3.3. In [7] Börger mentioned the following condition

(B): *If $e_1 e_2$ is a regular monomorphism, so is e_2.*

Condition (B) does not hold in general (see [22] and [8] for counterexample), but it is true under some conditions (see [7], 3.5(4) and Proposition 1.2(ii)), for example, when the category has (Epi, RegMono)-factorization system.

Theorem. *Let* $\mathbf{C}$ *be a category with pushouts. If* $\mathbf{C}$ *satisfies* (B), *then* $fg \in \mathcal{D}$ *implies* $g \in \mathcal{D}$ *for any morphisms* f, g *in* $\mathbf{C}$ *such that* fg *is defined.*

Proof. Suppose $g : A \to B$, $f : B \to C$ are morphisms such that fg is defined in $\mathbf{C}$. For any morphism $x : A \to X$ in $\mathbf{C}$, we take the following pushout diagrams (1) and (2):

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "x"'] \arrow[dr, phantom, "(1)"] & B \arrow[r, "f"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "(2)"] & C \arrow[d, "\sigma_2"'] \\ X \arrow[r, "\pi_1"'] & X +_A B \arrow[r, "\sigma_1"'] & (X +_A B) +_B C \end{tikzcd}$$

Then (1)+(2) becomes a pushout diagram and so

$$\sigma_1\pi_1 : X \to (X +_A B) +_B C$$

is a regular monomorphism since $fg \in \mathcal{D}_{\mathbf{C}}$. Hence π_1 is a regular monomorphism by (B) and therefore $g \in \mathcal{D}_{\mathbf{C}}$. □

4.3.4. Now we consider the effective codescent morphism case. Let $\mathbf{K}$ be a locally λ-presentable category and let (E, p) be the λ-directed colimit of $(E_i, p_i)_{i \in I}$ in $B/\mathbf{K}$ in which each $p_i \in \mathcal{E}_B$. By Theorem 4.3.2, $p \in \mathcal{D}_B$. For the effectivity of p, it suffices to prove that Φ_p is essentially surjective. We have:

Proposition. *Let* $\mathbf{K}$ *be a locally* λ*-presentable category and* $((E, p), (t_i)_{i \in I})$ *a* λ*-directed colimit of* $(E_i, p_i)_{i \in I}$ *in* $B/\mathbf{K}$ *in which each* $p_i \in \mathcal{E}_B$. *If* $(K, r; \theta) \in \mathrm{Des}^{\bullet}(p)$ *can be written as a* λ*-directed colimit of* $(K_i, r_i; \theta_i)$ *in* $\mathrm{Des}^{\bullet}(p)$ *such that all* K_i*'s*

are λ-presentable in $\mathbf{K}$, then there is $(L, b) \in B/\mathbf{K}$ such that

$$\Phi_p(L, b) \cong (K, r; \theta).$$

Proof. We distinguish the following cases:

Case 1. *K is λ-presentable.*

Since $\mathrm{colim}_i(K +_B E_i) = K +_B E$, there are $j_0 \in I$ and $\theta_j : K \to K +_B E_j$ for all $j \geq j_0$ such that

$$\begin{tikzcd}
K \arrow[rr, "\theta_j"] \arrow[dr, "\theta"'] & & K +_B E_j \arrow[dl, "1 + t_j"] \\
& K_B + E &
\end{tikzcd}$$

commutes. It follows that

$$(1 + t_j)\theta_j r t_j = \theta r t_j = \pi_2 t_j = (1 + t_j)\pi_{2j} :$$

$$\begin{tikzcd}
B \arrow[rr, "p"] \arrow[dr, "p_j"] \arrow[dd, "rp"'] & & E \arrow[dd, "\pi_2"] \\
& E_j \arrow[ur, "t_j"'] \arrow[dd, "\pi_{2j}"] & \\
K \arrow[rr, "\pi_1"] \arrow[dr, "\pi_{1j}"'] & & K +_B E \\
& K +_B E_j \arrow[ur, "1 + t_j"'] &
\end{tikzcd}$$

By [2, Proposition 1.62], $1 + t_j$ is a monomorphism. So $\theta_j rt_j = \pi_{2j}$ for all $j \geq j_0$.

Now it is routine to check $(K,\ rt_j;\ \theta_j) \in \mathrm{Des}^{\bullet}(p_j)$ by looking at the following commutative diagrams:

$$\begin{tikzcd}
B \arrow[r, "p_j"] \arrow[d, "rp"'] & E_j \arrow[r, "rt_j"] \arrow[d, "\pi_{2j}"'] & K \arrow[d, "1_K"] \\
K \arrow[r, "\pi_{1j}"'] & K +_B E_j \arrow[r, "< 1, rt_j >"'] & K
\end{tikzcd}$$

and

$$\begin{tikzcd}
K \arrow[rrr, "\theta"] \arrow[drr, "\theta_j"] \arrow[ddr, "\theta_j"'] \arrow[ddd, "\theta"'] & & & K +_B E \arrow[ddd, "\theta + 1"] \\
& & K +_B E \arrow[ur, "1 + t_j"'] \arrow[d, "\theta_j + 1"'] & \\
& K +_B E_j \arrow[r, "\pi_{1j} + 1"'] \arrow[dl, "1 + t_j"] & K +_B E_j +_B E_j \arrow[dr, "1 + t_j + t_j"'] & \\
K +_B E \arrow[rrr, "\pi_1 + 1"'] & & & K +_B E +_B E
\end{tikzcd}$$

Since $(K,\ rt_j;\ \theta_j) \in \mathrm{Des}^{\bullet}(p_j)$ and Φ_{p_j} is an equivalence of categories, there is $(L_j,\ l_j)$ in $B/\mathbf{K}$ such that

$$\Phi_{p_j}(L_j,\ l_j) \cong (K,\ rt_j;\ \theta_j),$$

where $(L_j,\ l_j)$ is defined by the following equalizer diagram:

$$
\begin{tikzcd}
 & B \arrow[dl, "l_j"'] \arrow[d, "rp"] \arrow[dr, "\pi_{2j}p_j"] & \\
L_j \arrow[r, "e_j"'] & K \arrow[r, shift left, "\theta_j"] \arrow[r, shift right, "\pi_{1j}"'] & K +_B E_j
\end{tikzcd}
$$

in $B/\mathbf{K}$.

Clearly, $(L_j, l_j)_{j \geq j_0}$ is directed. Taking the directed colimit in the above gives a new equalizer diagram:

$$
\begin{tikzcd}
 & B \arrow[dl, "l"'] \arrow[d, "rp"] \arrow[dr, "\pi_2 p"] & \\
L \arrow[r, "e"'] & K \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\pi_1"'] & K +_B E
\end{tikzcd}
$$

It follows that

$$\Phi(L, l) \cong (K, r; \theta).$$

CASE 2. *$(K, r; \theta)$ can be written as a λ-directed colimit of $(K_i, r_i; \theta_i)$ with λ-presentable K_i.*

By Case 1, for each i there is $(L_i, b_i) \in B/\mathbf{K}$ such that

$$\Phi(L_i, b_i) \cong (K_i, r_i; \theta_i).$$

Since Φ is full and faithful, (L_i, b_i) is λ-directed. Since Φ has the right adjoint, it preserves colimits. Then

$$\begin{aligned}(K,\ r;\ \theta) &= \text{colim } (K_i,\ r_i;\ \theta_i)\\ &\cong \text{colim } \Phi(L_i,\ b_i)\\ &= \Phi(\text{colim } (L_i,\ b_i))\\ &= \Phi((L,\ b)),\end{aligned}$$

as desired. □

A locally λ-presentable category $\mathbf{K}$ has a representative set of all λ-presentable objects. We denote any such set by $\mathbf{Pres}_\lambda\mathbf{K}$ and consider it as a full category of $\mathbf{K}$. Clearly, we have:

Proposition. *Let $B \in \mathrm{ob}(\mathbf{Pres}_\lambda\mathbf{K})$. Suppose $(E,\ p)$ is a λ-directed colimit of $(E_i, p_i)_{i\in I}$ in $B/\mathbf{K}$ such that $E, E_i \in \mathrm{ob}(\mathbf{Pres}_\lambda\mathbf{K})$. If each*

$$(p_i)_* : B/\mathbf{Pres}_\lambda\mathbf{K} \to E_i/\mathbf{Pres}_\lambda\mathbf{K}$$

is comonadic, so is $p_* : B/\mathbf{Pres}_\lambda\mathbf{K} \to E/\mathbf{Pres}_\lambda\mathbf{K}$.

4.3.5. Let $\mathbf{E}$ be a category and $\mathbf{K}$ a locally λ-presentable category. Suppose that the cofibration $P : \mathbf{E} \to \mathbf{K}$ satisfies

(♣) $\mathrm{Des}^*_{\mathbf{E}}(p)$ can be viewed as the category of Eilenberg-Moore coalgebra category of $p_* \dashv p!$ and p is codescent (effective codescent) with respect to P if and only if p_* is precomonadic (comonadic) for each $p : B \to E$ in $\mathbf{K}$.

For example, if P is basic cofibration or if P^{op} satisfies the Beck-Chevalley condition for each $p : E \to B$ in $\mathbf{K}^{\mathrm{op}}$, then P satisfies the above hypotheses.

Clearly, if the cofibration $P : \mathbf{E} \to \mathbf{K}$ satisfies ($\clubsuit$), all results in this section remain true with respect to the cofibration P.

4.3.6. Conjecture. *If* $\mathbf{K}$ *is a locally* λ*-presentable category (*λ*-accessible category with pushouts) and* B *is a fixed object of* $\mathbf{K}$*, then* $\mathcal{E}_B$ *is closed under* λ*-directed colimits. In particular,* λ*-pure morphisms in* $\mathbf{K}$ *are effective codescent morphisms.*

By the process we used in 4.3.5, we see that the above conjecture may depend on an answer to the following:

Question. *Let* $\mathbf{K}$ *be a locally* λ*-presentable category (locally finitely presentable category or* λ*-accessible category with pushouts), and let* T *be a* λ*-accessible comonad over* $\mathbf{K}$*. Is* $\mathbf{K}^T$ *a locally* λ*-presentable category (locally finitely presentable category or* λ*-accessible category)? If yes, what do* λ*-presentable objects of* $\mathbf{K}^T$ *look like?*

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