Applications of De Morgan toposes and the Gleason cover

by

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Abstract

In intuitionistic propositional logic, one of the so-called De Morgan's laws is not valid. This thesis studies the non intuitionistically valid one, namely, $\neg(\phi \wedge \psi) = \neg\phi \vee \neg\psi$, (denoted by (DML)), with examples and applications in topology, algebra, analysis, logic and topos theory. In particular, we recall the Gleason cover of a topos which is a universal construction of a De Morgan topos covering the given one. This construction is then used in connection with the Hahn-Banach theorem in any topos of sheaves on a locale, and in order to obtain the real closure of an ordered field in any topos of sheaves on a Boolean space. We also show that an algebraic analogue of (DML) may be related to the Zariski spectrum of a ring. Finally, we examine (DML) in the contexts of model theory and locale theory.

Résumé

Dans la logique propositionelle intuitioniste, une des lois de De Morgan n'est pas valable. Cette thèse étudie la loi qui est non intuitionellement valide, c'est à dire, $\neg(\phi \wedge \psi) = \neg\phi \vee \neg\psi$, (dénoté par (DML)), avec des exemples et des applications en topologie, en algèbre, en analyse, en logique, et dans la théorie des topos. En particulier, nous rappelons le recouvrement de Gleason d'un topos qui est une construction universelle d'un topos de De Morgan recouvrement le topos donné. Ensuite, cette construction est utilisé en connection avec le théorème d'Hahn-Banach dans un topos de faisceaux sur un locale et aussi pour obtenir la clôture réelle d'un corps ordonné dans un topos de faisceaux sur un espace Booleén. De plus, nous montrons que l'analogue algébrique de la (DML) peut être relié au spectre de Zariski d'un anneau. Finalement, nous examinons la (DML) dans des contextes de la théorie des modèles et de la théorie des locales.

Contents

Introduction

In propositional logic, there are two identities that are called De Morgan's laws. In intuitionistic logic, only one of the De Morgan's laws is valid. When we refer to De Morgan's law (denoted by (DML)) here, we mean the non intuitionistically valid one, namely, $\neg(\phi \wedge \psi) = \neg\phi \vee \neg\psi$. A property that is equivalent to (DML) is $\neg\phi \vee \neg\neg\phi = 1$. In topology, we say that a space is extremally disconnected (i.e., the closure of every open subset in the space is open, i.e., clopen) if this property holds for all open subsets in the space. This thesis presents various aspects of De Morgan's law with applications in topology, algebra, analysis, logic and topos theory. In *Chapter 1*, we present the basic notions on lattices, locales, toposes and logic that will be needed in the proceeding chapters. We refer the reader to the sources [25],[18],[33], and [29] for more details. We have also relied on the notes for the course on topos theory given by M. Bunge [4].

In *Chapter 2*, we present conditions equivalent to (DML) in a topos, due to Johnstone [19]. The advantage of working in a De Morgan topos is that the not necessarily valid Law of the Excluded Middle $\vdash \phi \vee \neg\phi$ at least can be used in the form $\vdash \neg\phi \vee \neg\neg\phi$ in the internal logic of the topos. We then proceed to analyse two examples that illustrate some consequences of (DML). The first result, due to Mulvey and Pelletier [36], shows that if one has a quotient map $\pi : L \to K$ of compact regular locales in a De Morgan topos defined localically over *Sets*, any point of the locale K may be lifted to a point of the locale L. The second example uses an algebraic analogue of (DML) to connect the notion of extremal disconnectedness to ring theory. In particular, it was shown by Niefield and Rosenthal [38], that if R is a commutative ring with identity and having no nilpotents, then the space $Spec(R)$, known as the Zariski spectrum of R, is extremally disconnected iff R satisfies the algebraic analogue of (DML). In *Chapter 3*, we recall the construction and basic properties of the Gleason cover of an arbitrary topos, taken from Johnstone [22]. The Gleason cover of a topos is a De Morgan topos by construction and has the corresponding universal property that given any geometric morphism with codomain the given topos and with domain a De Morgan topos, it factors uniquely through

the Gleason cover. Its use will be apparent in the applications presented in *Chapter 4* and *Chapter 5*. We then connect the Ore condition (i.e., any diagram of two arrows in a category with common codomain embeds into a commutative square) to (DML) in a topos, due to Johnstone [19], and to the notion of a relatively De Morgan topos, due to Kock and Reyes [30]. A relatively De Morgan topos can be defined in terms of its relatively complemented elements (Jibladze), i.e., the notion of clopens and regular elements in a topological space or frame. We end this chapter with a brief look at conditions equivalent to Booleaness and the notion of complemented elements in connection to zero-dimensionality, both due to Bunge and Funk ([5],[6]). In topology, a topological space is said to be zero-dimensional if the clopen subsets of the space form a base for the topology.

In *Chapter 4*, we review the Dedekind real numbers in a topos and its completion, called the MacNeille real numbers. We proceed to show that in a De Morgan topos, the Dedekind reals and the MacNeille reals coincide. We then give an application of the Hahn-Banach theorem, following Mulvey and Pelletier [37]. In *Chapter 5*, we study the categorical logical analogue of (DML) and connect it to the notions of model completion and the amalgamation property, due to Joyal and Reyes [26]. We then proceed with an application, due to Bunge [3], namely, that the real closure of an ordered field exists in any topos of sheaves on a Boolean space. Lastly, we briefly present the conditions needed for a classifying topos to satisfy (DML), due to Bagchi [1].

(DML) is an unusually interesting condition in that, while taking on quite seemingly unrelated forms in different fields of mathematics, each of these equivalent formulations have been independently recognized as an important one and used in a variety of applications. For this reason, we believe that (DML) will continue to be useful, particularly when the interactions between the many different forms that it can take become better known. In this thesis, by bringing together, for the first time, these different aspects of (DML), we hope to contribute to this worthwhile pursuit.

Chapter 1

Preliminaries

1.1 Lattices

Definition 1.1

1. Let A be a partially ordered set (denoted poset), S be a subset of A. An element $a \in A$ is a *join* (or *least upper bound*) for S, denoted by $a = \bigvee S$, if

   (a) a is an upper bound for S, i.e., $s \leq a$ for all $s \in S$, and

   (b) if b satisfies $\forall s \in S (s \leq b)$, then $a \leq b$.

   Dually, in any poset, we can consider the notion of *meet* (*greater lower bound*), defined by reversing all the inequalities in the definition of join. We say $(A, \vee, 0)$ and $(A, \wedge, 1)$ is called *join-semilattice* and *meet-semilattice*, respectively.

2. A *lattice* A is a set with two binary operations $\vee, \wedge$ and two distinguished elements $0, 1$, such that $\vee$ (respectively, $\wedge$) is associative, commutative and idempotent and has 0 (respectively, 1) as unit element, and such that $\vee$ and $\wedge$ satisfy, for all $a, b \in A$, the *absorptive laws*

$$a \wedge (a \vee b) = a, \; a \vee (a \wedge b) = a.$$

Moreover, A is called a *distributive lattice* if in addition $\vee$ and $\wedge$ satisfy, for all $a, b, c \in A$, the *distributive laws*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

A *homomorphism* is a structure-preserving map. A lattice is said to be *complete* if it has arbitrary joins and arbitrary meets(finite and infinite). Though a homomorphism of complete lattices need not preserve meets. Note that a lattice can be considered as a category with finite limits and all finite colimits.

3. A *Boolean algebra* is a distributive lattice A equipped with an additional unary operation $\neg : A \to A$ such that $\neg a$ is a *complement* of a (i.e., $\neg a \wedge a = 0$ and $\neg a \vee a = 1$), for all $a \in A$. Note that a Boolean homomorphism commutes with $\neg$, and that $\neg\neg a = a$ for all $a \in A$.

4. A *Heyting algebra* is a distributive lattice A equipped with an additional binary operation $\to$ with the property that

$$(a \wedge b) \leq c \;\; iff \;\; a \leq (b \to c),$$

for all $a, b, c \in A$. The operation $a \to (-)$ is right adjoint to $a \wedge (-)$. The *negation* (or pseudocomplement) of a is $\neg a = (a \to 0)$ such that $a \wedge \neg a = 0$ but in general we do not have $a \vee \neg a = 1$. This follows from the fact that $\neg a$ is not necessarily the set-theoretic complement of a and $\neg\neg a$ need not equal a, for all $a \in A$. For example, it is known that the set $\mathcal{O}(X)$ of all open subsets of a topological space X is a Heyting algebra in which the negation of the open set $U \in \mathcal{O}(X)$ is the union of all open subsets of X which do not meet U, so is the interior of the set-theoretic complement of U, i.e., the set-theoretic closure of U, and so, $\neg\neg U$ is the interior of the closure of U. Hence $\neg\neg U$ may be larger than U.

5. Given a Heyting algebra A, we say $a \in A$ is *regular* ($\neg\neg$-stable element) if $\neg\neg a = a$. The set of all regular elements of A, with its induced order, is denoted $A_{\neg\neg}$, and it is easy to verify that $A_{\neg\neg}$ is a Boolean algebra.

A Boolean algebra B is a Heyting algebra since for all $a, b, c \in B$

$$(a \wedge b) \leq c \;\; iff \;\; a \leq \neg b \vee c,$$

so that one can take $(b \to c) = \neg b \vee c$. The converse does not hold; for example, let A be a totally ordered set with least and greatest elements 0 and 1. A is a Heyting algebra, with implication defined by

$$\begin{aligned} a \to b \;\; &= 1 \quad if\ a \leq b \\ &= b \quad otherwise. \end{aligned}$$

But A is not normally Boolean; in fact every $a \neq 0$ in A satisfies $\neg\neg a = 1$.

Definition 1.2

1. A subset I of a lattice A is said to be an *ideal* if

   (a) $0 \in I$, and $a, b \in I$ implies $a \vee b \in I$; and

   (b) I is a *lower set*; i.e., $a \in I$ and $b \leq a$ implies $b \in I$.

   For any $a \in A$, we say the ideal $\downarrow (a) = \{b \in A \mid b \leq a\}$ is the *principal* ideal generated by a. An ideal is said to be *proper* provided that $1 \notin I$. A *prime ideal* is a proper ideal satisfying $(a \wedge b \in I)$ implies either $a \in I$ or $b \in I$. Dually,

2. A subset F of a lattice A is said to be a *filter* if

   (a) $1 \in F$, and $a, b \in F$ implies $a \wedge b \in F$; and

   (b) $a \in F$ and $a \leq b$ implies $b \in F$.

   A filter F is said to be *proper* provided that $0 \notin F$. A *prime filter* is a proper filter satisfying $(a \vee b \in F)$ implies either $a \in F$ or $b \in F$. A *completely prime filter* F is a proper filter satisfying $\bigvee B \in F$ implies that there exists $b \in B$ such that $b \in F$. A filter F is *maximal* if it is contained in no other filter, i.e., for any filter L such that $F \subseteq L$ one has $F = L$.

We recall that a *Stone space* of a Boolean algebra B is a topological space with the set

$$D(a) = \{F \mid a \in F, F \text{maximal filter}\},$$

for all $a \in B$, forming a basis for the topology on the set of all maximal filters on B. It is well known that the Stone space of B is a compact Hausdorff space. The Stone space can equivalently be defined as the space of prime ideals P in the Boolean algebra B with basic open sets of the form $\{P \mid a \notin P\}$.

1.2 Frames and locales

Definition 1.3

1. The category Frm of *frames* is the category whose objects are lattices A with all finite meets $\wedge$ and arbitrary joins $\bigvee$ satisfying the infinite distributive law

   $$a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$$

   for each $a \in A$ and each subset $\{b_i \mid i \in I\}$ of A. A *morphism of frames* is a function preserving finite meets and arbitrary joins.

2. The category *Loc* of *locales* is the opposite of the category Frm. The objects in *Loc* are the same as in Frm; but a *morphism of locales* $\pi : B \to A$ is then a mapping

   $$\pi^* : A \to B$$

   in Frm, called the *inverse image* mapping. A map of locales is said to be a *quotient map* provided that its inverse image mapping is injective, which is equivalent to its reflecting the order relation of the locales concerned.

It is easy to verify that for any topological space X, the set $\mathcal{O}(X)$ of all subsets of X is a locale. Furthermore, for any continuous mapping $f : X \to Y$, the map

$$\mathcal{O}(f) : \mathcal{O}(X) \to \mathcal{O}(Y)$$

of locales, of which the inverse image mapping $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a map of frames. The locale $\mathcal{O}(X)$ yields a functor

$$\mathcal{O} : \text{Topological spaces} \to \text{Loc}$$

from the category of topological spaces to the category of locales. This functor admits a right adjoint (see [33])

$$Pts : \text{Loc} \to \text{Topological spaces}$$

which assigns to each locale A the topological space $Pts(A)$ obtained in the following way.

Definition 1.4 A *point* of a locale A is defined to be a map of locales

$$x : 1 \to A$$

to A from the locale 1 of open subsets of the singleton space.

The topological space $Pts(A)$ is then obtained by endowing the set of points of A with the topology of which the subsets

$$D(a) = \{x \in Pts(A) \mid x^*(a) = 1\}$$

for each $a \in A$ are the basic open subsets. Any map of locales $\pi : A \to B$ induces a continuous mapping

$$Pts(\pi) : Pts(A) \to Pts(B)$$

by composition. It may be verified that the points of a locale A correspond bijectively to the completely prime filters on A. A locale A is said to be *spatial* (i.e., A has enough points) provided that the coadjunction $\mathcal{O}(Pts(A)) \to A$ is an isomorphism of locales. A locale A is spatial exactly if it has enough completely prime filters, to ensure that

$$D(a) \leq D(b) \textit{ implies } a \leq b$$

for any $a, b \in A$.

Definition 1.5

1. A locale A is said to be *compact* provided that for any family $(a_i)_{i \in I}$ of elements of A with $\bigvee_{i \in I} a_i = 1$ there exists a finite subfamily with $a_{i_1} \vee \cdots \vee a_{i_n} = 1$.

2. A locale A is said to be *regular* provided that any $b \in A$ is the join of those $a \in A$ which are *rather below* $b \in A$, denoted by $a \lhd b$, in the sense that there exists $c \in A$ such that $a \wedge c = 0$ and $c \vee b = 1$.

3. A locale A is said to be *completely regular* if each $b \in A$ is the join of those $a \in A$ which are completely below $b \in A$, where a is said to be *completely below* b, denoted by $a \lhd\!\lhd b$, provided that there exists an interpolation $d_{ik} \in A$, for $i = 0, 1, \ldots$, and $k = 0, 1, \ldots, 2^i$, dependent on i, such that for all appropriate i, k

   (a) $d_{00} = a$ and $d_{01} = b$;

   (b) $d_{ik} \lhd d_{ik+1}$;

   (c) $d_{ik} = d_{i+1\,2k}$.

Definition 1.6 A *nucleus* on a locale A is defined to be a map $j : A \to A$ satisfying 1. $j(a \wedge b) = j(a) \wedge j(b)$, 2. $a \leq j(a)$, and 3. $j(j(a)) = j(a)$,
for all $a, b \in A$.

If j is a nucleus on A, we define the set of *fixed points* $A_j = \{a \in A \mid j(a) = a\}$. Since $jj = j$, the image of j is precisely A_j. It is known that A_j is a frame, and $j : A \to A_j$ is a frame homomorphism, whose right adjoint is the inclusion $A_j \to A$. A *sublocale* of a locale A is defined to be a subset of the form A_j, for some nucleus j. Finally, we observe that for any element a in a locale A:

1. The map $j = a \vee (-) : A \to A$ is a nucleus and its corresponding sublocale is $\uparrow (a)$. Sublocales of this form are called *closed sublocales.*

2. The map $j = a \to (-) : A \to A$ is a nucleus and its corresponding sublocale is isomorphic to $\downarrow (a)$. Sublocales of this form are called *open sublocales.*

3. A sublocale A_j is said to be *dense* if it contains 0_A (i.e., if j(0)=0).

4. $\neg\neg : A \to A$ is a nucleus and its corresponding sublocale is $A_{\neg\neg} = \{a \in A \mid \neg\neg a = a\}$, i.e., the set of all regular elements of A.

1.3 The $\neg\neg$-topology

We begin by reminding the reader of some basic sheaf theory. Let $\mathcal{C}$ be a small category, and let $Sets^{\mathcal{C}^{op}}$ be the corresponding functor category. An object $P : \mathcal{C}^{op} \to Sets$ of $Sets^{\mathcal{C}^{op}}$ is called a *presheaf* on $\mathcal{C}$.

Definition 1.7 Let $\mathcal{C}$ be a small category with pullbacks. A *Grothendieck pretopology* on $\mathcal{C}$ is defined as follows: for each object U of $\mathcal{C}$, a set $P(U)$ of families of morphisms of the form $\{U_i \to U \mid i \in I\}$, called *covering families* of the pretopology, such that

1. For any U, the family whose only member $(U \xrightarrow{1} U) \in P(U)$.

2. If $V \to U$ is a morphism of $\mathcal{C}$ and $\{U_i \to U \mid i \in I\} \in P(U)$, then $\{V \times_U U_i \to V \mid i \in I\} \in P(V)$.

3. If $\{U_i \xrightarrow{f_i} U \mid i \in I\} \in P(U)$ and $\{V_{ij} \xrightarrow{g_{ij}} U_i \mid j \in J_i\} \in P(U_i)$ for each i, then $\{V_{ij} \xrightarrow{f_i g_{ij}} U \mid i \in I, j \in J_i\} \in P(U)$.

A presheaf $F : \mathcal{C}^{op} \to Sets$ is called a *sheaf* for the pretopology P such that the diagram

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is an equalizer for every covering family $\{U_i \to U \mid i \in I\}$. A *sieve* S on U is defined to be a family of morphisms in $\mathcal{C}$, all with codomain U, such that$(V \xrightarrow{f} U) \in S$ implies $(W \xrightarrow{fg} U) \in S$ for any $(W \xrightarrow{g} V)$.

Definition 1.8 Let $\mathcal{C}$ be a small category. A *site* $(\mathcal{C}, J)$ is defined to be a small category equipped with a Grothendieck topology. A *Grothendieck topology* on $\mathcal{C}$ is

defined by specifying, for each U of $\mathcal{C}$, a set $J(U)$ of sieves on U, called *covering sieves* of the topology, such that

1. For any U, the maximal sieve $\{f \mid codomain(f) = U\} \in J(U)$.

2. If $S \in J(U)$ and $V \xrightarrow{f} U$ is a morphism of $\mathcal{C}$, then the sieve $f^*(S) = \{W \xrightarrow{g} V \mid fg \in S\}$.

3. If $S \in J(U)$ and R is a sieve on U such that, for each $V \xrightarrow{f} U$ in S, we have $f^*(R) \in J(V)$, then $R \in J(U)$.

We denote the full subcategory of $Sets^{\mathcal{C}^{op}}$ whose objects are J-sheaves by $Sh(\mathcal{C}, J)$.

Definition 1.9 *(Lawvere-Tierney)* Let $\mathcal{E}$ be a topos. A *topology* in $\mathcal{E}$ is a morphism $j : \Omega \to \Omega$ such that the diagrams

$$\begin{tikzcd} 1 \arrow[r, "true"] \arrow[dr, "true"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd} \quad , \quad \begin{tikzcd} \Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd} \quad and \quad \begin{tikzcd} \Omega \times \Omega \arrow[r, "\wedge"] \arrow[d, "j \times j"'] & \Omega \arrow[d, "j"] \\ \Omega \times \Omega \arrow[r, "\wedge"] & \Omega \end{tikzcd}$$

commute. If j is a topology, we write $J \rightarrowtail \Omega$ for the subobject classified by j, and $\Omega_j \rightarrowtail \Omega$ for the equalizer of j and 1_Ω (equivalently, the image of j, since j is idempotent).

Remark 1.1 It is well known that if j is a topology in $\mathcal{E}$ and $X' \overset{\sigma}{\rightarrowtail} X$ is a monomorphism with classifying map $X \xrightarrow{h} \Omega$, then σ is *j-dense* iff h factors through $J \rightarrowtail \Omega$, and *j-closed* iff h factors through $\Omega_j \rightarrowtail \Omega$.

Definition 1.10 Let j be a topology in a topos $\mathcal{E}$, F an object of $\mathcal{E}$.

1. F is said to be (j)-*separated* if, given any j-dense $X' \overset{\sigma}{\rightarrowtail} X$ and any pair $X \underset{g}{\overset{f}{\rightrightarrows}} F$ such that $f\sigma = g\sigma$, we have $f = g$.

2. F is said to be a (j)-*sheaf* if, given any j-dense $X' \overset{\sigma}{\rightarrowtail} X$ and any $X' \xrightarrow{f} F$ there exists a unique $X \xrightarrow{g} F$ such that $g\sigma = f$.

Lemma 1.1 *Ω_j is a sheaf.*

Proof. By Remark 1.1, morphisms $X \to \Omega_j$ correspond to closed subobjects of X; so it suffices to prove that if $X' \rightarrowtail X$ is dense and $Y' \rightarrowtail X'$ is closed, there is a unique closed subobject $Y \rightarrowtail X$ such that $Y \cap X' \cong Y'$. But if we define Y to be the closure of the composite $Y' \rightarrowtail X' \rightarrowtail X$, it is immediate that $Y \cap X' \cong Y'$; and conversely, if $Z \rightarrowtail X$ is any closed subobject with $Z \cap X' \cong Y'$, then $Y' \rightarrowtail Z$ is dense (being the pullback of $X' \rightarrowtail X$), and so Z is the closure of Y' in X. □

We write $Sh_j(\mathcal{E})$ for the full subcategory of the topos $\mathcal{E}$ whose objects are sheaves. The topos $Sh_j(\mathcal{E})$ has a subobject classifier, namely the sheaf Ω_j.

Theorem 1.1 *(Lawvere-Tierney) Let $\mathcal{E}$ be a topos. Then $\neg\neg$ is a topology in $\mathcal{E}$, and $Sh_{\neg\neg}(\mathcal{E})$ is Boolean.*

Proof. It is known that there is a bijection between topologies in $\mathcal{E}$ and universal closure operations on $\mathcal{E}$. Thus we need only show that $\neg\neg$ induces a closure operation on subobjects of X. Let X be any object of $\mathcal{E}$. Then it is easily seen that the unary operation on subobjects of X induced by $\neg$ is order-reversing, and that $X' \leq \neg X''$ iff $X'' \leq \neg X'$. From this it follows that $X' \leq \neg\neg X'$ (take $X'' = \neg X'$) and that $\neg X' \cong \neg\neg\neg X'$. It can be verified that the closure operation $\neg\neg$ is universal; so $\neg\neg$ is a topology. Now to show that $Sh_{\neg\neg}(\mathcal{E})$ is Boolean, we need only show that $\Omega_{\neg\neg}$ is an internal Boolean algebra. This follows from the definition of $\Omega_{\neg\neg}$, i.e., that it is the equalizer of 1_Ω and $\neg\neg$. □

Theorem 1.1 can also be explained by recalling the known result that a closure operation is equivalent to a nucleus. And we have already seen in the previous two sections that $\neg\neg$ is a nucleus (in this case, on Ω) and the set $\Omega_{\neg\neg}$, of regular elements of Ω, is Boolean.

The importance of the next result will be apparent when we recall the proof, due to Johnstone [19], that certain conditions are equivalent to De Morgan's law in a topos.

Proposition 1.1 *Let F be an object of $\mathcal{E}$. The following are equivalent:*

1. *F is separated.*

2. *The diagonal $F \overset{\Delta}{\rightarrowtail} F \times F$ is closed.*

3. *There exists a monomorphism $F \rightarrowtail G$, where G is a sheaf.*

Proof. $1 \Rightarrow 2$. Let $\bar{F} \overset{(a,b)}{\rightarrowtail} F \times F$ be the closure of Δ. Then a and b are equalized by the dense subobject $F \rightarrowtail \bar{F}$, so they are equal; hence $\bar{F} \cong F$.
$2 \Rightarrow 3$. Since Δ is closed, its classifying map $F \times F \xrightarrow{\epsilon} \Omega$ factors through $\Omega_j \rightarrowtail \Omega$; so the singleton map $F \overset{\{\}}{\rightarrowtail} \Omega^F$ factors through Ω_j^F. In general, it can be shown that if H is a sheaf and X is any object of $\mathcal{E}$, then H^X is a sheaf. Since Ω_j is a sheaf by Lemma 1.1, Ω_j^F is a sheaf.
$3 \Rightarrow 1$. It can be shown that a subobject of a separated object is again separated.
$\square$

Definition 1.11

1. (Lawvere) By a *natural number object* in a topos $\mathcal{E}$, we mean an object N together with morphisms $1 \xrightarrow{0} N \xrightarrow{s} N$ such that, for any diagram $1 \xrightarrow{x} X \xrightarrow{u} X$ in $\mathcal{E}$, there exists a unique $N \xrightarrow{f} X$ such that

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & N & \xrightarrow{s} & N \\ & {\scriptstyle x}\searrow & \downarrow_f & & \downarrow_f \\ & & X & \xrightarrow{u} & X \end{array}$$

commutes.

2. Let $\mathcal{E}$ be a topos. $\mathcal{E}$ is said to satisfy the *axiom of choice* (AC) if every object of $\mathcal{E}$ is projective, or every epimorphism in $\mathcal{E}$ splits.

3. By a *Boolean-valued model of Set Theory*, we mean a topos $\mathcal{E}$ which satisfies (AC) and has a natural number object.

Lastly, we need to define what is meant by an internally complete poset. Let $P = (P_1 \rightrightarrows P)$ be an internal poset in a topos. Then we have morphisms $\uparrow(-) : P \to \Omega^P$, and $\downarrow(-) : P \to \Omega^P$ whose exponential transposes are respectively the classifying maps of $P_1 \rightarrowtail P \times P$ and $P_1^{op} \rightarrowtail P \times P$. An internal poset P is *internally complete* if there exists an order-preserving map $\Omega^P \xrightarrow{\cup} P$ which is internally left adjoint to the order-preserving map $\downarrow(-)$. Recall that if $P \xrightarrow{f} Q$ and $Q \xrightarrow{g} P$ are order-preserving maps between internal posets, then f is said to be *internally left adjoint* to g if $P \xrightarrow{(1,gf)} P \times P$ factors through $P_1 \rightarrowtail P \times P$ and $Q \xrightarrow{(fg,1)} Q \times Q$ through Q_1. It can be proved that

1. Ω is an internally complete poset in any topos.

2. Let $\mathcal{E}, \mathcal{F}$ be toposes, and $\mathcal{E} \xrightarrow{T} \mathcal{F}$ a functor having a left adjoint L which preserves pullbacks. If P is an internally complete poset in $\mathcal{E}$, then TP is internally complete in $\mathcal{F}$.

1.4 Logic and toposes

Definition 1.12

1. Let the connective $\vee$ be taken to be the supremum (of subobjects). A *logical category* $\mathcal{T}$ is a cartesian category with

   (a) images which are stable under pullbacks,

   (b) finite sups of subobjects of a given object which are stable under pullbacks. We say that $\bigvee_{i\in I} A_i = A$ is stable if for every $B \to A$, $\bigvee_{i\in I} B \times_A A_i \simeq B$.

2. A *logical morphism* between logical categories $f : \mathcal{T} \to \mathcal{T}'$ is a functor which preserves finite inverse limits, images and finite sups. It is known that $Mod_{\mathcal{T}'}(\mathcal{T})$ is the full subcategory of the functor category $Func(\mathcal{T}', \mathcal{T})$ with logical morphisms as objects. In particular, if $\mathcal{T}' = Sets$, then a model of a theory $\mathcal{T}$ is an *interpretation* $\mathcal{T} \xrightarrow{I} Sets$ and clearly the category of all models

of $\mathcal{T}$ in the topos *Sets*, denoted by $Mod_{Sets}(\mathcal{T})$, is a full subcategory of the functor category $Sets^{\mathcal{T}}$.

3. A theory $\mathcal{T}$ in a first-order language $\mathcal{L}$ is said to be a *geometric theory* if all its axioms are of the form

$$\forall x(\phi(x) \to \psi(x)) \tag{1.1}$$

where ϕ, ψ are *geometric formulas*, i.e., are built from the atomic formulas by means of conjunction, disjunction, and existential quantification. An interpretation M in a topos $\mathcal{E}$ assigns to each geometric formula $\phi(x)$ an object in $\mathcal{E}$, denoted by $\{x \mid \phi(x)\}^M$. An axiom of the form 1.1 is "true" for an interpretation M if $\{x \mid \phi(x)\}^M$ is a subobject of $\{x \mid \psi(x)\}^M$.

4. Let $\mathcal{T}$ and $\mathcal{T}'$ be two geometric theories in the same language. Then $\mathcal{T}$ is a *quotient theory* of $\mathcal{T}'$ means that the axioms of $\mathcal{T}'$ are a subset of those of $\mathcal{T}$. This clearly implies that $Mod_{\mathcal{E}}(\mathcal{T})$ is a full subcategory of $Mod_{\mathcal{E}}(\mathcal{T}')$ for any topos $\mathcal{E}$.

5. *Coherent theories* (or pretoposes) are categories having finite limits and finite universal (or stable) disjoint sums (or coproducts). In addition these theories have stable images and effective equivalence relations. It is said that a first-order theory is *coherent* if it can be presented with axioms of the form 1.1 where ϕ, ψ are *coherent formulas*, i.e., are built from the atomic formulas by means of $\wedge, \vee$ and $\exists$, $\uparrow$ (denotes *true*), $\downarrow$ (denotes $false$). Negation is not allowed.

6. *Robinson theories* are pretoposes such that for every $X \rightarrowtail Y$, $\neg X \rightarrowtail Y$ exists with the property $X' \wedge X = \emptyset \leftrightarrow X' \leq \neg X$, for all X' in the ordered set of subobjects of Y.

A quotient $S^{-1}\mathcal{T}$, called a category of fractions, may be constructed in the following way

1. The (usual) calculus of fractions allows one to construct a category of fractions. In particular, for a coherent theory, a *calculus of fractions* consists of a set S of monomorphisms, to be inverted, and satisfying the following conditions

   (a) S includes all isomorphisms and is closed under composition and inverse image.

   (b) If $A \xrightarrow{f} \!\!\!\!\!\rightarrow B$ and $B' \rightarrowtail B$, then $f^{-1}(B') \in S \Rightarrow B' \in S$.

   (c) If $A_1 \rightarrowtail B_1$, $A_2 \rightarrowtail B_2 \in S$, then $A_1 + A_2 \to B_1 + B_2 \in S$.

2. One adds "formally" quotients of the new equivalent relations of the category of fractions.

We end this chapter with the construction of a locale L given a propositional geometric theory $\mathcal{T}$. This locale L is called the *locale of the theory* $\mathcal{T}$. A *propositional geometric theory* $\mathcal{T}$ is a set whose members are called *primitive propositions* together with a set of axioms, each of the form

$$\phi_1 \wedge \cdots \wedge \phi_n \vdash \bigvee_i (\psi_{i1} \wedge \cdots \wedge \psi_{in})$$

in which ϕ_i and ψ_{ij} denote primitive propositions and in which the symbols $\wedge$ and $\bigvee$ are to be interpreted as conjunction and disjunction. We obtain the locale L by considering the propositions which may be formed by taking arbitrary disjunctions of finite conjunctions of primitive propositions, modulo the equivalence relation of provable equivalence in the theory, together with the partial ordering given by provable entailment in the theory. The following remarks establish that L is a locale.

1. L has finite meets and arbitrary joins from the operations of finite conjunction and arbitrary disjunctions in the theory $\mathcal{T}$.

2. The identity and zero elements of L correspond to the logical constants *true* and *false*.

3. It can be verified that finite conjunctions provably distribute over arbitrary disjunctions.

This method of obtaining a locale L from a propositional theory $\mathcal{T}$ describes constructively the nature of the points of L without supposing it has any. Recall that a point $x : 1 \to A$ of a locale is an $\wedge\bigvee$-homomorphism $x^* : A \to \Omega$ into the lattice Ω of subsets of the singleton set 1. The lattice Ω can also be considered to have identity element *true* and zero element *false*. Thus the points of the locale L of the theory $\mathcal{T}$ correspond exactly to the models of the theory. Hence, L is spatial exactly if it has enough points, or equivalently the theory $\mathcal{T}$ has enough models.

An example of the construction of a locale from a propositional geometric theory is the locale $R_{\mathcal{E}}$ of real numbers in a topos $\mathcal{E}$, due to Joyal and Tierney. It may be described as follows. The primitive propositions which generate the propositional geometric theory $R_{\mathcal{E}}$ are those given by

$$x \in (q, r)$$

for each pair q, r of rationals, together with the following axioms which express that the symbol x is to denote a real number lying in the open interval (q, r):

1. $x \in (q, r) \vdash$ false whenever $q \geq r$;
2. $x \in (q, r) \ \wedge \ x \in (q', r') \vdash x \in (q \vee q', r \wedge r')$;
3. $x \in (q, r) \vdash x \in (q, r') \ \vee \ x \in (q', r)$ whenever $q < q' < r' < r$;
4. $x \in (q, r) \dashv\vdash \bigvee_{q<q'<r'<r} x \in (q', r')$;
5. true $\vdash \bigvee_{q<r} x \in (q, r)$.

Thus the locale $R_{\mathcal{E}}$ of reals is clearly the locale of the theory $R_{\mathcal{E}}$. Furthermore, taking a model of the theory is equivalent to assigning to each rational q and to each rational r a truth value for the proposition that $q < x$ and the proposition $x < r$.

The axioms precisely express that one obtains a Dedekind cut on the rationals in the topos $\mathcal{E}$. Thus, the space of points of the theory will be exactly the Dedekind

reals in the topos, given the topology induced by rational open intervals. This description of the locale $R_{\mathcal{E}}$ as a propositional geometric theory $R_{\mathcal{E}}$ as well as the following consequence will be needed in the application of the Hahn-Banach theorem in Chapter 4. Any $\wedge\bigvee$-homomorphism from the locale L into a locale A may be considered to be a model of $\mathcal{T}$ in that locale. The maps of locales from A to L are therefore obtained by considering assignments to each primitive proposition of $\mathcal{T}$ an element of the locale A, in such a way that the axioms of $\mathcal{T}$ are realized in the locale.

Chapter 2

DeMorgan's law in toposes

In this chapter, we review De Morgan's laws and the connection to extremal disconnectedness and then proceed to give a list of conditions equivalent to De Morgan's law in a topos, due to Johnstone [19]. We end this chapter by considering two results to illustrate topos-theoretical and algebraic aspects of De Morgan's law. Both results use the concept of a locale. The first result, due to Mulvey and Pelletier [36], shows that if one has a quotient map

$$\pi : L \to K$$

of compact regular locales in a De Morgan topos $\mathcal{E}$ defined localically over *Sets*, any point of the locale K may be lifted to a point of the locale L. The second result, due to Niefield and Rosenthal [38], uses ring theoretic results about De Morgan's law and relates them to extremal disconnectedness in ring theory.

There are two so-called De Morgan's laws in propositional logic, namely

$$\neg(a \vee b) = \neg a \wedge \neg b \tag{2.1}$$

$$\neg(a \wedge b) = \neg a \vee \neg b. \tag{2.2}$$

The first De Morgan's law 2.1 always holds in intuitionistic propositional logic, thus in a locale in any topos. This follows from the adjointness of $\wedge$ and $\to$ in any Heyting algebra H. However, the second De Morgan's law 2.2, which we will

denote by (DML), may fail to hold in general ($\geq$ is always true). This follows from the fact that $\neg a$ is not necessarily the set-theoretic complement of a and $\neg\neg a$ need not equal a, for all $a \in H$ (see Chapter 1).

Definition 2.1 A locale A is said to be *extremally disconnected* if it satisfies the following equivalent conditions:

1. The identity $\neg(a \wedge b) = \neg a \vee \neg b$ holds for all $a, b \in A$.

2. The identity $\neg a \vee \neg\neg a = 1$ holds for all $a \in A$.

3. Every $\neg\neg$-stable element of A has a complement, i.e., $A_{\neg\neg}$ coincides with the subset A^c of complemented elements of A.

4. $A_{\neg\neg}$ is a sublattice of A.

Remark 2.1

1. Note that condition 1. in Definition 2.1 is precisely (DML) and condition 2. in Definition 2.1 is equivalent to saying that a locale A is a *Stone algebra.*

2. A space X is said to be extremally disconnected if the closure of every open set in X is open (i.e. clopen).

3. In terms of locales, a topological space X is extremally disconnected iff the locale $\mathcal{O}(X)$, of open subsets of X, is.

One example, due to Gleason, of the notion of extremal disconnectedness in topology is the following theorem. Recall that for X, Y topological spaces, a topology for a function space $\mathcal{G} \subset Y^X$ is *admissible* iff the map $P : \mathcal{G} \times X \to Y$ defined by $P(g, x) = g(x)$ is continuous.

Theorem 2.1 *([13]) The projective objects in the category of compact Hausdorff spaces are precisely the extremally disconnected spaces.*

Proof. In ([13], Theorem 1.2), it was proved that in any category of topological spaces and maps for which the following conditions were satisfied

1. All admissible maps are continuous.

2. If A is an admissible space and $\{p, q\}$ is a two-element space, then $A \times \{p, q\}$ and the projection maps of this space onto A are admissible.

3. If A is an admissible space and B is a closed subspace of A, then B and the inclusion map of B into A are admissible.

a projective space is extremally disconnected. Thus conditions 1, 2, 3. above need only be verified. Conversely, let A be an extremally disconnected compact space, let B and C be compact spaces, let f_1 be a continuous map of B onto C, and let f_2 be a continuous map of A into C. We must prove that there exists a continuous map g of A into B such that $f_2 = f_1 \circ g$.

In the space $A \times B$ consider $D = \{< a, b >| \ f_2(a) = f_1(b)\}$. This set is clearly closed and therefore compact. Since f_1 is onto, the projection π_1 of $A \times B$ onto A carries D onto A. It can be shown that since A and D are compact spaces and D is mapped continuously onto A, there is a compact subset E of D such that $\pi_1(E) = A$ but $\pi_1(E_0) \neq A$ for any proper closed subset E_0 of E. By the assumption that A is an extremally disconnected compact space and E is clearly a compact space, it can be shown that h is a homeomorphism where h is the restriction of π_1 to E and $h(E_0) \neq A$ for any proper closed subset E_0 of E. Let $g = \pi_2 \circ h^{-1}$, where π_2 is the projection of $A \times B$ into B; this is the required map. Say $a \in A$; since $h^{-1}(a) \in D$,

$$f_1(\pi_2(h^{-1}(a))) = f_2(\pi_1(h^{-1}(a))) = f_2(a).$$

Thus $f_2 = f_1 \circ \pi_2 \circ h^{-1} = f_1 \circ g$. $\square$

We now review the proof that certain conditions are equivalent to (DML) in a topos. Recall that an object X is said to be *decidable* if the diagonal subobject $X \xrightarrow{\Delta} X \times X$ has a complement.

Theorem 2.2 *([19], Theorem 1) The following conditions are equivalent in any topos $\mathcal{E}$:*

1. *(DML) holds;*
2. *The logical principle*

$$\neg\phi \vee \neg\neg\phi = t$$

   holds; i.e., Ω is an internal Stone algebra in $\mathcal{E}$;

3. *The subobject $1 \overset{f}{\rightarrowtail} \Omega$ has a complement;*
4. *A subobject is $\neg\neg$-closed iff it has a complement;*
5. *An object is $\neg\neg$-separated iff it is decidable;*
6. *Every $\neg\neg$-sheaf is decidable;*
7. *$\Omega_{\neg\neg}$ is decidable;*
8. *$\begin{pmatrix} t \\ f \end{pmatrix} : 2 \to \Omega_{\neg\neg}$ is an isomorphism (2 denotes the coproduct of two copies of 1);*
9. *2 is an internally complete poset in $\mathcal{E}$;*
10. *2 is injective in $\mathcal{E}$;*
11. *2 is a retract of Ω;*
12. *2 is a $\neg\neg$-sheaf;*
13. *$\Omega_{\neg\neg}$ is a sublattice of Ω;*
14. *$\neg\neg : \Omega \to \Omega$ is a Heyting algebra homomorphism.*

Proof. $1 \Rightarrow 2$. Suppose (DML) holds. Let $\psi = \neg\phi$ and so $\neg\phi \vee \neg\neg\phi = t$.
$2 \Rightarrow 3$. Let $\phi = 1_\Omega$. Since ϕ classifies t and $\neg\phi$ classifies $1 \overset{f}{\rightarrowtail} \Omega$, and so it is complemented, as $\neg\phi \vee \neg\neg\phi = t$.

$3 \Rightarrow 4$. Every $\neg\neg$-closed subobject is the negation of something. Suppose we have a $\neg\neg$-closed subobject $X' \rightarrowtail X$ of the form $\neg Y \rightarrowtail X$. We have for $\psi : X \to \Omega$, classifying $Y \rightarrowtail X$, that $\neg\psi$ classifies $\neg Y \rightarrowtail X$. Since any $\neg Y \rightarrowtail X$ can be expressed as a pullback of false and $1 \overset{f}{\rightarrowtail} \Omega$ is complemented, then so is $\neg Y \rightarrowtail X$. The converse is true in any topos.

$4 \Rightarrow 5$. By Proposition 1.1, an object is $\neg\neg$-separated iff its diagonal is $\neg\neg$-closed.

$5 \Rightarrow 6$. By Proposition 1.1, there exists a monomorphism $F \rightarrowtail G$, where F is a sheaf iff F is $\neg\neg$-separated iff F is decidable.

$6 \Rightarrow 7$. By Lemma 1.1, $\Omega_{\neg\neg}$ is a $\neg\neg$-sheaf and so is decidable.

$7 \Rightarrow 8$. The global element $1 \overset{t}{\rightarrowtail} \Omega_{\neg\neg}$ has a complement, which must be $1 \overset{f}{\rightarrowtail} \Omega_{\neg\neg}$ since this is the largest subobject of Ω intersecting true trivially.

$8 \Rightarrow 9$. By 1. and 2. at the end of section 1.3 in Chapter 1, $\Omega_{\neg\neg}$ is an internally complete poset in $Sh_{\neg\neg}(\mathcal{E})$ and hence in $\mathcal{E}$. By assumption, $2 \simeq \Omega_{\neg\neg}$ and therefore 2 is internally complete in $\mathcal{E}$, as required.

$9 \Rightarrow 10$. Since 2 is an internally complete poset, we have the map

$$2 \overset{\downarrow(-)}{\rightarrowtail} \Omega^2 \overset{\cup}{\to} 2$$

such that $\cup \downarrow (-) \simeq id$. Thus 2 is a retract of Ω^2. Since Ω is injective, so is Ω^2 and thus so is 2.

$10 \Rightarrow 11$. The inclusion $\begin{pmatrix} t \\ f \end{pmatrix} : 2 \to \Omega_{\neg\neg}$ splits and so 2 is a retract of Ω.

$11 \Rightarrow 10$. Since 2 is a retract of Ω and Ω is injective, then so is 2.

$10 \Rightarrow 12$. Since 2 is decidable then it is $\neg\neg$-separated. The injectivity of 2 implies that it is a $\neg\neg$-sheaf.

$12 \Rightarrow 8$. The diagram $1 \overset{t}{\to} \Omega_{\neg\neg} \overset{f}{\leftarrow} 1$ is a coproduct in $Sh_{\neg\neg}(\mathcal{E})$, so $\Omega_{\neg\neg}$ is the associated $\neg\neg$-sheaf of 2 (since $\Omega_{\neg\neg}$ is a sheaf and $\begin{pmatrix} t \\ f \end{pmatrix}$ is $\neg\neg$-dense). But since 2 is a $\neg\neg$-sheaf, we must have that $2 \simeq \Omega_{\neg\neg}$.

$8 \Rightarrow 13$. Since $2 \rightarrowtail \Omega$ is a sublattice, so is $\Omega_{\neg\neg} \rightarrowtail \Omega$.

$13 \Rightarrow 14$. By Theorem 1.1, $\neg\neg$ is a topology and so it commutes with $\wedge, t$ and f. It remains to see that it commutes with $\vee$, i.e., that

$$\neg\neg(\phi \vee \psi) = \neg\neg\phi \vee \neg\neg\psi. \tag{2.3}$$

But $\Omega_{\neg\neg} \rightarrowtail \Omega$ is a sublattice, so if $\phi, \psi \in \Omega_{\neg\neg}$, i.e., $\phi = \neg\neg\phi$ and $\psi = \neg\neg\psi$, then their sup in Ω is already the sup in $\Omega_{\neg\neg}$, i.e., $\phi \vee \psi \in \Omega_{\neg\neg}$. This says that 2.3 holds. Hence $\neg\neg : \Omega \to \Omega$ is a Heyting homomorphism.

$14 \Rightarrow 1$. Given that $\neg\neg : \Omega \to \Omega$ is a Heyting homomorphism and $\neg\neg\neg = \neg$, we have

$$\begin{aligned} \neg\phi \vee \neg\psi &= \neg\neg\neg\phi \vee \neg\neg\neg\psi \\ &= \neg\neg(\neg\phi \vee \neg\psi) && \textit{by } 14. \\ &= \neg(\neg\neg\phi \wedge \neg\neg\psi) && \textit{by } 2.1 \\ &= \neg\neg\neg(\phi \wedge \psi) && \textit{by } 14. \\ &= \neg(\phi \wedge \psi). \end{aligned}$$

□

This leads us to a useful result.

Theorem 2.3 *([19]) Let X be a topological space. Then the topos $Sh(X)$ of sheaves on X satisfies (DML) iff X is extremally disconnected.*

Proof. The idea behind the proof is as follows. A topological space X is extremally disconnected iff $\mathcal{O}(X)$ is extremally disconnected by 3. in Remark 2.1 iff $\mathcal{O}(X)$ satisfies (DML) by condition 1. in Definition 2.1 iff for every $U \in \mathcal{O}(X)$, $\neg U \cup \neg\neg U = 1$ by condition 2. in Definition 2.1. It follows that Ω is an internal Stone algebra in $\mathcal{E}$, i.e., the topos $Sh(X)$ satisfies (DML) by Theorem 2.2. □

Remark 2.2

1. An application of Theorem 2.3 in functional analysis is Burden's proof of the Hahn-Banach theorem for normed linear spaces in the category of sheaves on a topological space X [8]. In particular, Mulvey and Pelletier point out that if X is extremally disconnected, one has the Hahn-Banach theorem in its naive form, that any linear functional

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \\ R_X & & \end{array}$$

on a subspace extends to the space while preserving the norm [37]. A brief proof of the latter result will be shown in Chapter 4 and it depends on certain properties of extremally disconnected spaces which turn out to be consequences of De Morgan's law in the internal logic.

2. Theorem 2.3 can easily be generalized to the concept of locales, i.e., if $\mathcal{E}$ is the topos of sheaves on a locale L then the topos $\mathcal{E}$ is a De Morgan topos iff the locale L is extremally disconnected. Again, this is applied to the Hahn-Banach theorem [37] and will be reviewed in Chapter 4.

We now look into the first result mentioned in the beginning of this chapter, i.e., we will review the following theorem.

Theorem 2.4 *([36], Theorem 3.1) In any De Morgan topos $\mathcal{E}$ defined localically over* Sets, *any quotient map*

$$\pi : L \to K$$

of compact regular locales determines canonically a continuous mapping

$$Pts(\pi) : Pts(L) \to Pts(K)$$

which is surjective.

In other words, under the conditions stated in Theorem 2.4 , any point of the locale K may be lifted to any point of the locale L.

We first need to recall a few definitions and results. A *regular filter* F on a locale L is a filter on L satisfying the condition that $a \in F$ implies that there exists $b \in F$ such that $b \lhd a$. A regular filter is said to be *maximal* provided that F is proper, and that any proper regular filter which contains it is equal to it.

Proposition 2.1 *Let L be a regular locale in any topos $\mathcal{E}$. Then any completely prime filter P on L is necessarily a maximal regular filter on L.*

Proposition 2.2 *Let P be a regular filter on a compact regular locale L in any topos $\mathcal{E}$. Then P is completely prime if and only if P is prime.*

The proof of Proposition 2.1 and Proposition 2.2 are straightforward in the sense that it is a matter of interpreting the definitions of the terms.

Proposition 2.3 *([36], Proposition 2.4) Let P be a maximal regular filter on a compact regular locale L in a De Morgan topos $\mathcal{E}$. Then P is prime.*

Proof. Suppose that $a_1 \vee a_2 \in P$. It must be proved that $a_1 \in P$ or $a_2 \in P$. It can be verified that the compact regularity of the locale L implies that there exist $b_1 \lhd a_1$, $b_2 \lhd a_2$ with $b_1 \vee b_2 \in P$. Next it will be shown that this allows regular filters F_1, F_2 to be constructed on the locale L, containing the filter P and containing $a_1, a_2 \in L$ respectively, with the property that it is not the case that both $0 \in F_1$ and $0 \in F_2$. By De Morgan's law in the topos $\mathcal{E}$, it follows that F_1 is false or F_2 is false, i.e., that F_1 is proper or F_2 is proper. By the maximality of the regular filter P, one concludes respectively that $0 \in F_1$ or $0 \in F_2$ equals P. Hence, that $a_1 \in P$ or $a_2 \in P$, as required.

The subsets F_1, F_2 of the locale L, whose existence has just been asserted, are defined by

$$F_i = \{x \in L \mid \exists a \in L\, \exists c \in P\ a \wedge c \leq x \textit{ and } b_i \lhd a\}$$

for each i. It can be verified that each F_i is a regular filter on L.

One of the F_i must be a proper filter, by an argument indicated above depending on De Morgan's law. For suppose $0 \in F_1$ and $0 \in F_2$. Then there exists $a, a' \in L$ and $c, c' \in P$ for which $a \wedge c \leq 0, a' \wedge c' \leq 0$ and $b_1 \lhd a, b_2 \lhd a'$. Then $b_1 \vee b_2 \in P$ implies that $a \vee a' \in P$, and $c, c' \in P$ implies that $c \wedge c' \in P$. Hence, $(a \vee a') \wedge (c \wedge c') \in P$. But $(a \vee a') \wedge (c \wedge c') \leq (a \wedge c) \vee (a' \wedge c') \leq 0$, contradicting the properness of the filter P. It is therefore not the case that both $0 \in F_1$ and $0 \in F_2$. Thus, one of F_1 and F_2 is a proper regular filter containing P, hence, by the maximality of P, equal to P. From which, again, it follows that $a_1 \in P$ or $a_2 \in P$, as required. □

Theorem 2.5 *([36], Theorem 2.1) In any topos $\mathcal{E}$ in which (DML) is satisfied, the completely prime filters on a compact regular locale L are precisely the maximal regular filters on L.*

Proof. ($\Rightarrow$) It follows exactly from Proposition 2.1.

($\Leftarrow$) It follows from Proposition 2.2 and Proposition 2.3. □

We remind the reader that a filter P on L is said to *extend* a filter Q on K provided that $\pi^*(x) \in P$ for each $x \in Q$.

Lemma 2.1 *([36], Lemma 3.2) Let $\pi : L \to K$ be a quotient map of locales in any topos $\mathcal{E}$. Then any regular filter Q on K can be extended to a regular filter P on L, which is proper provided that Q is proper.*

Proof. Take $P = \{x \in L \mid \exists a \in Q \ \pi^*(a) \leq x\}$ to be the filter generated by the inverse image of the filter Q. The subset P is closed under finite meets, by the fact that Q is and that π^* preserves them. Equally, P is regular, by the fact that Q is and that π^* preserves the rather below relation. Finally, $0 \in P$ implies that $0 \in Q$, since the inverse image mapping is an embedding. So, Q proper implies P proper. □

Lemma 2.2 *([36], Lemma 3.3) Let $\mathcal{E}$ be a topos defined over Sets. Then any proper regular filter P on a locale L in the topos $\mathcal{E}$ is contained in a proper regular filter P' on L which is externally maximal.*

Proof. One is asserting that there exists a proper regular filter P' on L which is maximal in the partially ordered set of proper regular filters having the extent of P and containing P. This is proved simply by observing that this set is inductive in the topos *Sets*, when ordered by inclusion, by taking unions of regular filters on the locale L. Applying Zorn's lemma externally in the topos *Sets*, there exists a maximal element P'. □

Lemma 2.3 *([36], Lemma 3.4) Let $\mathcal{E}$ be a topos defined localically over Sets. Then any proper regular filter P on a locale L which is maximal externally is a maximal regular filter on L.*

Proof. Since the topos $\mathcal{E}$ is defined to be localically over *Sets*, it suffices to consider a proper regular filter P on L whose extent is a subobject of the terminal object 1 of the topos $\mathcal{E}$. It is then enough to show that any proper regular filter F defined over a subobject of the extent of P, on which it contains P, is actually equal to P. But the union of such a filter with P is again a proper regular filter whose extent is that of P. Hence, it equals P by the external maximality of P. Thus, F is equal to P over its extent, which proves its maximality in the topos $\mathcal{E}$. $\square$

Proof. (Theorem 2.4) Any point of the compact regular locale K is given by a completely prime filter on K, hence by a maximal regular filter on K, by Theorem 2.5. This may be extended to a proper regular filter on L, by Lemma 2.1, hence to a maximal regular filter on L, by Lemma 2.2 and Lemma 2.3. $\square$

Remark 2.3

1. Theorem 2.4 requires that $\mathcal{E}$ be a De Morgan topos. It is Proposition 2.3 that makes use of this requirement.

2. Theorem 2.4 also requires that $\mathcal{E}$ be defined localically over *Sets*. It is this condition that allows Zorn's lemma to be applied in the topos concerned and it is Lemma 2.3 that makes use of this condition.

3. It is clear that if the locales L and K are spatial, i.e., have enough points, any point of the locale K may be lifted to a point of the locale L. Theorem 2.4 holds even when this spatiality is not assumed.

4. Theorem 2.4 will be needed in the application of the Hahn-Banach theorem in Chapter 4.

5. Theorem 2.4 is a consequence of De Morgan's law in the internal logic of the topos.

We devote the rest of this chapter to the algebraic analogue of extremal disconnectedness, due to Niefield and Rosenthal [38]. That is,

Theorem 2.6 *([38], Theorem 2) Let R be a commutative ring with identity and having no nilpotents. Then $Spec(R)$ is extremally disconnected iff R satisfies (DML).*

In this case, R is a *Baer ring*, i.e., a commutative ring such that the annihilator $Ann(A)$ of A is a principal ideal generated by an idempotent $e \in R$ for all ideals A of R.

The first step is to define the algebraic analogue of De Morgan's laws. This can be done by showing that the set of ideals of the ring R is a locale. Then with the construction of $Spec(R)$, known as the Zariski spectrum of the ring R, algebraic and ideal theoretic properties of R are connected to topological properties of $Spec(R)$. Let R denote a commutative ring with identity and let $Idl(R)$ and $Rad(R)$ denote the lattice of ideals of R and the lattice of radical ideals of R, respectively. Recall that an ideal A is *radical* iff $x^n \in A$ implies $x \in A$.

$Idl(R)$ is a locale, since if $A, B, C \in Idl(R)$, then

$$A \cdot B \subseteq C \ iff \ A \subseteq C : B$$

where $C : B = \{r \in R \mid rB \subseteq C\}$. The analogue of $\neg b = b \to 0$ for any element b in a locale is the operation $0 : B = \{r \in R \mid rB = 0\} = Ann(B)$. Thus, it is clear that for any $A, B \in Idl(R)$, the following may be considered to be the algebraic analogues of De Morgan's laws for a ring R:

$$Ann(A + B) = Ann(A) \cap Ann(B) \tag{2.4}$$

$$Ann(A \cap B) = Ann(A) + Ann(B) \tag{2.5}$$

A related but weaker condition than (DML) is

$$Ann(a) + Ann(b) = Ann(ab) \tag{2.6}$$

where $a, b \in R$ and (WDML) denotes *Weak De Morgan's law.*

We proceed to define the *Zariski spectrum* of the ring R. Let $X = X(R)$ denote the set of prime ideals of R. There is a natural way of introducing a topology on

the set X. If $A, B, C \in Rad(R)$, then $\sqrt{AB} = \sqrt{A \cap B}$ and $C : B$ will again be radical, then it follows that $Rad(R)$ is a locale with

$$A \cap B \subseteq C \ iff \ A \subseteq C : B.$$

Let $V(A)$ be the subset of X consisting of the prime ideals P containing a subset A of R, i.e.,

$$V(A) = \{P \mid A \subseteq P, P\, prime\}.$$

It is known that the radical of an ideal A is the intersection of the prime ideals containing A, thus $V(A) = V(rad(A)) = V(\sqrt{A})$. Also, if P is a prime ideal containing A_1A_2 for $A_1, A_2 \in Idl(R)$ then either $P \supset A_1$ or $P \supset A_2$. Hence $V(A_1A_2) = V(A_1) \cup V(A_2)$. It can be checked that the sets $V(A)$, A a subset of R, satisfy the axioms for closed sets in a topological space.

Every open subset of X is the complement of the closed set $V(A)$ and is of the form

$$D(A) = \{P \mid A \not\subset P, P\, prime\}$$

for all $A \in Idl(R)$.

The primes of $Rad(R)$ are precisely the primes of ideals of R. Thus we may define a topological space X, whose elements are the primes of $Rad(R)$, and such that X is equipped with the above topology. Hence the corresponding space X is denoted $Spec(R)$.

The next result gives conditions under which R satisfies algebraic (DML).

Theorem 2.7 *([38], Theorem 1) The following are equivalent for a commutative ring R with identity.*

1. *$Ann(AB) = Ann(A) + Ann(B)$, for all $A, B \in Idl(R)$.*

2. *R satisfies (DML) and R has no nilpotents.*

3. *$Ann(A) \oplus Ann(Ann(A)) = R$, for every $A \in Idl(R)$.*

4. R is a Baer ring.

5. R satisfies (WDML) and $Ann(A)$ is principal, for every $A \in Idl(R)$.

Proof. $1 \Rightarrow 2$. Assume 1. holds. It is straightforward to prove that if R satisfies (WDML), then R has no nilpotents. R satisfies (DML) since

$$Ann(A \cap B) \subset Ann(AB) \subset Ann(A) + Ann(B) \subset Ann(A \cap B)$$

(the last containment always holds).

$2 \Rightarrow 3$. Suppose R satisfies 2. Since R has no nilpotents, $A \cap Ann(A) = 0$. Applying (DML), we get

$$Ann(A) + Ann(Ann(A)) = Ann(A \cap Ann(A)) = Ann(0) = R.$$

Thus, $Ann(A) \oplus Ann(Ann(A)) = R$.

$3 \Rightarrow 4$. Suppose $Ann(A) \oplus B = R$. Then $1 = x + y$, where $x \in A$ and $y \in B$. A straightforward calculation shows that $x^2 = x$ and $Ann(A) = Rx$.

$4 \Rightarrow 5$. If R is a Baer ring then for every $A \in Idl(R)$, there exists an idempotent $e \in R$ such that $Ann(A) = Re$. Since $Re = Ann(1 - e)$ and $1 - e$ is idempotent when e is, it follows that R is Baer iff for every ideal A, there exists an idempotent $e \in R$ such that $Ann(A) = Ann(e)$. Clearly $Ann(A)$ is principal for every ideal A of R. It can be shown, by a straightforward proof, that if e, e' are idempotent then

$$Ann(ee') = Ann(e) + Ann(e').$$

Thus R satisfies (WDML).

$5 \Rightarrow 1$. Suppose 5. holds. It can be proved that if $Ann(A) = Ann(A')$ and $Ann(B) = Ann(B')$, then $Ann(AB) = Ann(A'B')$. Since R satisfies (WDML) and using the preceding result, we need to show that for every ideal A, $Ann(A) = Ann(a)$, for some $a \in R$. Since the annihilator of every ideal is principal, we can write $Ann(A) = Rx$ and $Ann(x) = Ra$. Then

$$Ann(a) = Ann(Ra) = Ann(Ann(x)) = Ann(Ann(Rx)) = Ann^3(A) = Ann(A).$$

$\square$

Lemma 2.4 *([38], Lemma 4) If R has no nilpotents, then the closure of $D(A)$ in $Spec(R)$ is given by $\overline{D(A)} = V(Ann(A))$.*

Proof. Let $P \in D(A)$, i.e., $A \not\subset P$, P prime. Then $Ann(A) \subset P$ since $A \cdot Ann(A) = 0$ and $P \in V(Ann(A))$. Hence $D(A) \subset V(Ann(A))$, and so $\overline{D(A)} \subset V(Ann(A))$. Conversely, if $P \in V(Ann(A))$, i.e., $Ann(A) \subset P$, then we must show that every open neighbourhood of P meets $D(A)$. Let $D(B)$ be such an open set, i.e., $B \not\subset P$. Then $B \not\subset Ann(A)$. Hence, $AB \neq 0$. Since R has no nilpotents, $0 \subset \sqrt{AB}$, where $\sqrt{\ }$ denotes the prime radical of an ideal. Therefore,

$$D(A) \cap D(B) = D(\sqrt{AB}) \neq D(\sqrt{0}) = \emptyset.$$

□

Finally, we present the proof of Theorem 2.6.

Proof. It is well known that if R has no nilpotents, then A is a direct summand of R iff $V(A)$ is an open subset of $Spec(R)$. Using this and Theorem 2.7, $2 \Leftrightarrow 3.$, R satisfies (DML) iff $Ann(A)$ is a direct summand of R iff $V(Ann(A))$ is open, for all A. By Lemma 2.4, $V(Ann(A))$ is open, for all A, iff $\overline{D(A)}$ is open, for all A, iff $Spec(R)$ is extremally disconnected. □

In Chapter 4, an application of the Hahn-Banach theorem in functional analysis will be shown to be a consequence of the algebraic (DML).

Chapter 3

The Gleason cover and relatively De Morgan toposes

In this chapter, we recall the construction and the basic properties of the Gleason cover of an arbitrary topos, as shown by Johnstone [22]. We then review the Ore condition in the instance that the base topos is Boolean, due to Johnstone [19], and in the instance that the base topos is arbitrary, due to Kock and Reyes [30]. The first result is a consequence of De Morgan's law (DML) while the latter uses the notion of relatively complemented elements of a frame, given by Jibladze (March 1990), and is a consequence of the notion of relatively De Morgan, as studied by Kock and Reyes [30]. We end this chapter with a brief look at conditions equivalent to Booleaness in terms of weakly closed sublocales and closed sublocales for any locale in the Boolean topos and the identification of the frame of nuclei on a locale X with the opposite of weakly closed sublocale of X (originally proved by Jibladze), due to Bunge and Funk [5]. Lastly, zero-dimensionality is defined using the relatively complemented elements of a frame, due to Bunge and Funk [6].

The Gleason cover $\gamma\mathcal{E}$ of a topos $\mathcal{E}$ has two main properties. One is that the map $e : \gamma\mathcal{E} \to \mathcal{E}$ is a surjective geometric morphism and the other is that the topos $\gamma\mathcal{E}$ satisfies (DML).

Definition 3.1 *([22])* The *Gleason cover* $\gamma\mathcal{E}$ *of a topos* $\mathcal{E}$ is defined to be the topos

of $\mathcal{E}$-valued sheaves for the finite cover topology on the internal Boolean algebra $\Omega_{\neg\neg}$, or equivalently the topos $\mathcal{E}[Idl(\Omega_{\neg\neg})]$ of canonical sheaves on the internal locale of ideals of $\Omega_{\neg\neg}$, i.e., $\mathcal{E}[Idl(\Omega_{\neg\neg})] = Sh(\mathcal{E}^{Idl(\Omega_{\neg\neg})^{op}})$.

We recall Theorem 2.2 that a topos $\mathcal{E}$ satisfies (DML) iff the internal Boolean algebra $\Omega_{\neg\neg}$ coincides with 2. Also note that the Gleason cover of $\mathcal{E}$ may be defined as the internal "Stone space" of the locale $\Omega_{\neg\neg}$ (see Chapter 1). We begin by explaining some of the terms in the definition above.

The localic $\mathcal{E}$-topos corresponding to the internal locale A, $\mathcal{E}[A]$ is a sheaf subtopos of the presheaf topos $\mathcal{E}^{A^{op}}$. We need to define the Lawvere-Tierney topology and the subobject classifier in $\mathcal{E}^{A^{op}}$ [21].

Define a map $j : Sv(A) \to Sv(A)$ by

$$j(< R, a >) =< \downarrow (\bigvee_A R), a >,$$

where $Sv(A)$ is the object of sieves on A, i.e., the monomorphism $Sv(A) \rightarrowtail \Omega^A \times A$ and for $< R, a > \in \Omega^A \times A$,

$$< R, a > \in Sv(A) \leftrightarrow (\forall x \in R)((x \leq a) \wedge (\forall b \in A)(b \leq x \to b \in R)).$$

The topology j is called the internal canonical topology on A and $\mathcal{E}[A]$ is defined to be $Sh_j(\mathcal{E}^{A^{op}})$. The map $\alpha : A_1 \times_A Sv(A) \to Sv(A)$ makes $Sv(A) \to A$ into an internal presheaf on A; specifically,

$$\alpha(<< b, a >, < R, a >>) =< R \cap \downarrow(b), b >,$$

and this presheaf is the subobject classifier in $\mathcal{E}^{A^{op}}$. Thus the subobject classifier in $\mathcal{E}[A]$ is the presheaf on A defined by $A_1 \xrightarrow{d_1} A$. Since the subobject classifier of $\mathcal{E}[A]$ is the equalizer of j and id, the canonical topology j splits as the top line of

$$\begin{array}{ccccc}
Sv(A) & \twoheadrightarrow & A_1 & \rightarrowtail & Sv(A) \\
\downarrow & & \downarrow & < d_0, d_1 > & \downarrow \\
\Omega^A \times A & \twoheadrightarrow & A \times A & \rightarrowtail & \Omega^A \times A \\
 & \bigvee_A \times 1 & & \downarrow (-) \times id &
\end{array}$$

where the vertical arrows are monomorphisms.

Let A be a meet-semilattice. A *coverage* C on A will mean a function which assigns to each $a \in A$ a set $C(a)$ of subsets of $\downarrow(a)$, called *coverings of* a, with the following "meet-stability" property:

$$S \in C(a) \to \{s \wedge b \mid s \in S\} \in C(b)$$

for all $b \leq a$. A *C-ideal of* A, I, is defined to be a lower set and satisfies

$$\exists S \in C(a)(S \subseteq I) \to a \in I$$

for all $a \in A$. Notice that for a distributive lattice A, taking $C(a)$ to be the set of all finite subsets of A with join a, is a coverage on A and that a C-ideal is the same thing as an ideal of A.

Proposition 3.1 *For any site (A, C), meaning a meet-semilattice A equipped with a coverage C, C-$Idl(A)$, ordered by inclusion, is a frame, and it is the free frame on (A, C), i.e., there is a meet-lattice homomorphism*

$$A \xrightarrow{f} C\text{–}Idl(A),$$

which "transforms covers to joins", in the sense that, for every $a \in A$ and $S \in C(a)$, $f(a) = \bigvee_{C\text{–}Idl(A)}\{f(s) \mid s \in S\}$ and f is universal among such maps.

Proof. First we show that C-$Idl(A)$ is a sublocale of the free frame DA where DA denotes the set of all lower subsets of A and let $\eta : A \to DA$ send a to $\downarrow(a)$. It is clear that an arbitrary intersection of C-ideals is a C-ideal, so if we define $j : DA \to DA$ by

$$j(S) = \bigcap\{I \in C - Idl(A) \mid I \supseteq S\}$$

then we have $S \subseteq j(S) = j(j(S))$ for any S, and the image of j is precisely C-$Idl(A)$. So we need only show that j preserves finite intersections.

Let $S, T \in DA$ and write I for $j(S \cap T)$. Consider

$$J = \{a \in A \mid (\forall s \in S)(a \wedge s \in I)\};$$

it is clear that $T \subseteq J$, since $S \cap T \subseteq I$. We will show that J is a C-ideal. Suppose $U \in C(a), U \subseteq J$; then for every $s \in S$ we have $\{u \wedge s \mid u \in U\} \in C(a \wedge s)$ by meet-stability of C, and $\{u \wedge s \mid u \in U\} \subseteq I$ by the definition of J. Since I is a C-ideal, we deduce $a \wedge s \in I$ for all $s \in S$, and hence $a \in J$.

Now if we define

$$K = \{a \in A \mid (\forall t \in J)(a \wedge t \in I)\},$$

then a similar argument shows K is a C-ideal, and $S \subseteq K$ since $S \cap J \subseteq I$. But now we have $j(S) \cap j(T) \subseteq K \cap J \subseteq I = j(S \cap T)$; the reverse inclusion is trivial since j is order-preserving. So j is a nucleus, and $C - Idl(A)$ is a sublocale of DA.

Now, it is clear that for any $S \in C(A)$ we have

$$a \in j(\bigcup\{\downarrow(s) \mid s \in S\})$$

so that the composite map

$$A \xrightarrow{\downarrow(-)} D \xrightarrow{j} (DA)_j = C - Idl(A)$$

transforms covers to joins. It is straightforward to verify that f is universal among such maps. □

Corollary 3.1 *The set $Idl(A)$ of ideals of a distributive lattice A is a frame under the inclusion ordering. Moreover, the assignment $A \mapsto Idl(A)$, is a left adjoint to the forgetful functor $Frm \to DLat$ where Frm is the category of frames and $DLat$ is the category of distributive lattices and homomorphisms.*

Proof. Take C to be the coverage on A defined by finite joins. Then a C-ideal of A is just an ideal in the usual sense; and a meet-semilattice homomorphism $A \to B$ transforms covers in C to joins iff it is a lattice homomorphism. □

Proposition 3.2 *Let B be an internal Boolean algebra in a topos $\mathcal{E}$, and let $Idl(B)$ be the locale of ideals of B. Then $Idl(B)$ is a Stone algebra iff B is internally complete.*

Proof. ([22]) Assume that B is complete. If I is an ideal of B, then the negation of I in $Idl(B)$ is seen to be

$$J = \neg I = \{b \in B \mid \forall i \in I(i \wedge b = 0)\} \rightarrowtail B.$$

Since B is complete (and hence a frame), then $J \rightarrowtail B$ above is closed under arbitrary joins in B and so $J = \downarrow (\bigvee_B J)$. Therefore, $J = \neg I$ is a principal ideal. In particular, every $\neg\neg$-closed ideal in B is principal, hence complemented in $Idl(B)$. Thus $Idl(B)$ is a Stone algebra. Conversely, if $Idl(B)$ is a Stone algebra, then the principal ideal map

$$B \xrightarrow{\downarrow(-)} Idl(B)$$

identifies B with the subframe of $Idl(B)$ consisting of the complemented elements of $Idl(B)$, and by "Stone algebra" these agree with the $\neg\neg$-closed elements of $Idl(B)$, i.e., with $(Idl(B))_{\neg\neg}$. Since the latter is a complete Heyting algebra, so is B. Therefore, B is a complete Boolean algebra. $\square$

In other words, for a Boolean algebra B, the locale $Idl(B)$ (or equivalently the space $SpecB$) is extremally disconnected iff B is complete.

Corollary 3.2 *([22], Corollary 1.2) For any topos $\mathcal{E}$, $\gamma\mathcal{E}$ satisfies (DML).*

Proof. The locale $Idl(\Omega_{\neg\neg})$ is an internal Stone algebra in $\mathcal{E}$ by Proposition 3.2 iff $\mathcal{E}[Idl(\Omega_{\neg\neg})]$ is De Morgan by Theorem 2.2. $\square$

Our next step is to show that the canonical geometric morphism $e : \gamma\mathcal{E} \to \mathcal{E}$ is surjective. But first we need to study certain properties of the lattice $\Omega_{\neg\neg}$.

Given a locale A, we have an order preserving map

$$\rho_A : A \to \Omega,$$

namely the classifying map of $1_A : 1 \rightarrowtail A$ and an order-preserving map

$$\lambda_A : \Omega \xrightarrow{\theta_A} \Omega^A \xrightarrow{\bigvee_A} A$$

where θ_A is the transpose of the classifying map of $(true, 1_A) : 1 \rightarrowtail \Omega \times A$, i.e., $\lambda_A(p) = \bigvee_A\{a \in A \mid (a = 1_A) \wedge p\}$. It can be shown that λ_A preserves finite meets and is internally left adjoint to ρ_A.

Definition 3.2 A locale A is said to be *nontrivial* if $\models \neg(0_A = 1_A)$, i.e., $2 \stackrel{(0,1)}{\to} A$ is a monomorphism or equivalently $\rho_A(0_A)$ =false.

An internal locale satisfying the following conditions is called *consistent.*

Lemma 3.1 *([21], Lemma 2.8) The following conditions on a locale A are equivalent:*

1. $\mathcal{E}[A] \to \mathcal{E}$ *is surjective;*
2. $\rho_A\lambda_A = 1_\Omega$*;*
3. λ_A *is monomorphic;*
4. ρ_A *is epimorphic;*
5. *The following diagram is a pullback*

$$\begin{array}{ccc} 1 & \to & 1 \\ true \downarrow & & \downarrow 1_A \\ \Omega & \xrightarrow[\lambda_A]{} & A \end{array} .$$

Proof. 1 $\Leftrightarrow$ 2. The localic morphism $\mathcal{E}[A] \to \mathcal{E}$ is surjective iff $\rho_A\lambda_A = 1$ is trivial.
2 $\Rightarrow$ 3 and 4. is trivial, and the converse follows from the adjunction $(\lambda_A \dashv \rho_A)$.
5. follows from 3. since the given diagram always commutes.
5 $\Rightarrow$ 2. Since we can deduce that $\rho_A\lambda_A$ classifies *true.* □

Lemma 3.2 *([21], Lemma 4.4) Let A be a distributive lattice in* $\mathcal{E}$*. The following are equivalent:*

1. *A is nontrivial;*

2. *$Idl(A)$ is nontrivial;*

3. *$Idl(A)$ is consistent.*

Proof. $3 \Rightarrow 2.$ is trivial; and $2 \Rightarrow 1.$ since A is (isomorphic to) a sublattice of $Idl(A)$.

$1 \Rightarrow 3.$ Consider the map $I : \Omega \to \Omega^A$ defined by

$$I(p) = \{a \in A \mid (a = 0_A) \vee p\}.$$

So $I(p)$ is an ideal of $Idl(A)$. Then $\models I(p) = A \leftrightarrow p$ since, if $I(p) = A$ then $1_A \in I(p)$ but A is nontrivial, i.e., $\models \neg(0_A = 1_A)$, then $I(p) = A \to p$. Conversely, if we have p then a could be any $a \in A$. Thus, $\rho_{Idl(A)}(I(p)) = p$. Therefore, $\rho_{Idl(A)}$ is epimorphic, as desired. $\square$

Corollary 3.3 *([22], Corollary 1.4) The canonical map $e : \gamma\mathcal{E} \to \mathcal{E}$ is surjective.*

Proof. Since the locale $\Omega_{\neg\neg}$ is nontrivial, it follows that $Idl(\Omega_{\neg\neg})$ is consistent by Lemma 3.2 iff the localic morphism $\mathcal{E}[Idl(\Omega_{\neg\neg})] \to \mathcal{E}$ is surjective by Lemma 3.1. $\square$

We now introduce the notion of a *minimal* locale.

Lemma 3.3 *([21], Lemma 2.9) A locale A is called minimal if the following conditions are equivalent:*

1. *The following diagram is a pullback*

$$\begin{array}{ccc} 1 & \to & 1 \\ 0_A \downarrow & & \downarrow \mathit{false} \\ A & \xrightarrow{\rho_A} & \Omega \end{array}$$

2. *ρ_A commutes with negation.*

3. For any object X of $\mathcal{E}$, the only nontrivial closed sublocale of X^*A in $\mathcal{E}/X$ is the whole of X^*A.

Proof. $1 \Leftrightarrow 2$. In any Heyting algebra A, we have $\models (\neg a = 1_A) \leftrightarrow (a = 0_A)$. But condition 1. says $\models (a = 0_A) \leftrightarrow \neg(a = 1_A)$, and 2. says $\models (\neg a = 1) \leftrightarrow \neg(a = 1)$.
$1 \Leftrightarrow 3$. The closed sublocale of X^*A corresponding to an X-element $X \xrightarrow{a} A$ is nontrivial iff

$$\begin{array}{ccc} X & \rightarrow & 1 \\ a \downarrow & & \downarrow \mathit{false} \\ A & \rightarrow & \Omega \\ & \rho_A & \end{array}$$

commutes. Each of the two conditions says that the unique such a is $X \rightarrow 1 \xrightarrow{0} A$.
□

Lemma 3.4 *([21], Lemma 2.10) If A is a minimal locale, then ρ_A maps $A_{\neg\neg}$ isomorphically onto $\Omega_{\neg\neg}$.*

Proof. By condition 2. of Lemma 3.3, ρ_A certainly maps $A_{\neg\neg}$ into $\Omega_{\neg\neg}$. It does so monomorphically, since for variables a, b of type A, we have
$\models (a = b) \leftrightarrow (\neg a \wedge b = a \wedge \neg b = 0_A)$, and ρ_A preserves finite meets and negation. And it does so epimorphically, since from

$$\rho_A(\neg\neg\lambda_A(p)) \wedge \neg p \leq \rho_A(\neg\neg\lambda_A(p)) \wedge \rho_A\lambda_A(\neg p) = \rho_A(0_A) = \mathit{false}$$

we deduce $\models \rho_A(\neg\neg\lambda_A(p)) = \neg\neg p$. □

By construction of the Gleason cover, $Idl(\Omega_{\neg\neg})$ is a minimal locale by considering the Boolean algebra $\Omega_{\neg\neg}$ of $\neg\neg$-stable truth-values in $\mathcal{E}$. Since the negation of the maximal element $\mathit{true} : 1 \rightarrowtail \Omega_{\neg\neg}$ is $\mathit{false} : 1 \rightarrowtail \Omega_{\neg\neg}$, it is clear that the unique proper ideal of $\Omega_{\neg\neg}$ is the singleton $\{\mathit{false}\}$ [20]; so if $\rho : Idl(\Omega_{\neg\neg}) \rightarrow \Omega$ is the

classifying map of the maximal element $\ulcorner\Omega_{\neg\neg}\urcorner : 1 \rightarrowtail Idl(\Omega_{\neg\neg})$ then the diagram

$$\begin{array}{ccc} & 1 & \rightarrow & 1 & \\ \{false\} & \downarrow & & \downarrow & false \\ & Idl(\Omega_{\neg\neg}) & \xrightarrow{\rho} & \Omega & \end{array}$$

is a pullback. Furthermore, the surjection $e : \gamma\mathcal{E} \to \mathcal{E}$ is minimal in the sense that there is no proper closed subtopos of $\gamma\mathcal{E}$ which maps surjectively to $\mathcal{E}$. By the property that $Idl(\Omega_{\neg\neg})$ is minimal, the following diagram commutes

$$\begin{array}{ccc} (Idl(\Omega_{\neg\neg}))_{\neg\neg} & \xrightarrow{\sim} & \Omega_{\neg\neg} \\ \downarrow & & \downarrow \\ Idl(\Omega_{\neg\neg}) & \xrightarrow{\rho} & \Omega \end{array} .$$

Remark 3.1 Recall Theorem 2.1 that the projective objects in the category of compact Hausdorff spaces are precisely the extremally disconnected spaces. Gleason also showed that for every compact Hausdorff space X, there is a surjection $e : \gamma X \to X$ where γX is projective, i.e., γX is extremally disconnected. This can be seen by considering the Stone space $\gamma X = Spec((\mathcal{O}(X))_{\neg\neg})$ where the set $(\mathcal{O}(X))_{\neg\neg}$ of all regular elements of $\mathcal{O}(X)$, is a complete Boolean algebra. Furthermore, the surjection e is "minimal" in the sense that there is no proper closed subspace of γX which maps surjectively to X. The extremal disconnectedness of γX and the minimality of e characterizes γX up to homeomorphism over X. Johnstone's construction of the Gleason cover is the topos-theoretic analogue of the projective cover constructed in this remark.

The (2-) category $LTop/\mathcal{E}$ of localic $\mathcal{E}$-toposes is equivalent to the (2-) category $Loc(\mathcal{E})$ of internal locales in $\mathcal{E}$ (see [21]). Thus the above diagram of internal locales in $\mathcal{E}$ can be translated into a diagram of localic $\mathcal{E}$-toposes, and it can be deduced that this diagram

$$\begin{array}{ccc} Sh_{\neg\neg}(\gamma\mathcal{E}) & \xrightarrow{\sim} & Sh_{\neg\neg}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \gamma\mathcal{E} & \xrightarrow{e} & \mathcal{E} \end{array}$$

commutes in *Top* where the vertical arrows are the canonical inclusions.

We have already seen that the Gleason cover $\gamma\mathcal{E}$ may be constructed for any topos $\mathcal{E}$. Suppose now that we are given the following two cases, i.e., the topos $\mathcal{E}$ is De Morgan and the topos $\mathcal{E}$ is Boolean, respectively.

Corollary 3.4 *([22], Corollary 1.5)* $e : \gamma\mathcal{E} \to \mathcal{E}$ *is an equivalence iff* $\mathcal{E}$ *satisfies (DML).*

Proof. If $e : \gamma\mathcal{E} \to \mathcal{E}$ is an equivalence then $\mathcal{E}$ satisfies (DML) follows from Corollary 3.2. Conversely, $\mathcal{E}$ satisfies (DML) is equivalent to $\Omega_{\neg\neg} \cong 2$, and so $Idl(\Omega_{\neg\neg}) \cong Idl(2)$. By condition 3. of Lemma 3.1, $Idl(2) \cong \Omega$. But Ω is the terminal object in the category of internal locales in $\mathcal{E}$ and so $\mathcal{E}[\Omega] \simeq \mathcal{E}$. □

Corollary 3.5 *([22], Corollary 1.7)* $\gamma\mathcal{E}$ *is Boolean iff* $\mathcal{E}$ *is.*

Proof. If $\mathcal{E}$ is Boolean, then it satisfies (DML) and so $\gamma\mathcal{E}$ is Boolean by Corollary 3.4. Conversely, if $\gamma\mathcal{E}$ is Boolean, then the inclusion $Sh_{\neg\neg}(\mathcal{E}) \hookrightarrow \mathcal{E}$ is equivalent to the composite $Sh_{\neg\neg}(\gamma\mathcal{E}) \xrightarrow{\sim} \gamma\mathcal{E} \xrightarrow{e} \mathcal{E}$. Therefore $Sh_{\neg\neg}(\mathcal{E}) \hookrightarrow \mathcal{E}$ is surjective and so $\mathcal{E}$ is Boolean. □

We now look into the Ore condition and its connection to De Morgan's law and to the notion of relatively De Morgan. In order to define the latter, we will need to define the notion of relatively complemented elements of a frame.

We say that a category $\mathcal{C}$ satisfies the *Ore condition* if every diagram of two arrows with common codomain embeds into a common square.

Theorem 3.1 *([19], Proposition 1.1) Let* $\mathcal{C}$ *be a small category and let* $\hat{\mathcal{C}} = \mathcal{S}^{\mathcal{C}^{op}}$ *be the topos of presheaves where the base topos* $\mathcal{S}$ *is Boolean. The topos* $\hat{\mathcal{C}}$ *is De Morgan iff* $\mathcal{C}$ *satisfies the Ore condition.*

Proof. In $\hat{\mathcal{C}}$, Ω is the presheaf $C \mapsto \{$ sieves on $C\}$, and $false$ is the global element which picks out the empty sieve on each object. Condition 3. of Theorem 2.2 is thus equivalent to saying that the nonempty sieves form a subpresheaf of Ω, i.e., that

any pullback of a nonempty sieve is nonempty. Now if f and g are two morphisms of $\mathcal{C}$ with the same codomain, the pullback along g of the sieve generated by f is nonempty iff there is a morphism of $\mathcal{C}$ factoring through both f and g, i.e., iff there is a commutative square

$$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \downarrow & & \downarrow f \\ \cdot & \rightarrow & \cdot \\ & g & \end{array}$$

The converse is easy. $\square$

Consider now the case that the base topos $\mathcal{S}$ of the topos $\hat{\mathcal{C}}$ of $\mathcal{S}$-valued presheaves on $\mathcal{C}$ is arbitrary where $\mathcal{C}$ is a category object in $\mathcal{S}$. We will recall the proof, that in this case, $\mathcal{C}$ satisfies the Ore condition iff the presheaf topos $\hat{\mathcal{C}}$ is relatively De Morgan.

We need to define what is meant by a relatively De Morgan frame and thus a relatively De Morgan topos. We begin by recalling the notion of relatively complemented elements of a frame, i.e., the clopen and regular elements in a frame.

Let A be a frame. Given the map $\tau : D \to A$ where D is an arbitrary set and τ is often taken to be the inclusion of a subset, we consider the following two subsets of A,

$$Clp(A) = \{a \in A \mid 1_A = \bigvee_{\lambda \in D} (a \leftrightarrow \lambda)\} \tag{3.1}$$

$$Reg(A) = \{a \in A \mid a = \bigwedge_{\lambda \in D} (a \to \lambda) \to \lambda\}. \tag{3.2}$$

Remark 3.2

1. Note that $1 = \bigvee_\lambda (a \leftrightarrow \lambda)$ where λ ranges over the set Ω of truth values is a generalization of the law of the excluded middle $1 = a \vee \neg a$. This generalization is due to Jibladze.

2. For a fixed $d \in A$, $(- \to d) \to d$ is a nucleus on the frame A. Thus for $d = 0_A$, the set of fixpoints for this particular nucleus are the elements $a \in A$ such that $a = \neg\neg a$ (since $\neg a = a \to 0$ in any Heyting algebra) and so we get the usual notion of regular elements of a frame. For $\tau : D \to A$,

$$\bigwedge_{\lambda \in D} (- \to \lambda) \to \lambda \tag{3.3}$$

is a nucleus L_D and the set of fixpoints for L_D is by definition $Reg(A)$. It is easy to see that $a = \bigwedge_{\lambda \in D}(a \to \lambda) \to \lambda$ implies $a = \neg\neg a$.

3. If $A = \mathcal{O}(X)$, the frame of open subsets of a topological space X, and $D = \{0_A, 1_A\}$ then $Clp(A)$ and $Reg(A)$ are the *clopen* and the *regular open* subsets, respectively.

Proposition 3.3 *([30], Proposition 1.1) For any $\tau : D \to A$, $Clp(A) \subseteq Reg(A)$.*

Proof. Let $a \in Clp(A)$, so $1_A = \bigvee_\lambda (a \leftrightarrow \lambda)$ where λ ranges over D. We should prove $\bigwedge_\lambda (a \to \lambda) \to \lambda \le a$ (the other inequality always holds). It suffices to see that for any b with $b \le (a \to \lambda) \to \lambda$ for all $\lambda \in D$ we have $b \le a$. The assumption on b may be reformulated

$$b \wedge (a \to \lambda) \le \lambda \ \forall \lambda. \tag{3.4}$$

Since $1_A = \bigvee_\lambda (a \leftrightarrow \lambda)$, b is covered by the family $\{b \wedge (a \to \lambda) \mid \lambda \in D\}$, so it suffices to prove that $b \wedge (a \to \lambda) \le a$. But

$$b \wedge (a \leftrightarrow \lambda) = b \wedge (a \to \lambda) \wedge (a \leftrightarrow \lambda) \le \lambda \wedge (a \leftrightarrow \lambda) \le \lambda \wedge (\lambda \to a) \le a,$$

using 3.4 for the first inequality. □

Proposition 3.4 *([30], Proposition 1.2) For any $\tau : D \to A$, the following conditions are equivalent:*

1. *$Clp(A) \subseteq A$ has a left exact left adjoint (which is necessarily given by 3.3);*

2. *$Clp(A) = Reg(A)$;*

3. For all $a \in A$, $1_A = \bigvee_\lambda[\bigwedge_\delta(a \to \delta) \to \delta] \leftrightarrow \lambda$ where λ and δ range over D.

Proof. $1 \Rightarrow 2$. Assume 1. Combining the assumed left adjoint with the inclusion $Clp(A) \subseteq A$, we get a nucleus P on A with $Clp(A)$ as its fixpoint set. Since $Clp(A) \subseteq Reg(A)$, by Proposition 3.3, we have the opposite inequality for the corresponding nuclei, so $L_D \leq P$. To see $L_D = P$ (which implies 2.), it thus suffices to see that $P \leq L_D$. Since L_D is the largest nucleus fixing (the image under τ of) D, it suffices to see that P fixes that image, i.e., to see that $\tau(\delta) \in Clp(A)$ for every $\delta \in D$. But (omitting τ from notation),

$$\bigvee_\lambda (\delta \leftrightarrow \lambda) \geq \delta \leftrightarrow \delta = 1_A,$$

whence δ belongs to Clp.

$2 \Rightarrow 1$. Since $Reg(A) \subseteq A$ has a left exact left adjoint (given by the nucleus L_D).

$2 \Rightarrow 3$. Since $\bigwedge_\delta(a \to \delta) \to \delta \in Reg(A)$, it belongs also to $Clp(A)$.

$3 \Rightarrow 2$. Every element of the form $\bigwedge_\delta(a \to \delta) \to \delta$ is in $Clp(A)$; but every element of $Reg(A)$ is of this form, so $Reg(A) \subseteq Clp(A)$, hence by Proposition 3.3, $Reg(A) = Clp(A)$. □

Definition 3.3 *([30])* Let A be a frame in a topos $\mathcal{S}$ and $\tau : \Omega_\mathcal{S} \to A$ the unique frame map. Then A is *relatively De Morgan* if $Reg(A) = Clp(A)$ (with λ ranging over $\Omega_\mathcal{S}$ in 3.1 and 3.2).

Let $\beta : \mathcal{E} \to \mathcal{S}$ be a geometric morphism and $\tau : \beta^*\Omega_\mathcal{S} \to \Omega_\mathcal{E}$ as the canonical comparison map which classifies the monic $\beta^*(true) : \beta^*1 \rightarrowtail \beta^*\Omega_\mathcal{S}$. Now Definition 3.3 can be defined for a geometric morphism with $\tau : \beta^*\Omega_\mathcal{S} \to \Omega_\mathcal{E}$ as $\tau : \Omega_\mathcal{S} \to A$.

Definition 3.4 *([30])* Let $\beta : \mathcal{E} \to \mathcal{S}$ be a geometric morphism. We let $Clp(\Omega_\mathcal{E}) \subseteq \Omega_\mathcal{E}$ and $Reg(\Omega_\mathcal{E}) \subseteq \Omega_\mathcal{E}$ refer to the comparison map $\tau : \beta^*\Omega_\mathcal{S} \to \Omega_\mathcal{E}$. Then $\mathcal{E}$ is *relatively De Morgan over $\mathcal{S}$* if $Reg(\Omega_\mathcal{E}) = Clp(\Omega_\mathcal{E})$ (with λ ranging over $\beta^*\Omega_\mathcal{S}$ in 3.1 and 3.2).

Proposition 3.5 *([30], Proposition 2.1) The subobject $Clp(\Omega_{\mathcal{E}}) \subseteq \Omega_{\mathcal{E}}$ equals the image of $\beta^*\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$. More generally, for any $\tau : D \to \Omega_{\mathcal{E}}$ in a topos $\mathcal{E}$, the extension of the formula (with free variable ranging over $\Omega_{\mathcal{E}}$),*

$$true = \bigvee_{\lambda \in D} (a \leftrightarrow \tau(\lambda)) \tag{3.5}$$

equals the image of τ.

Proof. The element $true \in \Omega_{\mathcal{E}}$ is 'inaccessible by sup' in the sense that $true = sup\ U$ (for $U \subseteq \Omega_{\mathcal{E}}$) implies $true \in U$; for any subset $U \subseteq \Omega_{\mathcal{E}}$, $sup\ U$ equals the truth value of the statement $true \in U$. Thus the formula 3.5 is equivalent to the formula

$$\exists \lambda \in D \; true = (a \leftrightarrow \tau(\lambda))$$

and then again to

$$\exists \lambda \in D \; a = \tau(\lambda),$$

whose extension clearly is just the image of τ. □

Let us now look into the case that $\mathcal{C}$ satisfies the Ore condition where $\mathcal{C}$ is a category object in an arbitrary topos $\mathcal{S}$. The geometric morphism $\beta : \hat{\mathcal{C}} \to \mathcal{S}$ is open by ([23], Proposition 2.6) where $\hat{\mathcal{C}}$ is the topos of $\mathcal{S}$-valued presheaves on $\mathcal{C}$. Recall that an open geometric morphism $\beta : \mathcal{E} \to \mathcal{S}$ implies that $\tau : \beta^*\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$ is a monomorphism and therefore τ may be omitted from 3.1 and 3.2. Thus $\tau : \beta^*\Omega_{\mathcal{S}} \to \Omega_{\hat{\mathcal{C}}}$ is monic.

In general, an open geometric morphism $\beta : \mathcal{E} \to \mathcal{S}$ is relatively De Morgan iff the canonical comparison map $\tau : \beta^*\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$ has a left adjoint. This follows from the fact that if $\mathcal{E}$ is relatively De Morgan then $Reg(\Omega_{\mathcal{E}}) = Clp(\Omega_{\mathcal{E}})$ which is equivalent to $Clp(\Omega_{\mathcal{E}}) \subseteq \Omega_{\mathcal{E}}$ has a left adjoint (by Proposition 3.4) and thus τ has a left adjoint (by Proposition 3.5).

Theorem 3.2 *([30], Theorem 3.1) The open geometric morphism $\beta : \hat{\mathcal{C}} \to \mathcal{S}$ is relatively De Morgan iff $\mathcal{C}$ satisfies the Ore condition.*

Proof. We may argue as if $\mathcal{S}$ were *Sets*, provided the argument is positive and constructive. Let $\mathcal{C}$ be a category in $\mathcal{S}$. We describe the canonical $\tau : \beta^*\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$ where $\mathcal{E} = \hat{\mathcal{C}}$. For $C \in \mathcal{C}$ an object, τ_C is the unique frame map

$$\Omega_{\mathcal{S}} = \beta^*\Omega_{\mathcal{S}}(C) \to P(y(C)),$$

where $P(y(C))$ denotes the set (frame) of subfunctors of the representable functor $y(C) = hom_{\mathcal{C}}(-, C)$. We describe a left adjoint left inverse σ for τ_C, namely given by

$$\sigma(R) = \|R \text{ is inhabited}\| \tag{3.6}$$

for any subfunctor (sieve) $R \subseteq y(C)$, where $\|\ldots\|$ denotes 'truth value of ...'. Clearly $\sigma(\tau_C(\lambda)) = \lambda$; and if $f \in R(D)$, R is inhabited, so $f \in \tau_C(\|R \text{ is inhabited}\|)$, so $R \subseteq \tau_C(\sigma(R))$.

Assume that $\mathcal{C}$ satisfies the Ore condition. To prove that $\hat{\mathcal{C}} \to \mathcal{S}$ is De Morgan is equivalent to proving that $\tau : \beta^*\Omega_{\mathcal{S}} \to \Omega_{\hat{\mathcal{C}}}$ has a left adjoint. We have already *pointwise* a left adjoint, given by the description 3.6. It suffices to see that σ_C is natural in C, i.e., to prove that for each $f : D \to C$ the diagram

$$\begin{array}{cccccc} & P(y(C)) & \xrightarrow{\zeta} & P(y(D)) & \\ \sigma_C & \downarrow & & \downarrow & \sigma_D \\ & \Omega_{\mathcal{S}} & \xrightarrow[id]{} & \Omega_{\mathcal{S}} & \end{array}$$

commutes, where the top map ζ to a sieve $R \subseteq y(C)$ associates the set of arrows g with codomain D and with $f \circ g \in R$. Now let $R \in P(y(C))$ be a sieve on C. Assume $\sigma_C(R)$ is *true*, so R is inhabited, say witnessed by $(h : C' \to C) \in R$. Completing the square

$$\begin{array}{cccc} & D & \xrightarrow{f} & C & \\ h' & \uparrow & & \uparrow & h \\ & D' & \to & C' & \end{array}$$

we get that $\zeta(R)$ is inhabited (witnessed by h'), so $\sigma_D(\zeta(R))$ is true. This implies that $\sigma_C(R) \leq \sigma_D(\zeta(R))$. The other inequality $\sigma_D(\zeta(R)) \leq \sigma_C(R)$ is trivial: if

$\sigma_D(\zeta(R))$ is true, $\zeta(R)$ is inhabited which implies that R is inhabited, so $\sigma_C(R)$ is true.

Conversely, if $\hat{\mathcal{C}} \to \mathcal{S}$ is relatively De Morgan, $\tau : \beta^*\Omega_{\mathcal{S}} \to \Omega_{\hat{\mathcal{C}}}$ has a left adjoint, hence the pointwise left adjoint σ_C of τ_C is natural in C. Contemplating the naturality square above for $f : D \to C$ and applying it to the principal sieve generated by h, we get that the sieve of those h', which fit in the Ore square above, is inhabited. Thus $\mathcal{C}$ satisfies the Ore condition. □

Remark 3.3

1. The notion of relatively Boolean can also be defined in terms of relatively complemented elements. For a frame A in $\mathcal{S}$, A is *relatively Boolean* if $A = Clp(A)$. Moreover, for a geometric morphism $\beta : \mathcal{E} \to \mathcal{S}$ with the comparison map $\tau : \beta^*\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$, $\mathcal{E}$ is *relatively Boolean over* $\mathcal{S}$ if $\Omega_{\mathcal{E}} = Clp(\Omega_{\mathcal{E}})$ or equivalently the canonical map $\beta^*\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$ is an isomorphism.

2. If $\mathcal{S}$ is a Boolean topos, i.e., $\Omega_{\mathcal{S}} = 1 + 1$ (so also $\beta^*\Omega_{\mathcal{S}} = 1 + 1$), the notion of relatively Boolean and relatively De Morgan can be defined as the usual notion of Boolean and De Morgan.

3. It is clear that if $\mathcal{S}$ is a Boolean topos in Theorem 3.2 then we get exactly the conditions in Theorem 3.1.

We now look into conditions equivalent to Booleaness, by Bunge and Funk ([5], Theorem 3.4).

We need to recall some new concepts. Let $\mathcal{O}(X)$ denote the frame of opens for a locale X. Let $Sub(X)$ denote the coframe of sublocales of a locale X and let $Cl(X)$ denote the coframe of closed sublocales of X. A sublocale $B \in Sub(X)$ is said to be *weakly closed* if every strongly dense inclusion $B \hookrightarrow B'$, $B' \in Sub(X)$, is an isomorphism. Recall that a morphism of locales $Y \xrightarrow{f} X$ is said to be *strongly dense* if f is dense under pullbacks along every closed sublocale of the terminal locale 1 [24]. Let $W(X)$ denote the poset of weakly closed sublocales of X. $W(X)$ is a

subcoframe of $Sub(X)$ [17], and it contains $Cl(X)$, i.e., $\mathcal{O}(X)^{op}$, as a subcoframe. It was shown that $W(X) \simeq hom(\mathcal{O}(X), Sub(1))$ and that $W_o(X) \simeq hom(\mathcal{O}(X), \Omega)$ where $W_o(X)$ denotes the poset of weakly closed sublocales of X with *open domain* ([5], Theorem 3.2 and Theorem 3.3, respectively).

Theorem 3.3 *([5], Theorem 3.4) Let $\mathcal{S}$ denote an elementary topos. Then the following are equivalent:*

1. *$\mathcal{S}$ is Boolean;*
2. *For all locales X in $\mathcal{S}$, $Cl(X) = W(X)$;*
3. *For all objects T in $\mathcal{S}$, $Cl(T) = W(T)$;*
4. *For all locales X in $\mathcal{S}$, $W_o(X) = W(X)$;*
5. *$W_o(1) = W(1)$.*

Proof. cf. [5]. □

The next theorem, due to Bunge and Funk, identifies the frame of nuclei on a locale X with $W(X)^{op}$, i.e., the frame of weakly closed nuclei on $\mathcal{O}(X)$. It is a new proof of Jibladze's theorem [17] (with no appeal to Wigner [44]).

In [34], it was shown that nuclei j on the frame $\mathcal{O}(X)$, for an arbitrary locale X, are characterized by the identity

$$\forall U, V \in \mathcal{O}(X), (U \to jV) = (jU \to jV).$$

Restricting to Ω, we obtain,

$$\forall \omega, \omega' \in \Omega, (v^*\omega \to k\omega') = (k\omega \to k\omega'), \tag{3.7}$$

where $v^* : \Omega \to \mathcal{O}(X)$ (since $v : X \to 1$) and $k : \Omega \to \mathcal{O}(X)$ is the composite jv^*. Let N_X denote the collection, ordered pointwise, of all functions k that satisfy 3.7. Such a function is called an *Ω-nucleus on $\mathcal{O}(X)$* [17].

Theorem 3.4 *([5], Theorem 4.1)(Jibladze) For any locale X (with structural morphism v), the map*

$$W(X)^{op} \to N_X; j \mapsto jv^*,$$

is an isomorphism of posets. In particular, N_X is a frame.

Proof. cf. [5]. □

We end this chapter with the following result. In [6], Bunge and Funk use the notion of a relatively complemented element of a frame to prove that any spread (defined shortly) for a geometric morphism over a base topos $\mathcal{S}$ is zero-dimensional over $\mathcal{S}$ in the (2-) category of elementary toposes, bounded geometric morphisms and natural transformations (between inverse image functors of geometric morphisms) and denote this (2-) category by Top. Recall that in topology, a topological space X is said to be *zero-dimensional* if the clopen subsets of X form a base for the topology. Consider a diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\phi} & \mathcal{F} \\ u & \searrow\swarrow & v \\ & \mathcal{S} & \end{array}$$

in Top where ϕ is localic, i.e., that $\mathcal{E}$ is equivalent to the topos of sheaves over $\mathcal{F}$ on the frame $\phi_*\Omega_{\mathcal{E}}$. Consider the composite morphism

$$v^*\Omega_{\mathcal{S}} \to \Omega_{\mathcal{F}} \to \phi_*\Omega_{\mathcal{E}},$$

where the second morphism is the unique frame map. We now let the notion of Clp refer to this morphism; we have the object of $\mathcal{S}$-complemented elements of $\phi_*\Omega_{\mathcal{E}}$:

$$Clp_{\mathcal{S}}(\phi_*\Omega_{\mathcal{E}}) = \{U \in \phi_*\Omega_{\mathcal{E}} \mid 1_{\phi_*\Omega_{\mathcal{E}}} = \bigvee_{\omega \in v^*\Omega_{\mathcal{S}}} (U \leftrightarrow \omega)\}.$$

We define the terminology mentioned so far.

Definition 3.5

1. A geometric morphism $\mathcal{K} \xrightarrow{\psi} \mathcal{K}'$ is said to be *bounded* ([18],[40]) if there is a $K \in \mathcal{K}'$ and a morphism $D \to \psi^* K$ in $\mathcal{K}$, said to be a *generating family* (for $\mathcal{K}$ over $\mathcal{K}'$), such that every $X \in \mathcal{K}$, there is a morphism $I \xrightarrow{a} K$ and an epimorphism $A \twoheadrightarrow X$, where the following diagram is a pullback

$$\begin{array}{ccc} A & \to & D \\ \downarrow & & \downarrow \\ \psi^* I & \underset{\psi^* a}{\to} & \psi^* K \end{array} .$$

2. Let $\mathcal{E} \xrightarrow{u} \mathcal{S}$ denote an arbitrary geometric morphism in *Top*. A morphism $E \xrightarrow{b} E'$ in $\mathcal{E}$ is called $\mathcal{S}$-*definable* [2], or just *definable*, if it arises as a pullback

$$\begin{array}{ccc} E & \xrightarrow{b} & E' \\ \downarrow & & \downarrow \\ u^* I & \underset{u^* a}{\to} & u^* J \end{array}$$

for some morphism $I \xrightarrow{a} J$ in $\mathcal{S}$.

3. In a diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\phi} & \mathcal{F} \\ u & \searrow\swarrow & v \\ & \mathcal{S} & \end{array}$$

in *Top*, ϕ is said to be a *spread over* $\mathcal{S}$ if there is a generating family $E \to \phi^* F$ for $\mathcal{E}$ over $\mathcal{F}$ which is definable.

4. If M is a complete join-semilattice in a topos $\mathcal{F}$, then a morphism $K \to M$ *sup-generates* M when $\Omega^K \to \Omega^M \xrightarrow{\bigvee} M$ is an epimorphism.

5. A localic geometric morphism $\mathcal{E} \xrightarrow{\phi} \mathcal{F}$ over $\mathcal{S}$ will be said to be *zero-dimensional over* $\mathcal{S}$ if the frame $\phi_* \Omega_{\mathcal{E}}$ is sup-generated by its sublattice $Clp_{\mathcal{S}}(\phi_* \Omega_{\mathcal{E}})$.

Theorem 3.5 *([6], Theorem 1.15). Any spread over $\mathcal{S}$ is zero-dimensional over $\mathcal{S}$.*

The proof uses two results. One is that a geometric morphism $\mathcal{E} \xrightarrow{\phi} \mathcal{F}$ over $\mathcal{S}$ is a spread iff ϕ is localic and the morphism

$$\phi_*\tau : \phi_* u^*\Omega_{\mathcal{S}} \to \phi_*\Omega_{\mathcal{E}}$$

sup-generates the frame $\phi_*\Omega_{\mathcal{E}}$ where $\tau : u^*\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$ classifies the monic $u^*(true) : u^*1 \to u^*\Omega_{\mathcal{S}}$ (see [6], Proposition 1.4). The second is that for any localic geometric morphism $\mathcal{E} \xrightarrow{\phi} \mathcal{F}$ over $\mathcal{S}$, the morphism $\phi_*\tau : \phi_*u^*\Omega_{\mathcal{S}} \to \phi_*\Omega_{\mathcal{E}}$ factors through $Clp_{\mathcal{S}}(\phi_*\Omega_{\mathcal{E}})$ (see [6], Proposition 1.14).

It was remarked in [6] that for any locale X and any morphism $I \xrightarrow{m} \mathcal{O}(X)$, the object of clopens $Clp(X)$ for m is contained in the object of weakly closed elements of $\mathcal{O}(X)$ for m, due to G.E. Reyes.

Chapter 4

Application I: The Hahn-Banach theorem

In this chapter, the following two results, in relation to the real numbers in a topos, will be revealed. The first result is that in a De Morgan topos $\mathcal{E}$ the object of Dedekind reals in $\mathcal{E}$, denoted $R_{\mathcal{E}}$, coincides with its order-completion the MacNeille reals, denoted $^*R_{\mathcal{E}}$ [19]. The second result is that the object $^*R_{\mathcal{E}}$ of MacNeille reals in $\mathcal{E}$ is isomorphic to $e_*(R_{\gamma\mathcal{E}})$, where $e : \gamma\mathcal{E} \to \mathcal{E}$ is the canonical geometric morphism and $\gamma\mathcal{E}$ is the Gleason cover of a topos $\mathcal{E}$ [22]. We will then use these two results to show that the Hahn-Banach theorem may serve as an example of the occurrence of extremal disconnectedness in functional analysis.

We dedicate the first part of this chapter to review the real numbers in a topos $\mathcal{E}$.

The Dedekind reals $R_{\mathcal{E}}$ of the topos $\mathcal{E}$ are constructed from the rational numbers $Q_{\mathcal{E}}$ of $\mathcal{E}$ by considering the subset $\Omega^{Q_{\mathcal{E}}} \times \Omega^{Q_{\mathcal{E}}}$ consisting of all pairs $x = (L, U)$ satisfying the following conditions:

1. $\exists p \in Q_{\mathcal{E}} \;\; p \in L$ and $\exists q \in Q_{\mathcal{E}} \;\; q \in U$ (L and U are inhabited);

2. $p \in L \leftrightarrow \exists p' > p \;\; p' \in L$ (L is an open lower section)
   $q \in U \leftrightarrow \exists q' < q \;\; q' \in U$ (U is an open upper section);

3. $p \in L \wedge q \in U \to p < q$ (L and U are disjoint);

4. $p < q \to p \in L \vee p \in U$ (L and U are adjacent).

The MacNeille reals, $^*R_\mathcal{E}$ are constructed explicitly from the rationals $Q_\mathcal{E}$ by considering the subset $\Omega^{Q_\mathcal{E}} \times \Omega^{Q_\mathcal{E}}$ consisting of all points $x = (L, U)$ satisfying the following conditions:

1. $\exists p \in Q_\mathcal{E} \;\; p \in L$ and $\exists q \in Q_\mathcal{E} \;\; q \in U$;

2. $p \in L \leftrightarrow \exists p' > p \;\; \forall q \in U \;\; q > p'$
   $q \in U \leftrightarrow \exists q' < q \;\; \forall p \in L \;\; p < q'$.

In general, for any poset A, the MacNeille completion of A can be constructed and is denoted $M(A)$. The construction of $M(A)$ is as follows: if A is a poset and $S \subseteq A$, we will write $u(S)$ for the set of upper bounds for S and $l(S)$ for the set of lower bounds for S. We define a cut in A to be a pair of subsets (L, U) such that $L = l(U)$ and $U = u(L)$. The set of all cuts in A, ordered by $(L_1, U_1) \leq (L_2, U_2)$ iff $L_1 \subseteq L_2$ (or equivalently $U_1 \supseteq U_2$) is denoted by $M(A)$. For every element $a \in A$, the pair $(\downarrow(a), \uparrow(a))$ is a cut in A, which we denote by $m(a)$; a cut (L, U) is of this form iff $L \cap U$ is nonempty. We are interested in the following proof.

Theorem 4.1 *For any poset A, $M(A)$ is a complete lattice and the embedding $m : A \to M(A)$ preserves all joins and meets which exist in A.*

Proof. Let S be a subset of $M(A)$. Consider the set

$$L_0 = \bigcap\{L \mid (L, U) \in S\}.$$

Since $L_0 \subseteq L$ for every $(L, U) \in S$, we have $u(L_0) \supseteq U$ for every $(L, U) \in S$, and hence $l(u(L_0)) \subseteq \bigcap\{L \mid (L, U) \in S\} = L_0$. So $(L_0, u(L_0))$ is a cut in A and is clearly the greatest lower bound of S. To construct the joins in $M(A)$, consider the set

$$U_0 = \bigcup\{U \mid (L, U) \in S\}.$$

Since $U_0 \supseteq U$ for every $(L, U) \in S$, we have $l(U_0) \subseteq L$ for every $(L, U) \in S$, and hence $u(l(U_0)) \supseteq \bigcup\{U \mid (L, U) \in S\} = U_0$. So $(l(U_0), U_0)$ is a cut in A and

it is clearly the least upper bound of S. Now if $a = \bigwedge S$ in A, then we have $\downarrow(a) = \bigcap\{\downarrow(s) \mid s \in S\}$; so m preserves all such meets. Similarly if $a = \bigvee S$ in A, then we have $\uparrow(a) = \bigcup\{\uparrow(s) \mid s \in S\}$; so m preserves all such joins. $\square$

Thus the lattice of MacNeille reals, ${}^*R_{\mathcal{E}}$, is the order-completion of the Dedekind reals, $R_{\mathcal{E}}$ in the topos $\mathcal{E}$. It may also be seen that any Dedekind real is again a MacNeille real since there is an embedding

$$m : R_{\mathcal{E}} \to {}^*R_{\mathcal{E}}$$

where m preserves all meets and all joins which exist in $R_{\mathcal{E}}$.

In addition, if $\mathcal{E}$ is a De Morgan topos then we have the following theorem.

Theorem 4.2 *([19], Prop 1.3) In a De Morgan topos $\mathcal{E}$ with a natural number object, the object $R_{\mathcal{E}}$ of Dedekind real numbers in $\mathcal{E}$ coincides with the object ${}^*R_{\mathcal{E}}$ of MacNeille real numbers.*

Proof. By Theorem 4.1, every Dedekind real is a MacNeille real in $\mathcal{E}$. To prove the converse. Let (L, U) be a MacNeille real. Since L is a lower section, we have that

$$\begin{aligned} q'' \in L \wedge q' \leq q'' &\rightarrow q' \in L \\ \neg(q' \in L) \wedge q'' \in L &\rightarrow \neg(q' \leq q'') \\ &\rightarrow q' > q''. \end{aligned}$$

Since Q satisfies trichotomy,

$$\neg(q' \in L) \rightarrow \forall q'' \in L(q'' < q').$$

Hence from condition 2. in which the MacNeille reals satisfy, we deduce

$$q' < q \wedge \neg(q' \in L) \rightarrow q \in U.$$

Similarly, since U is an upper section, we have that

$$\begin{aligned} q'' \in U \wedge q'' \leq q &\rightarrow q \in U \\ \neg(q \in U) \wedge q'' \in U &\rightarrow \neg(q'' \leq q) \\ &\rightarrow q'' > q \\ \neg(q \in U) &\rightarrow \forall q'' \in U(q < q''). \end{aligned}$$

Again from condition 2. in which the MacNeille reals satisfy, we deduce

$$q' < q \wedge \neg(q \in U) \rightarrow q' \in L.$$

Now suppose that $q' < q$ and define $q'' = \frac{1}{2}(q + q')$. Then since $\vdash \neg\phi \vee \neg\neg\phi$ is satisfied in $\mathcal{E}$, we have

$$\neg(q'' \in L) \vee \neg\neg(q'' \in L).$$

But from the first half of this disjunction, we deduce $(q \in U)$ and from the second half, we deduce $\neg(q'' \in U)$. Since L and U are disjoint, we get $(q' \in L)$. Then (L, U) is a Dedekind real as required. By applying this argument to generalized elements of ${}^*R_{\mathcal{E}}$, we deduce that the inclusion $m : R_{\mathcal{E}} \rightarrow {}^*R_{\mathcal{E}}$ is an isomorphism. □

Notice that since the Gleason cover $\gamma\mathcal{E}$ of the topos $\mathcal{E}$ satisfies (DML), the object $R_{\gamma\mathcal{E}}$ of Dedekind real numbers in $\gamma\mathcal{E}$ coincides with the object ${}^*R_{\gamma\mathcal{E}}$ of MacNeille real numbers in $\gamma\mathcal{E}$.

In [22], it was stated that the Dedekind real numbers in $\gamma\mathcal{E}$ correspond to the classifying topos for the propositional theory of real numbers (see Chapter 1), or equivalently to the locale morphism

$$Idl(\Omega_{\neg\neg}) \rightarrow L(R)$$

in $\mathcal{E}$, where $L(R)$ is the locale of formal real numbers in $\mathcal{E}$ [12].

The locale $L(R)$ is generated by the formal rational intervals (q, r), $(q \in Q \sqcup \{-\infty\}, r \in Q \sqcup \{\infty\})$, subject to the relations:

1. $(-\infty, \infty) = 1$;
2. $(q, r) = 0$ if $q \geq r$;
3. $(q_1, r_1) \wedge (q_2, r_2) = (max(q_1, q_2), min(r_1, r_2))$;
4. $(q_1, r_1) \vee (q_2, r_2) = (q_1, r_2)$ if $q_1 \leq q_2 < r_1 \leq r_2$;
5. $(q, r) = \bigvee\{(q', r') \mid q < q' < r' < r\}$.

Proposition 4.1 *([22], Proposition 2.3) There is a bijection between Dedekind real numbers in $\gamma\mathcal{E}$ and MacNeille reals in $\mathcal{E}$.*

Proof. Given a Dedekind real number x in $\gamma\mathcal{E}$, regard it as a locale map $Idl(\Omega_{\neg\neg}) \to L(R)$ in $\mathcal{E}$. By the definition of $L(R)$, such a map is determined by the effect of its inverse image x^* on the rational intervals (q, r); and x^* must preserve the relations given above. Define

$$L = \{q \in Q \mid x^*(q, \infty) = \Omega_{\neg\neg}\}, U = \{r \in Q \mid x^*(-\infty, r) = \Omega_{\neg\neg}\};$$

we shall show that (L, U) is a MacNeille real in $\mathcal{E}$.

From relations 1 and 5. above, and compactness of $Idl(\Omega_{\neg\neg})$, we deduce

$$\exists q, r \in Q(x^*(q, r) = \Omega_{\neg\neg}),$$

whence

$$\exists q \in Q(q \in L) \wedge \exists r \in Q(r \in U).$$

Again from relation 5. above, we deduce,

$$q \in L \to \exists q' > q(q' \in L)$$

and its dual, the converse implications being an easy consequence of relation 3. above. The disjointness of L and U follows from 2 and 3., since from $(q \in L) \wedge (r \in U)$ we deduce $x^*(q, r) = \Omega_{\neg\neg}$ and hence $q < r$. Finally, we observe that for ideals I, J of $\Omega_{\neg\neg}$ we have

$$(I \vee J = \Omega_{\neg\neg}) \wedge \neg(I = \Omega_{\neg\neg}) \to (J = \Omega_{\neg\neg})$$

since $\neg(I = \Omega_{\neg\neg}) \to (I = 0)$, and so (taking $I = x^*(q, \infty), J = x^*(-\infty, r)$, and using relation 4. above) we deduce

$$q < r \wedge \neg(q \in L) \to r \in U,$$

and dually $q < r \wedge \neg(r \in U) \to q \in L$. So (L, U) is a MacNeille real.

Conversely, suppose given a MacNeille real (L, U) in $\mathcal{E}$; then we define $x : Idl(\Omega_{\neg\neg}) \to L(R)$ by

$$x^*(q,r) = \{p \in \Omega_{\neg\neg} \mid (\exists q' > q)(\exists r' < r)(p \to \neg\neg(q' \in L \wedge r' \in U))\}.$$

First we have to show that $x^*(q,r)$ is indeed an ideal of $\Omega_{\neg\neg}$; it is obviously downward-closed. Suppose p_1, p_2 are such that

$$(\exists q_i > q)(p_i \to \neg\neg(q_i \in L))$$

for $i = 1, 2$. Then since $(q_1 \in L \vee q_2 \in L) \to min(q_1, q_2) \in L$, we have

$$\neg\neg(p_1 \vee p_2) \to \neg\neg(min(q_1, q_2) \in L);$$

and clearly $q < min(q_1, q_2)$. From this and the dual argument, we deduce

$$p_1 \in x^*(q,r) \wedge p_2 \in x^*(q,r) \to (\neg\neg(p_1 \vee p_2)) \in x^*(q,r),$$

so $x^*(q,r)$ is an ideal of $\Omega_{\neg\neg}$.

Next, it must be verified that x^* preserves the relations 1-5. above (see [22]).

Lastly, we have to show that the two constructions we have defined are inverse to each other. One way round is easy; for the other, we have to show that if (L, U) is a MacNeille real, then

$$q \in L \leftrightarrow (\exists q' > q)\neg\neg(q' \in L).$$

But this follows since $\neg\neg(q' \in L)$ implies $\neg(q' \in U)$. $\square$

Finally, we obtain the second result mentioned in the beginning of this chapter.

Corollary 4.1 *([22], Corollary 2.4) The object* $^*R_{\mathcal{E}}$ *of MacNeille reals in* $\mathcal{E}$ *is isomorphic to* $e_*(R_{\gamma\mathcal{E}})$*, where* $e : \gamma\mathcal{E} \to \mathcal{E}$ *is the canonical geometric morphism.*

Proof. Proposition 4.1 establishes a bijection between the global elements of these two objects; to show that they are isomorphic, we have to extend this bijection (naturally in X) to their X-elements for an arbitrary object X of $\mathcal{E}$. But we may

do this simply by repeating the argument of Proposition 4.1 in the topos $\mathcal{E}/X$, bearing in mind that the Gleason cover of $\mathcal{E}/X$ is $(\gamma\mathcal{E})/e^*X$. $\square$

We recall from functional analysis, the Hahn-Banach theorem in its classical form:

Theorem 4.3 *(Hahn-Banach) If A is a subspace of a normed linear space B and if μ is a bounded linear functional on A, then μ can be extended to a bounded linear functional ν on B so that $||\nu|| = ||\mu||$, where the norm $||\nu||$ and $||\mu||$ are computed relative to the domain of ν and μ; explicitly,*

$$||\mu|| = sup\left\{\frac{|\mu(x)|}{||x||} \mid x \in A\right\}, \quad ||\nu|| = sup\left\{\frac{|\nu(x)|}{||x||} \mid x \in B\right\}.$$

We are interested in the constructive proof of the Hahn-Banach theorem within a geometric context in any Grothendieck topos $\mathcal{E}$. That is, we will study the Hahn-Banach theorem in the following form:

Theorem 4.4 *([37]) Let A be a subspace of a seminormed space B in a topos $\mathcal{E}$ of sheaves on a locale. Then any linear *functional on the subspace A may be extended*

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \mu & \searrow & \downarrow \quad \nu \\ & & {}^*R_{\mathcal{E}} \end{array}$$

*to a linear *functional on B having identical norm.*

Remark 4.1

1. The observation that the Hahn-Banach theorem may be examined within a geometric context and that working with locales, which classically generalizes the lattice of open sets of a topological space without reference to its points, both contribute to the constructive context mentioned above. In fact, working within a constructive context avoids the dependence of the Axiom of Choice as in the proof of the Hahn-Banach theorem in its classical form.

2. The concept of locales allows one to apply the Hahn-Banach theorem in topology and algebra. For example, one may take the locale in Theorem 4.4 to be $\mathcal{O}(X)$, the set of all open subsets of a topological space X, in topology, and $Idl(R)$, the lattice of ideals of a commutative ring R with identity, in algebra. These notions will be pointed out towards the end of this chapter.

We will first need to define what is meant by a seminormed space and a linear functional. It is known that given a propositional geometric theory $\mathcal{T}$, one may construct a locale, and this locale is called the locale of the theory $\mathcal{T}$ (see Chapter 1). This construction will then be applied to obtain a description of the unit ball of the dual of a seminormed space by considering the theory of linear functional of norm not exceeding one on the seminormed space. We will then recall that the dual locale of a seminormed space is a compact, completely regular locale. Finally, we will recall the proof of Theorem 4.4. It will be assumed that $\mathcal{E}$ is a Grothendieck topos.

Definition 4.1 *([37])* A *seminormed space* B in the topos $\mathcal{E}$ is a linear space B over the field of rationals in $\mathcal{E}$, together with a mapping

$$N : Q_{\mathcal{E}}{}^{+} \to \Omega^{B}$$

from the positive rationals in $\mathcal{E}$ to the set of subsets of B, satisfying the following conditions:

1. $a \in N(q) \leftrightarrow \exists q' < q \;\; a \in N(q')$;
2. $\exists q \;\; a \in N(q)$;
3. $a \in N(q) \;\wedge\; a' \in N(q') \to a + a' \in N(q + q')$;
4. $a \in N(q') \to qa \in N(qq')$;
5. $a \in N(q) \to -a \in N(q)$;
6. $0 \in N(q)$;

whenever $a, a' \in B$ and $q, q' \in Q_{\mathcal{E}}{}^{+}$.

In other words, the seminorm is defined in terms of the open balls about zero and hence the mapping

$$N : Q_{\mathcal{E}}{}^{+} \to \Omega^{B}$$

assigns to each positive rational q the open ball around zero of radius q.

In particular, the real numbers of the topos $\mathcal{E}$, i.e., the Dedekind reals $R_{\mathcal{E}}$, are a seminormed space with respect to the open balls which may be defined in terms of the absolute value which exists on $R_{\mathcal{E}}$ by writing

$$N_{R}(q) = \{x \in R_{\mathcal{E}} \mid |x| < q\}$$

for each $q \in Q_{\mathcal{E}}{}^{+}$.

Definition 4.2 *([37])* A *linear functional* of norm not exceeding one on a seminormed space B in the topos $\mathcal{E}$ is a linear map

$$\mu : B \to R_{\mathcal{E}}$$

to the seminormed space $R_{\mathcal{E}}$ of real numbers, satisfying the condition:
$\forall a \in B\ \forall q \in Q^{+}\ (a \in N(q) \to \mu(a) \in N_{R}(q))$.

Definition 4.3 *([37])* The *theory of linear functionals of norm not exceeding one* on a seminormed space B in a topos $\mathcal{E}$ is the propositional geometric theory $\mathcal{F}\mathrm{n}B$ obtained by taking a primitive proposition

$$a \in (r, s)$$

for each $a \in B$ and each pair $r, s \in Q_{\mathcal{E}}$ together with the following axioms:

1. $true \vdash 0 \in (r, s)$ whenever $r < 0 < s$;

2. $0 \in (r, s) \vdash false$ otherwise;

3. $a \in (r, s) \vdash -a \in (-s, -r)$;

4. $a \in (r,s) \vdash ta \in (tr,ts)$ whenever $t > 0$;

5. $a \in (r,s) \ \wedge \ a' \in (r',s') \vdash a + a' \in (r+r', s+s')$;

6. $a \in (r,s) \vdash a \in (r,s') \ \vee \ a \in (r',s)$ whenever $r < r' < s' < s$;

7. true $\vdash a \in (-1,1)$ whenever $a \in N(1)$;

8. $a \in (r,s) \dashv\vdash \bigvee_{r<r'<s'<s} a \in (r',s')$.

The dual locale *Fn B* of the seminormed spaces B is the locale of the theory $\mathcal{F}\text{n}B$ of linear functionals on B of norm not exceeding 1. The locale *Fn B* is the constructive generalization of that of the *weak** topology on the unit ball of the dual of the seminormed space B.

In general, the points of the locale of the theory correspond exactly to the models of the theory. In the case of the dual locale *Fn B*, the correspondence between the models M of the theory $\mathcal{F}\text{n}B$ and their linear functionals $\mu : B \to R_{\mathcal{E}}$ of norm not exceeding one is given by the relationship

$$M \models a \in (r,s) \ \ iff \ \ r < \mu(a) < s$$

for each $a \in B$ and each pair r, s of rationals in the topos $\mathcal{E}$ [37]. It was shown that each linear functional

$$\mu : B \to R_{\mathcal{E}}$$

of norm ≤ 1 on the seminormed space B gives a model of the theory by making the primitive propositions $a \in (r,s)$ true. That is, the statement $r < \mu(a) < s$ follows from the observation that axioms 1-5. in Definition 4.3 are a direct consequence of the linearity of μ, that axiom 7. holds since μ is norm-decreasing, and that axioms 6. and 8. are valid since values of μ are Dedekind reals.

Now to show that any model M of the theory $\mathcal{F}\text{n}B$ arises from a unique linear functional $\mu : B \to R_{\mathcal{E}}$ of norm ≤ 1 on the seminormed space B. This was done by showing first that any model M of the theory $\mathcal{F}\text{n}B$ assigns to each $a \in B$ a Dedekind

real $\mu(a) \in R_{\mathcal{E}}$ by defining its lower and upper cuts to consist, respectively, of those $r, s \in Q_{\mathcal{E}}$ for which $\exists r', s' \in Q_{\mathcal{E}}$ such that the propositions

$$a \in (r, s') \;\; \textit{and} \;\; a \in (r', s)$$

are validated in the model and then verifying the axioms 1-4. for Dedekind reals. Secondly, that the mapping $\mu : B \to R_{\mathcal{E}}$ determined by the model of the theory $\mathcal{F}\mathrm{n}B$ is a linear functional on a seminormed space B of norm ≤ 1. Thus the points of the dual locale $Fn\ B$ are exactly the linear functionals on the seminormed space B of norm ≤ 1.

At this point, we recall Theorem 2.4. In the case of the Hahn-Banach theorem, the canonical map

$$Fn\ B \to Fn\ A$$

of dual locales is a quotient map, for any subspace A of a seminormed space B in the topos. Thus if $Fn\ B$ is a compact regular locale and the topos satisfies the required conditions in Theorem 2.4, then this theorem may be used. It turns out that the dual locale $Fn\ B$ is indeed compact regular, in fact it is compact, completely regular by a constructive form of Alaoglu's Theorem.

Theorem 4.5 *([37], Theorem 3) For a seminormed space B in a topos $\mathcal{E}$, the dual locale* Fn B *is a compact, completely regular locale.*

Proof. To establish the complete regularity of $Fn\ B$, we first observe that since each element of the locale may be expressed in the form

$$\bigvee a_1 \in (r_1, s_1) \wedge \cdots \wedge a_n \in (r_n, s_n)$$

and since the completely below relation distributes over finite conjunctions, it suffices to show that each proposition $a \in (r, s)$ is the join of elements completely below it. But by axiom 8. in Definition 4.3, $a \in (r, s)$ is provably equivalent to

$$\bigvee_{r<r'<s'<s} a \in (r', s').$$

So, by establishing that

$$a \in (r', s') \lhd\!\lhd a \in (r, s) \ \ whenever \ \ r < r' < s' < s,$$

we will have accomplished our goal. In fact we note that we need only prove that

$$a \in (r', s') \lhd a \in (r, s) \ \ whenever \ \ r < r' < s' < s,$$

since an interpolation indexed by i, k can then be obtained to show that $a \in (r', s') \lhd\!\lhd a \in (r, s)$ by defining

$$r_{ik} = (k/2^i)r + (1 - (k/2^i))r'$$
$$s_{ik} = (k/2^i)s + (1 - (k/2^i))s'$$

and letting the i, kth element of the interpolation be $a \in (r_{ik}, s_{ik})$ for each $i = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^i$.

To prove this assertion involving the rather below relation, let $r < r' < s' < s$ be given, and choose a positive rational t such that $a \in N(t)$. Evidently, it may be assumed that $-t < r$ and $s < t$. Then it is asserted that the proposition

$$a \in (-t, r') \vee a \in (s', t)$$

plays the role of the element of the locale required in proving that

$$a \in (r', s') \lhd a \in (r, s).$$

Firstly, its conjunction with $a \in (r', s')$ is provably false, by observing that

$$a \in (r', s') \wedge a \in (s', t) \vdash false.$$

Equally, one has that

$$a \in (-t, r') \wedge a \in (r', s') \vdash false,$$

yielding the required result. However, on the other hand, one has that the disjunction of the proposition with $a \in (r, s)$ is provably *true*, by observing that

$$true \vdash a \in (-t, t),$$

by axioms 7 and 4., so that axiom 6. applied successively yields that

$$true \vdash a \in (-t, r') \vee a \in (r, s) \vee a \in (s', t),$$

since $-t < r < r' < s' < s < t$, giving the required result. Hence,

$$a \in (r', s') \lhd a \in (r, s)$$

whenever $r < r' < s' < s$. The locale $Fn\ B$ is therefore completely regular, by the preceding remarks.

In showing that $Fn\ B$ is compact, we examine the way in which the dual locale is obtained: we are given firstly the finitary part of the theory by means of axioms 1-7. and then an additional infinitary axiom 8. Consider now the finitary geometric theory $\mathcal{F}n_f B$ obtained from $\mathcal{F}\mathrm{n}B$ by replacing axiom 8. by its finitary half:

8$'$. $a \in (r', s') \vdash a \in (r, s)$ *whenever* $r < r' < s' < s$.

Clearly, the locale $Fn_f\ B$ obtained from $\mathcal{F}n_f B$, being the lattice of ideals of the distributive lattice determined by $\mathcal{F}n_f B$, is compact. Now there is an inclusion map of locales

$$Fn\ B \to Fn_f\ B,$$

of which the inverse image mapping is the canonical homomorphism corresponding to adding the axiom

8$''$. $a \in (r, s) \vdash \bigvee_{r<r'<s'<s} a \in (r', s')$

to the finitary theory. The fact that $Fn\ B$ is compact will be a consequence of showing that $Fn\ B$ is actually a retract of $Fn_f\ B$, since any retract of a compact locale is again compact. The retraction map

$$Fn_f\ B \to Fn\ B$$

is the map of which the inverse image assigns to each proposition $a \in (r, s)$ the proposition $\bigvee_{r<r'<s'<s} a \in (r', s')$.

Evidently, if this is indeed a map of locales, then it provides a retraction of the inclusion $Fn\ B \to Fn_f\ B$, for the inverse image of any element of $Fn\ B$ of the form

$a \in (r,s)$ under its composite with the retraction is exactly $\bigvee_{r<r'<s'<s} a \in (r',s')$. But, in the locale $Fn\ B$ this is equal to the element $a \in (r,s)$, since these are provably equivalent in the theory $\mathcal{F}\mathrm{n}B$ by axiom 8.

To see that this assignment does determine a map of locales, it is only necessary to prove that each axiom of the theory $\mathcal{F}\mathrm{n}B$ is validated in the locale $Fn_f\ B$ when $a \in (r,s)$ is interpreted by $\bigvee_{r<r'<s'<s} a \in (r',s')$. The calculations involved are straightforward. □

We need to make the following observations before we may recall the proof of the Hahn-Banach theorem. One is that the topos $\mathcal{F}\mathrm{n}\ B$ of sheaves on the locale $Fn\ B$ is the classifying topos (see [18j]) of the theory $\mathcal{F}\mathrm{n}B$. Secondly, reconsider Definition 4.2 of a linear functional. It was observed by Burden [8] that the Hahn-Banach theorem fails to hold in its naive formulation if this definition of linear functional is to be used. The problem arises because the Dedekind reals are not constructively complete with respect to their partial ordering ([9],[42]), as seen in the beginning of this chapter. For instance, there exist bounded inhabited subsets of $R_{\mathcal{E}}$ which fail to have a least upper or a greatest lower bound. It is for this reason that a seminorm does not exist on a seminormed space B in the conventional sense, i.e., the formula

$$\|a\| = inf\{q \in Q^+ \mid a \in N(q)\}$$

because it does not describe, in general, a real number in the topos. Recall that the Dedekind reals $R_{\mathcal{E}}$ in a topos $\mathcal{E}$ admit an order-completion, the MacNeille reals ${}^*R_{\mathcal{E}}$. Thus this problem may be corrected provided that one considers functionals from the seminormed space B into the MacNeille reals ${}^*R_{\mathcal{E}}$ in the topos $\mathcal{E}$. Note that the linear functionals in this case are written as linear *functionals. The advantage here is that any non-empty subset of the MacNeille reals which is bounded above will have a supremum and dually for infima of subsets which are bounded below.

We are now ready to prove our main application in this chapter, i.e., Theorem 4.4, the Hahn-Banach theorem.

Proof. (Theorem 4.4) The proof depends on the existence of the Gleason cover of the topos $\mathcal{E}$. Recall Chapter 3, that given any topos $\mathcal{E}$, there exists a cover

$$e : \gamma\mathcal{E} \to \mathcal{E}.$$

Consider a linear *functional of norm ≤ 1 on a subspace A of a seminormed space B in the topos $\mathcal{E}$

$$\begin{array}{ccc} & A & \hookrightarrow & B \\ \mu & \downarrow & & \\ & {}^*R_{\mathcal{E}} & & \end{array}$$

By Corollary 4.1, we know that ${}^*R_{\mathcal{E}} = e_* R_{\gamma\mathcal{E}}$. It is equivalent to consider a linear functional of norm ≤ 1 on the inverse image of e^*A which is a subspace of a seminormed space e^*B in the topos $\gamma\mathcal{E}$ by the adjointness of the direct and inverse image functors. Thus we obtain the following diagram

$$\begin{array}{ccc} e^*A & \to & e^*B \\ \downarrow & & \\ R_{\gamma\mathcal{E}} & & \end{array}$$

in $\gamma\mathcal{E}$, where, because $\gamma\mathcal{E}$ is a De Morgan topos, the Dedekind reals in $\gamma\mathcal{E}$, $R_{\gamma\mathcal{E}}$ coincide with the MacNeille reals, ${}^*R_{\gamma\mathcal{E}}$ as seen in the beginning of this chapter. Passing back along the adjointness, it may be seen that the extension of the linear *functional in $\mathcal{E}$ is equivalent to the extension of the linear functional in the Gleason cover of $\mathcal{E}$.

Applying the Hahn-Banach theorem to the subspace e^*A of the seminormed space e^*B in the Grothendieck topos $\gamma\mathcal{E}$, we obtain a quotient map

$$Fn\, e^*B \to Fn\, e^*A$$

of compact completely regular locales in $\gamma\mathcal{E}$.

Now Theorem 2.4 can be applied to this quotient map

$$Fn\, e^*B \to Fn\, e^*A$$

of dual locales in the topos $\gamma\mathcal{E}$. The point of the locale $Fn\ e^*A$ which corresponds to a linear functional

$$^*\mu : e^*A \to R_{\gamma\mathcal{E}}$$

may be lifted to a point of the locale $Fn\ e^*B$ which corresponds to a linear functional

$$^*\nu : e^*B \to R_{\gamma\mathcal{E}}$$

which is the required extension. Thus we obtain the following diagram

$$\begin{array}{ccccc} e^*A & \hookrightarrow & e^*B & \\ ^*\mu & \searrow & \downarrow & ^*\nu \\ & & R_{\gamma\mathcal{E}} & \end{array}$$

in $\gamma\mathcal{E}$. By adjointness, this provides an extension of the linear *functional of norm ≤ 1 to a linear *functional of norm ≤ 1 on the seminormed space B. Thus we obtain the following diagram

$$\begin{array}{ccccc} A & \hookrightarrow & B & \\ \mu & \searrow & \downarrow & \nu \\ & & {}^*R_{\mathcal{E}} & \end{array}$$

in $\mathcal{E}$, as required. $\square$

Since the conditions in which the Hahn-Banach theorem holds has been established in the topos of sheaves on a locale, one may apply these results in the topos of sheaves on a topological space X where $\mathcal{O}(X)$ is the locale of open subsets.

Example : Burden's proof of the Hahn-Banach theorem [8] in the category of sheaves on a topological space X considered functionals from the seminormed space B into the MacNeille reals though there was no mention of the notion of extremally disconnectedness. In [37], the connection to extremally disconnected spaces was investigated and Burden's proof of the Hahn-Banach theorem was proved in the following form:

Theorem 4.6 *([37], Theorem 1) If A is a subspace of the seminormed space B in the category of sheaves on a topological space X, then any linear *functional*

$$\mu : A \to {}^*R_X$$

*may be extended to a linear *functional*

$$\nu : B \to {}^*R_X$$

on the seminormed space B having identical norm.

It may be remarked that the sheaf R_X of continuous real functions on X are not constructively complete with respect to their partial ordering and so as explained earlier in this chapter, the MacNeille reals ${}^*R_X$ in the category of sheaves on the topological space X must be considered. In this case, the MacNeille reals are given by the pairs $f = (\check{f}, \hat{f})$ consisting of a lower semicontinuous real function $\check{f}$ and an upper semicontinuous real function $\hat{f}$ satisfying the condition that $\check{f}$ is the greatest lower semicontinuous function less than $\hat{f}$, and $\hat{f}$ is the least upper semicontinuous function greater than $\check{f}$. The proof of the Hahn-Banach theorem in this form, again depends on the Gleason cover. In other words, given any topological space X, there exists the canonical geometric morphism $e : \gamma X \to X$ where the Gleason cover γX is extremally disconnected (see Remark 3.1). Thus the topos of sheaves on the topological space γX satisfies (DML) and so the Dedekind reals $R_{\gamma X}$ and the MacNeille reals ${}^*R_{\gamma X}$ coincide. By Corollary 4.1, the MacNeille reals ${}^*R_X$ are the direct image of the Dedekind reals $R_{\gamma X}$. Algebraically, if the topological space γX is extremally disconnected, then the ring $C(\gamma X)$ of continuous real-valued functions on γX satisfies (DML), i.e., $C(\gamma X)$ is a *Baer* ring ([38], Theorem 3). This concludes our example.

In conclusion, if $\mathcal{E}$ is a topos of sheaves on a locale L, then every linear functional on a subspace of a seminormed space in $\mathcal{E}$ has an extension precisely if the locale L is extremally disconnected. Of course this occurs exactly if the topos $\mathcal{E}$ defined localically over *Sets* is a De Morgan topos, and so the Hahn-Banach theorem applies naively exactly if the locale is extremally disconnected.

Chapter 5

Application II: The real closure of an ordered field, De Morgan's law and classifying toposes

In this chapter we will define what is meant by De Morgan's law (DML) in categorical logic, due to Joyal and Reyes [26]. Our first application will be the proof of the existence of the real closure of an ordered field in any topos $\mathcal{E}$ of sheaves on a Boolean space as shown by Bunge [3]. The proof in [3] implicitly makes use of the Gleason cover of $\mathcal{E}$ as remarked by Johnstone. The second application will be the conditions needed for a classifying topos of a geometric theory $\mathcal{T}$ to satisfy (DML), due to Bagchi [1].

Notions in model theory such as existentially complete structures, model companions and model completions of a theory $\mathcal{T}$ may be studied in the framework of categorical logic. But first, we remind the reader of the definitions of these well known notions in model theory. Let $\mathcal{L}$ be a first order language, and let $\mathcal{T}$ and $\mathcal{T}^*$ be two theories in $\mathcal{L}$.

Definition 5.1

1. The map $M \xrightarrow{f} N \in Mod_{\mathcal{E}}(T)$ is an *extension* if for every primitive relation

$\mathcal{R}$ (including $=$) of the language of $\mathcal{T}$, the diagram

$$\begin{array}{ccc} M^n & \stackrel{f^n}{\to} & N^n \\ \uparrow & & \uparrow \\ M(\mathcal{R}) & \to & N(\mathcal{R}) \end{array}$$

is a pullback.

2. Let M and M' be structures. M' is said to be an *elementary extension* of M, denoted $M \prec M'$, iff

   (a) M' is an extension of M, $M \subset M'$, and

   (b) M' is *elementarily equivalent* to M, denoted $M \equiv M'$, i.e., if for any sentence X, either $M \models X$ and $M' \models X$ or $M \models \neg X$ and $M' \models \neg X$.

3. A substructure M of a structure M' is said to be *existentially complete* in M' if every existential sentence which is defined in M and true in M' is true in M also.

4. The theory $\mathcal{T}^*$ is said to be *model-consistent* relative to $\mathcal{T}$ if each model of $\mathcal{T}$ can be embedded in a model of $\mathcal{T}^*$.

5. The theories $\mathcal{T}$ and $\mathcal{T}^*$ are said to be *mutually model-consistent* if each is model-consistent relative to the other.

6. The theory $\mathcal{T}^*$ is said to be *model-complete* relative to $\mathcal{T}$ if whenever M is a model of $\mathcal{T}$, M' and M'' are models of $\mathcal{T}^*$ and contain M as a substructure, and X is a sentence defined in M, then M' satisfies X iff M'' satisfies X. If $\mathcal{T}^*$ is model-complete relative to itself, then $\mathcal{T}^*$ is said to be *model-complete.* In other words, $\mathcal{T}^*$ is model-complete iff whenever a model M' of $\mathcal{T}^*$ is an extension of a model M of $\mathcal{T}^*$, then M' is an elementary extension of M.

7. The theory $\mathcal{T}^*$ is called a *model completion* of $\mathcal{T}$ if

   (a) $\mathcal{T}^*$ contains $\mathcal{T}$,

   (b) $\mathcal{T}^*$ is model-consistent relative to $\mathcal{T}$, and

(c) $\mathcal{T}^*$ is model-complete relative to $\mathcal{T}$.

It is known that if $\mathcal{T}^*$ is a model completion, then $\mathcal{T}^*$ is model-complete itself.

8. The theory $\mathcal{T}^*$ is called a *model companion* for $\mathcal{T}$ if

   (a) $\mathcal{T}$ and $\mathcal{T}^*$ are mutually model-consistent, and

   (b) $\mathcal{T}^*$ is model-complete.

9. A theory is said to have the *amalgamation property* if whenever M, M' and M''' are models of the theory and M is a substructure of both M' and M'', then there is a model M''' of the theory such that the following diagram commutes

$$\begin{tikzcd} & M' \arrow[dr] & \\ M \arrow[ur] \arrow[dr] & & M''' \\ & M'' \arrow[ur] & \end{tikzcd}$$

where $\to$ denotes $\subseteq$.

Let us now look at some examples. An algebraically closed field is existentially complete in the sense that every existential sentence defined in it and true in some extension of it is already true in it. Moreover, let $\mathcal{T}$ be the theory of fields and $\mathcal{T}^*$ be the theory of algebraically closed fields. Then, $\mathcal{T}^*$ is model-consistent relative to $\mathcal{T}$, for every field is contained in an algebraically closed field. Furthermore, $\mathcal{T}^*$ is model-complete relative to $\mathcal{T}$. Since $\mathcal{T}^*$ contains $\mathcal{T}$, $\mathcal{T}^*$ is a model completion of $\mathcal{T}$ (up to logical equivalence). The latter implies by definition that $\mathcal{T}^*$ is a model companion of $\mathcal{T}$ (the converse is not necessarily true, since the theory of real closed fields is not model-complete relative to the theory of formally real fields). By Proposition 5.1 (below), $\mathcal{T}$ has the amalgamation property. Another example is that the theory of real closed ordered fields is the model completion of the theory of ordered fields. Note that some theories may fail to have a model companion, as

for instance the theory of groups. In this situation, the notion of finite and infinite forcing must be examined.

In the case that the theory $\mathcal{T}$ is coherent, notions in model theory may be related to (DML). We want to show the categorical logic analogue to the classical

Proposition 5.1 *Let $\mathcal{T}^*$ be a model companion of $\mathcal{T}$. Then the following are equivalent:*

1. *$\mathcal{T}^*$ is a model completion of $\mathcal{T}$.*
2. *$\mathcal{T}$ has the amalgamation property.*

Let us recall that in categorical logic, a model of a theory $\mathcal{T}$ is an interpretation $\mathcal{T} \xrightarrow{I} Sets$. It is known that the category of all models of $\mathcal{T}$, Mod($\mathcal{T}$) is a full subcategory of the functor category $Sets^{\mathcal{T}}$.

A site is constructed consisting of a coherent theory $\mathcal{T}$ viewed as a category with a Grothendieck topology on $\mathcal{T}$ defined as follows: for each monomorphism $A \overset{j}{\rightarrowtail} B$ in $\mathcal{T}$, let F_j be the family of monomorphisms

$$\{A \overset{j}{\rightarrowtail} B\} \cup \{A' \rightarrowtail B \mid A' \wedge A = 0\}.$$

Let Σ be the Grothendieck topology generated by those families F_j, with j running through the monomorphisms of $\mathcal{T}$. A model of $\mathcal{T}$ is existentially closed iff it transforms families from Σ into surjective families in *Sets*.

In the case that the topology Σ is finitary, i.e., that the covering families

$$\{A \overset{j}{\rightarrowtail} B\} \cup \{A' \rightarrowtail B \mid A' \wedge A = 0\}$$

can be refined into finite covering families of the topology Σ, the coherent theory $\Sigma^{-1}\mathcal{T}$ (see Chapter 1) is the *model companion* of $\mathcal{T}$.

Remark 5.1

1. The Grothendieck topology Σ on a coherent theory $\mathcal{T}$ was motivated by the result that if M is a model of $\mathcal{T}$ then M is existentially closed iff for any monomorphism $A \rightarrowtail B$ in $\mathcal{T}$, we have

$$M(B) = M(A) \ \cup \bigcup_{A' \wedge A = 0, A' \rightarrowtail B} M(A').$$

2. Recall that given a propositional geometric theory, one may construct a locale. Similarly, any distributive lattice D generates a coherent theory $\mathcal{T}$. The models of $\mathcal{T}$ are exactly the prime filters of D and the existentially closed models are exactly the maximal filters of D.

The logical aspect of (DML) can be realized as a condition on a theory $\mathcal{T}$, that is, $\neg(A_1 \wedge A_2) = \neg A_1 \vee \neg A_2$ for $A_1, A_2 \rightarrowtail B \in \mathcal{T}$. An equivalent condition is that $\neg A \vee \neg\neg A = B$ for all $A \rightarrowtail B \in \mathcal{T}$.

Thus, the following result is obtained.

Proposition 5.2 *([26]) Let $\mathcal{T}$ be a coherent theory. Then $\mathcal{T}$ admits a model companion and $Mod(\mathcal{T})$ has the amalgamation property iff for each $B \in \mathcal{T}$, the lattice of subobjects of B admits a negation satisfying $\neg A \vee \neg\neg A = B$.*

Remark 5.2

1. It is exactly this case where $\mathcal{T}$ admits a *model completion*.

2. Recall Theorem 3.1 that for a Boolean topos $\mathcal{S}$ and an internal category $\mathcal{C}$ in the topos $\mathcal{S}^{\mathcal{C}^{op}}$, $\mathcal{C}$ satisfies the Ore condition iff $\mathcal{S}^{\mathcal{C}^{op}}$ is De Morgan. This is equivalent to saying that any diagram

$$\begin{array}{ccc} U & \rightarrow & V \\ \downarrow & & \\ W & & \end{array}$$

in $\mathcal{C}^{op}$ can be embedded in a communative square. This occurs exactly when $Mod(\mathcal{T})$ has the amalgamation property.

An interpretation $\mathcal{T} \xrightarrow{I} \mathcal{E}$ of $\mathcal{T}$ in an arbitrary topos $\mathcal{E}$ may be extended to the Robinson theory $\mathcal{T}_\neg$ generated by $\mathcal{T}$:

$$\begin{tikzcd} \mathcal{T} \arrow[r, "I"] \arrow[d] & \mathcal{E} \\ \mathcal{T}_\neg \arrow[ur, "K"'] & \end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "I"] \arrow[dr] & \mathcal{E} \arrow[r, "M"] & Sets \\ & \mathcal{T}_\neg \arrow[u, "K"] \arrow[ur, "M \circ K"'] & \end{tikzcd}$$

A model M of $\mathcal{E}$ is said to be *generic* if $M \circ K$ is an interpretation of $\mathcal{T}_\neg$ which preserves negation. In this case $M \circ K$ must coincide with the unique extension of $M \circ I$ to $\mathcal{T}_\neg$. A necessary and sufficient condition for M to be generic is that $M \circ K$ transforms dense monomorphisms of $\mathcal{T}_\neg$ into isomorphisms. Note that if M is a model of $\mathcal{E}$ the composite $M \circ K$ does not preserve negation in general. It is clear that $\mathcal{T}_\neg$ must be a theory that best approximates $\mathcal{T}$ and admits a model completion.

Next recall, from Chapter 3, the diagram

$$Sh_{\neg\neg}(\gamma\mathcal{E}) \sim Sh_{\neg\neg}(\mathcal{E}) \overset{i}{\hookrightarrow} \gamma\mathcal{E} \xrightarrow{e} \mathcal{E}$$

where i is the canonical inclusion, e is the canonical geometric morphism and, $\gamma\mathcal{E}$ is the Gleason cover of a topos $\mathcal{E}$. Note that $Sh_{\neg\neg}(\mathcal{E})$ is a Boolean-valued model of set theory (see Chapter 1). It is exactly this diagram that will be needed in the following results leading up to and including the proof of the first application. We will make use of the following observations:

1. Any Boolean algebra B may be extended to a complete Boolean algebra $^*B$, i.e., the MacNeille completion of B, such that the elements of $^*B$ are the $\neg\neg$–closed elements of the frame $Idl(B)$.'

2. The Gleason cover $\gamma\mathcal{E}$ of any topos $\mathcal{E} = Sh_{fc}(B)$ of sheaves on a Boolean algebra B for the finite cover topology is $\gamma(Sh_{fc}(B)) = Sh_{fc}(^*B)$.

3. $(Sh_{fc}(B))_{\neg\neg} = Sh_\infty({}^*B) \hookrightarrow Sh_{fc}(B)$ must factor through the Gleason cover, since it is Boolean and hence De Morgan, and the minimality of the latter says that in the factorization

$$(Sh_{fc}(B))_{\neg\neg} = Sh_\infty({}^*B) \overset{i}{\hookrightarrow} Sh_{fc}({}^*B) \overset{e}{\twoheadrightarrow} Sh_{fc}(B)$$

with e a surjection and i is flat, i.e., i is an inclusion for which i_* preserves first-order logic $(\forall, \rightarrow, \neg)$ [32].

Lemma 5.1 *Let $\mathcal{T}$ be a coherent theory, B be a complete Boolean algebra and let*

$$Sh_\infty(B) \underset{i}{\overset{a}{\leftrightarrows}} Sh_{fc}(B)$$

be the subtopos of $Sh_{fc}(B)$ given by the $\neg\neg$–topology. If $M \xrightarrow{f} N \in Mod_{Sh_\infty(B)}(\mathcal{T})$, then $iM \xrightarrow{if} iN \in Mod_{Sh_{fc}(B)}(\mathcal{T})$.

Proof. ([7]) Since $Sh_\infty(B)$ satisfies the axiom of choice (i.e., epimorphisms split), i preserves images and it is enough to show that i preserves $\vee$ (i preserves $\wedge$, $\top$ automatically since it has a left adjoint). But this follows by using characteristic morphisms, from the fact that $B \hookrightarrow \Omega \in Sh_{fc}(B)$ preserves $\vee$, whenever B is complete. (In topological terms: in an extremally disconnected Stone space, the regular open sets coincide with the clopen sets and the supremum of two clopen is just their union). □

The following definitions and results will be needed for our first application in which the existence of the real closure of an ordered field in a topos of sheaves on a Boolean algebra B is proved.

Definition 5.2 *([3], Definition 1.2)* Let $\mathcal{M}$ and $\mathcal{N}$ be categories of structures for a language $\mathcal{L}$, with $\mathcal{M}$ a subcategory of $\mathcal{N}$. It is said that $\mathcal{M}$ has the *prime model extension property* in $\mathcal{N}$ if: given $K \in \mathcal{N}$, there exists an extension $K \xrightarrow{f} \bar{K} \in \mathcal{N}$, with $\bar{K} \in \mathcal{M}$ such that for any extension $K \xrightarrow{g} K' \in \mathcal{N}$, with $K' \in \mathcal{M}$, there exists a (not necessarily unique) extension $\bar{g} : \bar{K} \rightarrow K'$, such that the following diagram

commutes

$$\begin{array}{ccc} & & K \\ g & \swarrow & \downarrow\ f \\ K' & \leftarrow & \bar{K} \\ & \bar{g} & \end{array} .$$

Let $\mathcal{T} \to \bar{\mathcal{T}}$ be a quotient of coherent theories. Let $\mathcal{E}$ be a Grothendieck topos.

Definition 5.3 *([3], Definition 2.1)* A pair $< \psi(y), \phi(x,y) >$ of formulas (of the language of $\mathcal{T} \to \bar{\mathcal{T}}$) is called a *$\bar{\mathcal{T}}$-defining pair* if

$$\bar{\mathcal{T}} \vdash \forall y[\psi(y) \to \exists! x \phi(x,y)].$$

Definition 5.4 *([3], Definition 2.2)* There is a *geometric equivalence on $\bar{\mathcal{T}}$-defining pairs in $\mathcal{E}$*, provided that given any two $\bar{\mathcal{T}}$-defining pairs $< \psi, \phi >$ and $< \psi', \phi' >$, there exists a coherent formula $\theta^{<\psi,\phi>}_{<\psi',\phi'>}$, such that for any $K \in Mod_{\mathcal{E}}(\mathcal{T})$, if $K \models \psi(a)$ and $K \models \psi'(a')$, then it follows that

$$K \models \{\theta^{<\psi,\phi>}_{<\psi',\phi'>}(a,a') \leftrightarrow \forall x[\phi(x,a) \leftrightarrow \phi'(x,a')]\}.$$

Definition 5.5 *([3], Definition 2.3)* The quotient $\mathcal{T} \to \bar{\mathcal{T}}$ has the *Sturm property in $\mathcal{E}$* if given that

$$\bar{\mathcal{T}} \vdash \forall y[\psi(y) \to \exists x \phi(x,y)],$$

with ψ, ϕ open and coherent, and given $K \in Mod_{\mathcal{E}}(\mathcal{T})$, with $K \models \psi(a)$, there exists a coherent formula $\tilde{\phi}(x,y)$ with $< \psi, \tilde{\phi} >$ $\bar{\mathcal{T}}$-defining, such that

$$K \models \forall x[\tilde{\phi}(x,a) \to \phi(x,a)].$$

Theorem 5.1 *([3], Theorem 2.4) Let $\mathcal{T} \to \bar{\mathcal{T}}$ be a quotient of coherent theories such that $\bar{\mathcal{T}}$ is consistent and can be axiomatized by sequents of the form $\psi(y) \to \exists x \phi(x,y)$, with ψ, ϕ open and coherent. Assume that $\mathcal{T} \to \bar{\mathcal{T}}$ has the Sturm property in a Grothendieck topos $\mathcal{E}$ and that there is a geometric equivalence of $\bar{\mathcal{T}}$-defining pairs in $\mathcal{E}$. Then, $Mod_{\mathcal{E}}(\bar{\mathcal{T}})$ has the prime model extension property in $Mod_{\mathcal{E}}(\mathcal{T})$.*

Proof. For $K \in Mod_{\mathcal{E}}(\mathcal{T})$, form the coproduct

$$K^* = \amalg_{<\psi,\phi>\bar{\mathcal{T}}-defining}(||\psi(y)||_K),$$

On K^*, an equivalence relation is defined via the subobject E of $K^* \times K^*$ given by

$$E = || \bigvee_{<\psi,\phi>,<\psi',\phi'>} \theta^{<\psi,\phi>}_{<\psi',\phi'>}(y,y')||_{K^* \times K^*}.$$

Let $E \overset{\to p_1}{\underset{\to p_2}{}} K^* \xrightarrow{\pi} \bar{K}$ be a coequalizer diagram. We make $\bar{K}$ into a structure in the obvious way. For $b \in \bar{K}$, the notation $b = [a_{<\psi,\phi>}]$ makes explicit a representative for the equivalence class b. If $\alpha(y_1, \ldots, y_n)$ is an n-ary relation symbol, and if $b_1, \ldots, b_n \in \bar{K}$ are such that $b_i = [a_{i<\psi_i,\phi_i>}]$, then letting $\bar{K} \models \alpha(b_1, \ldots, b_n)$ iff $K \models \forall y_1 \cdots \forall y_n[(\bigwedge_{i=1\ldots n} \phi_i(y_i, a_i)) \to \alpha(y_1, \ldots, y_n)]$, is independent of the choice of representatives and extends to all open coherent formulas $\alpha(y_1, \ldots, y_n)$.

The crucial part is to prove that $\bar{K} \in Mod_{\mathcal{E}}(\bar{\mathcal{T}})$. Consider an axiom for $\bar{\mathcal{T}}$ as in the statement of the theorem, and suppose that $\bar{K} \models \psi(b)$, where $b = b_1, \ldots, b_n$, with $b_i = [a_{i<\psi_i,\phi_i>}]$. Define new formulas

$$\psi^*(z) = \forall y_1 \cdots \forall y_n[(\bigwedge_{i=1\ldots n} \psi_i(z_i) \wedge \bigwedge_{i=1\ldots n} \phi_i(y_i, z_i)) \to \psi(y_1, \ldots, y_n)]$$

and

$$\phi^*(x, z) = \forall y_1 \cdots \forall y_n[(\bigwedge_{i=1\ldots n} \psi_i(z_i) \wedge \bigwedge_{i=1\ldots n} \phi_i(x, z_i)) \to \phi(x, y_1, \ldots, y_n)],$$

where $z = z_1 \cap \cdots \cap z_n$ corresponds to $a = a_1 \cap \cdots \cap a_n$.

Note that

$$\bar{\mathcal{T}} \vdash \forall z[\psi^*(z) \to \exists x \phi^*(x, z)],$$

and that $K \models \psi^*(a)$.

By the Sturm property, a coherent formula $\tilde{\phi}^*(x, z)$ exists for which $< \psi^*, \tilde{\phi}^* >$ is $\bar{\mathcal{T}}$-defining and also $K \models \forall x[\tilde{\phi}^*(x, a) \to \phi^*(x, a)]$.

Denoting a by a_0, ψ^* and $\tilde{\phi}^*$, respectively, by ψ_0 and ϕ_0, the above may be rewritten as follows

$$K \models \forall x \forall y_1 \cdots \forall y_n[(\bigwedge_{i=0\ldots n} \psi_i(a_i) \wedge \bigwedge_{i=0\ldots n} \phi_i(y_i, a_i)) \to \phi(x, y_1, \ldots, y_n)]$$

and hence expresses the relation

$$\bar{K} \models \phi(b_0, b_1, \ldots, b_n)$$

by letting $b_0 = [a_{0<\psi_0,\phi_0>}]$. Therefore, $\bar{K} \models \exists x\phi(x, b)$, as required.

A morphism $K \xrightarrow{f} \bar{K}$ can be defined as the composite $K \xrightarrow{f'} K^* \xrightarrow{\pi} \bar{K}$, where f' is the injection into the coproduct corresponding to the $\bar{\mathcal{T}}$-defining pair $< y = y, x = y >$, and is easily shown to be an extension. To verify that it is a prime extension to a model of $\bar{\mathcal{T}}$ in $\mathcal{E}$, let $K \xrightarrow{g} K'$ be an extension with K' a model of $\bar{\mathcal{T}}$ in $\mathcal{E}$. Define $\bar{K} \xrightarrow{\bar{g}} K'$ as follows. Given $b \in \bar{K}$, say $b = [a_{<\psi,\phi>}]$, let $\bar{g}b$ be the unique x for which $K' \models \phi(x, ga)$. If also $b = [a_{<\psi',\phi'>}]$, from $K \models \forall x[\phi(x, a) \leftrightarrow \phi'(x, a')]$ follows from $K' \models \forall x[\phi(x, ga) \leftrightarrow \phi'(x, ga')]$ and, therefore, $\bar{g}b$ is well defined. Finally, $\bar{g} \circ f = g$ since, given any $a \in K$, $\bar{g}(fa) = \bar{g}([a_{<y=y,x=y>}])$, and the unique x for which $K' \models (x = ga)$ is $\bar{g}(fa) = ga$. $\square$

It was remarked ([3], Remark 2.6) by Johnstone that the argument which will be used to prove our first application works for any topos $\mathcal{E}$ for which we know how to construct real closures in $Sh_{\neg\neg}(\mathcal{E})$, e.g., for any localic Grothendieck topos $\mathcal{E}$, since then $Sh_{\neg\neg}(\mathcal{E})$ is a Boolean-valued model of Set Theory. Thus for $\mathcal{E} = Sh_{fc}(B)$, the proof of the real closure of an ordered field make up an instance of the factorization of the inclusion $Sh_{\neg\neg}(\mathcal{E}) \rightarrow \mathcal{E}$ through the Gleason cover of $\mathcal{E}$. Also note that since $Sh_{\infty}({}^*B)$ is a Boolean-valued model of Set Theory, real closures exist there. We are now ready for the proof of our first application.

Theorem 5.2 *Let B be a Boolean algebra and let K be an ordered field in $Sh_{fc}(B)$. Then, K has a real closure.*

Proof. ([3]) Let $K \in Mod_{Sh_{fc}(B)}(\mathcal{T})$. The theory $\bar{\mathcal{T}}$ of real closed fields is the quotient of the (universal) theory $\mathcal{T}$ of ordered fields by the axioms

$$x \geq 0 \Rightarrow \exists y(y^2 = x)$$
$$true \Rightarrow \exists y(y^{n+1} + x_n y^n + \cdots + x_0), \text{for each } n \text{ even.}$$

Hence, it has the axiomatization required by Theorem 5.1. Embed B into a complete Boolean algebra $^*B$ and consider the geometric morphisms

$$Sh_\infty(^*B) = Sh_{\neg\neg}(Sh_{fc}(^*B)) \underset{i}{\overset{a}{\leftrightarrows}} Sh_{fc}(^*B) \underset{e_*}{\overset{e^*}{\leftrightarrows}} Sh_{fc}(B).$$

Since $Sh_\infty(^*B)$ is a Boolean-valued model of Set Theory, the ordered field $ae^*K \in Mod_{Sh_\infty(^*B)}(\mathcal{T})$ has a real closure

$$ae^*K \xrightarrow{f} \tilde{K}$$

with $\tilde{K} \in Mod_{Sh_\infty(^*B)}(\bar{\mathcal{T}})$. Since i is flat, it preserves finitary logic, and so $i\tilde{K}$ is a real closed field in $Sh_\infty(^*B)$. Moreover, since e^*K is an ordered field, we still have an extension

$$e^*K \xrightarrow{\hat{f}} i\tilde{K}.$$

Sturm's theorem ([11],[15]) gives an algorithm which makes sense in any topos, provided the ordered field one applies it to is already contained in some real closed field. Such being the case for e^*K above, we use it in order to establish what we have called the Sturm property [Definition 5.5] for $\mathcal{T} \to \bar{\mathcal{T}}$ in $Sh_{fc}(B)$. Let $K \models \psi(\hat{a})$, where $\bar{\mathcal{T}} \vdash \forall y[\psi(y) \to \exists x \phi(x,y)]$. Then, also $e^*K \models \psi(e^*a)$ as well as $i\tilde{K} \models \psi(e^*a)$, since ψ is coherent. By Sturm's theorem, there exists $k \geq 0$ and $\bar{\mathcal{T}}$-defining pairs $< \psi_j, \phi_j >_{j=1,\ldots,k}$, with ψ_j, ϕ_j coherent, such that for some $1 \leq j \leq k$, $\mathcal{T} \vdash \forall y[\psi(y) \to \psi_j(y)]$, while for all $1 \leq j \leq k$, $e^*K \models \forall x[\phi_j(x, e^*a) \to \phi(x, e^*a)]$. Let j_0 be the smallest j for which

$$\mathcal{T} \vdash \forall y[\psi(y) \to \psi_j(y)]$$

and let $\tilde{\phi} = \phi_{j_0}$. Then, $< \psi, \tilde{\phi} >$ is $\bar{\mathcal{T}}$-defining and

$$\|\tilde{\phi}(x, e^*a)\|_{e^*K} \leq \|\phi(x, e^*a)\|_{e^*K}$$

from which it follows [7] that

$$e^*(\|\tilde{\phi}(x,a)\|_K) \leq e^*(\|\phi(x,a)\|_K)$$

and therefore, since e^* is faithful, that

$$||\tilde{\phi}(x,a)||_K \leq ||\phi(x,a)||_K.$$

In other words,

$$K \models \forall x[\tilde{\phi}(x,a) \to \phi(x,a)],$$

as desired. The geometric equivalence on $\bar{\mathcal{T}}$-defining pairs in $Sh_{fc}(B)$ is established similarly. □

Remark 5.3 Sturm's algorithm gives the number of roots of relevant polynomials over e^*K in any real closed extension of e^*K and these roots are "definable" over e^*K. Now, $Sh_{fc}({}^*B)$ is the Gleason cover of $Sh_{fc}(B)$, thus the geometric morphism $e : Sh_{fc}({}^*B) \to Sh_{fc}(B)$ is a surjection, i.e., e^* is faithful and therefore e^* reflects the definability of these roots back to K. This fact allows us to add them to K formally in order to get the real closure $K \to \bar{K}$ in $Sh_{fc}(B)$.

We now look into our second application, by Bagchi [1], that the conditions under which the truth-value object of a classifying topos (see [18]) of a geometric theory $\mathcal{T}$ satisfies De Morgan's law (DML).

Let B_ω be the class of pseudocomplemented distributive lattices with 0 and 1. This class is called the class of distributive p-algebras. B_ω is the equational class axiomatized by the axioms for the class of distributive lattices with 0 and 1 together with the following axioms, due to Weispfenning [43],

1. $\neg 0 = 1$,
2. $\neg 1 = 0$,
3. $\forall x, y\ (x \wedge \neg(x \wedge y) = x \wedge \neg y)$.

Then the equational subclasses of B_ω are the members of the following ω-chain:

$$B_{-1} \subset B_0 \subset B_1 \subset \cdots \subset B_\omega,$$

where B_{-1} contains only the trivial algebra with $0 = 1$. B_0 is the class of Boolean algebras and is axiomatized by the axioms for distributive p-algebras and $\forall x(x \vee \neg x = 1)$. B_1 is the class of Stone algebras and is axiomatized by the axioms for distributive p-algebras and $\forall x(\neg x \vee \neg\neg x = 1)$. In general for all $r \geq 1$, B_r is axiomatized by the axioms for distributive p-algebras and by the axiom I_r [31] as follows:

$$I_r = \bigwedge_{i,j \in 0 \ldots r, i<j} (x_i \wedge x_j = 0) \rightarrow (\bigvee_{i \in 0 \ldots r} \neg x_i = 1).$$

I_r may be rewritten as the following equation

$$I'_r = \forall x_1, \ldots, x_r\ (\neg(\bigwedge_{i \in 1 \ldots r} x_i) \vee (\bigvee_{i \in 1 \ldots r} \neg(\bigwedge_{j \in 1 \ldots r \setminus \{i\}} x_j \wedge \neg x_i))) = 1$$

Remark 5.4

1. The axioms characterizing the class of Stone algebras, B_1 $(r = 1)$, are precisely (DML).

2. The axioms characterizing B_r, for $r \geq 2$, are weaker than De Morgan's law (WDML) [19].

3. Any complete Heyting algebra is a distributive p-algebra.

We need to introduce more new concepts. Assume the language $\mathcal{L}$ is a countable relational language. Let $E_n^+(Mod(\mathcal{T}))$ be the lattice of equivalence classes (with respect to $\mathcal{T}$-provability) of existential positive formulas for some theory $\mathcal{T}$ and let $DE_n^+(Mod(\mathcal{T}))$ be the lattice of equivalence classes (with respect to $\mathcal{T}$-provability) of arbitrary disjunctions of existential positive formulas for some theory $\mathcal{T}$ where the subscript n indicates the free variables of the formulas under consideration are among $x_1, \ldots, x_n$. For every $n \in \omega$, $E_n^+(Mod(\mathcal{T}))$ is a sublattice of $DE_n^+(Mod(\mathcal{T}))$ and both lattices are distributive lattices with 0 and 1 as follows.

Definition 5.6

1. $\phi^{Mod(\mathcal{T})} = \{\psi \in DE_n^+ \mid Mod(\mathcal{T}) \models \phi \leftrightarrow \psi\}$ where $Mod(\mathcal{T}) \models \theta$ means that θ holds in every model of $\mathcal{T}$.

2. For every $\Phi \subset DE_n^+(Mod(\mathcal{T}))$,

$$\Phi^{Mod(\mathcal{T})} = \{\phi^{Mod(\mathcal{T})} \mid \phi \in \Phi\}.$$

3. Let $\phi, \psi \in DE_n^+(Mod(\mathcal{T}))$ be given, the operations $\wedge, \vee, \rightarrow, \leftrightarrow$ refer to the logical operations of $\mathcal{L}$ and the subscript D refers to the lattice $DE_n^+(Mod(\mathcal{T}))$. We define
$0_D = \perp^{Mod(\mathcal{T})}$; $1_D = \top^{Mod(\mathcal{T})}$;
$\phi^{Mod(\mathcal{T})} \wedge_D \psi^{Mod(\mathcal{T})} = (\phi \wedge \psi)^{Mod(\mathcal{T})}$;
$\phi^{Mod(\mathcal{T})} \vee_D \psi^{Mod(\mathcal{T})} = (\phi \vee \psi)^{Mod(\mathcal{T})}$;
$\phi^{Mod(\mathcal{T})} \leq_D \psi^{Mod(\mathcal{T})} = Mod(\mathcal{T}) \models \phi \rightarrow \psi$.
Furthermore, $E_n^+(Mod(\mathcal{T})) = (E_n^+)^{Mod(\mathcal{T})}$ and $DE_n^+(Mod(\mathcal{T})) = (DE_n^+)^{Mod(\mathcal{T})}$.

4. For every $\Phi \subset E_n^+(Mod(\mathcal{T}))$,
$Isp(\Phi) =$ ideal spanned by Φ in $E_n^+(Mod(\mathcal{T}))$.
$Idl(Mod(\mathcal{T})) = \{\Phi^{Mod(\mathcal{T})} \subset E_n^+(Mod(\mathcal{T})) \mid \Phi^{Mod(\mathcal{T})}$ *is an ideal*$\}$.

Definition 5.7 Let L be a distributive lattice with 0 and 1. L is called a *distributive arithmetical lattice* iff

1. $Comp(L) = \{a \in L \mid a \textit{ is compact}\}$ is a sublattice of L containing 1, and

2. for every $a \in L$, $a = \bigvee\{b \in Comp(L) \mid b \leq a\}$.

Let L be a complete Heyting algebra. It is known that for every $a \in L$, $L_{\leq a} = \{b \in L \mid b \leq a\}$ is a complete Heyting algebra with the symbols $0', 1', \wedge', \vee', \rightarrow', \neg'$ defined as follows:
$0' = 0$; $1' = a$; $\forall y, z \in L_{\leq a}, y \wedge' z = y \wedge z$; $\forall y, z \in L_{\leq a}, y \vee' z = y \vee z$;
$\forall (y_i \in L_{\leq a} \mid i \in I)$, $\bigvee'_{i \in I} y_i = (\bigvee_{i \in I} y_i) \wedge a$;
$\forall y, z \in L_{\leq a}$, $y \rightarrow' z = \bigvee\{w \in L_{\leq a} \mid w \wedge y \leq z\} = (y \rightarrow z) \wedge a$;
$\neg'(y) = y \rightarrow' 0 = (y \rightarrow 0) \wedge a = \neg y \wedge a$.

Proposition 5.3 *Given* $\Phi^{Mod(\mathcal{T})}, \Psi^{Mod(\mathcal{T})} \subset E_n^+(Mod(\mathcal{T}))$,
$Isp(\Phi^{Mod(\mathcal{T})}) \subset Isp(\Psi^{Mod(\mathcal{T})}) \Leftrightarrow (\bigvee \Phi)^{Mod(\mathcal{T})} \leq_D (\bigvee \Psi)^{Mod(\mathcal{T})}$.

Proof. $Isp(\Phi^{Mod(\mathcal{T})}) \subset Isp(\Psi^{Mod(\mathcal{T})})$

$\Leftrightarrow \forall\phi^{Mod(\mathcal{T})} \in \Phi^{Mod(\mathcal{T})}$ we may choose $n \in \omega$ and $\psi_1^{Mod(\mathcal{T})}, \ldots, \psi_n^{Mod(\mathcal{T})} \in \Psi^{Mod(\mathcal{T})}$ such that $\phi^{Mod(\mathcal{T})} \leq \bigvee_{i\in 1\ldots n} \psi_i^{Mod(\mathcal{T})}$.

$\Leftrightarrow \forall\phi^{Mod(\mathcal{T})} \in \Phi^{Mod(\mathcal{T})}$, $\exists n \in \omega$ and $\psi_1^{Mod(\mathcal{T})}, \ldots, \psi_n^{Mod(\mathcal{T})} \in \Psi^{Mod(\mathcal{T})}$ such that $Mod(\mathcal{T}) \models \phi \rightarrow \bigvee_{i\in 1\ldots n} \psi_i$.

$\Leftrightarrow \forall\phi^{Mod(\mathcal{T})} \in \Phi^{Mod(\mathcal{T})}$, $Mod(\mathcal{T}) \models \phi \rightarrow \bigvee \Psi$ (by compactness)

$\Leftrightarrow Mod(\mathcal{T}) \models \bigvee \Phi \rightarrow \bigvee \Psi$

$\Leftrightarrow (\bigvee \Phi)^{Mod(\mathcal{T})} \leq_D (\bigvee \Psi)^{Mod(\mathcal{T})}$. □

Proposition 5.4 *In view of Proposition 5.3, we may define the map*

$$h : DE_n^+(Mod(\mathcal{T})) \rightarrow Idl(E_n^+(Mod(\mathcal{T}))) : (\bigvee \Phi)^{Mod(\mathcal{T})} \mapsto Isp(\Psi^{Mod(\mathcal{T})}).$$

h is a lattice isomorphism with inverse

$$h^{\leftarrow} : Idl(E_n^+(Mod(\mathcal{T}))) \rightarrow DE_n^+(Mod(\mathcal{T})) : I \mapsto \bigvee I.$$

Proof. h is clearly injective and surjective, and by Proposition 5.3, both h and $h^{\leftarrow}$ preserve order. Moreover $h^{\leftarrow}$ is clearly the inverse of h. □

We are now ready for the first conditions in which the r^{th} Lee identity I_r is satisfied.

Proposition 5.5 *([1], III.18) Let L be a distributive arithmetical lattice. Then, for every $a \in Comp(L)$, $L_{\leq a}$ satisfies $I_r \Leftrightarrow L$ satisfies I_r.*

Proof. ($\Rightarrow$) This is immediate as $1 \in Comp(L)$ and $L = L_{\leq 1}$.

($\Leftarrow$) Let $a \in Comp(L)$ and $(y_i \in L_{\leq a} \mid i \in 0, \ldots, r)$ be given such that for all $i, j \in 0, \ldots, r, i \neq j \Rightarrow y_i \wedge' y_j = 0$. Hence for all $i, j \in 0, \ldots, r, i \neq j \Rightarrow y_i \wedge y_j = y_i \wedge' y_j = 0$. Hence by assumption $\bigvee_{i\in 0,\ldots,r}(\neg y_i) = 1$. Hence $\bigvee_{i\in 0,\ldots,r}(\neg' y_i) = \bigvee_{i\in 0,\ldots,r}(\neg y_i \wedge a) = (\bigvee_{i\in 0,\ldots,r} \neg y_i) \wedge a = 1 \wedge a = a = 1'$. As $a \in Comp(L)$ was arbitrary, this establishes the result. □

Corollary 5.1 *([1], III.19) For every $n \in \omega$, $(DE_n^+(Mod(\mathcal{T}))$ satisfies $I_r')$ $\Leftrightarrow$ (for every $\phi^{Mod(\mathcal{T})} \in E_n^+(Mod(\mathcal{T})), DE_n^+(Mod(\mathcal{T}))_{\leq \phi^{Mod(\mathcal{T})}}$ satisfies $I_r')$).*

Proof. $DE_n^+(Mod(\mathcal{T}))$ is a distributive arithmetical lattice with

$$E_n^+(Mod(\mathcal{T})) = Comp(DE_n^+(Mod(\mathcal{T}))).$$

$\square$

Let $Sh(\mathcal{C}, J)$ be the classifying topos of a geometric theory $\mathcal{T}$. Let the sieve R on the object $\{x \mid X(x)\}$ be given. Then $\Omega\{x \mid X(x)\}$ denotes the collection of sieves on the object $\{x \mid X(x)\}$ and $\Omega_J\{x \mid X(x)\}$ denotes the collection of J-closed sieves on the object $\{x \mid X(x)\}$. Note ([1], Corollary IV.19) that $Idl(E_n^+(Mod(\mathcal{T}))_{\leq \phi^{Mod(\mathcal{T})}})$ and $\Omega_J\{x \mid X(x)\}$ are isomorphic as lattices.

It is precisely this next result that allowed Bagchi to reduce the problem of determining the conditions under which the truth-value object of a classifying topos of a geometric theory $\mathcal{T}$ satisfies the identity I_r to the problem of determining the conditions under which, for every $n \in \omega$, $DE_n^+(Mod(\mathcal{T}))$ satisfies I_r. Note that the *satisfaction* or *forcing* relation, denoted $\Vdash$, used in the next proof, is defined as follows [26]. Let $(\mathcal{C}, J)$ be a site and $\mathcal{L}$ be a first-order language. Let U be a set-valued functor on $\mathcal{C}$ and $U \xrightarrow{q} \tilde{U}$ its associated sheaf. The inverse image q^{-1} is a lattice isomorphism between the subsheaves of $\tilde{U}$ and the closed subfunctors of U. A formula ϕ whose free variables are among $x_1, \ldots, x_n$ will be interpreted as a monomorphic subfunctor $\bar{\phi} \rightarrowtail U^n$. For $(a_1, \ldots, a_n) \in \bar{\phi}(A)$, where each $a_i \in U(A)$, and is denoted by $A \Vdash \phi[a_1, \ldots, a_n]$.

Corollary 5.2 *([1], IV.20) $Sh(\mathcal{C}, J) \models I_r \Leftrightarrow$ for every $n \in \omega$, $DE_n^+(Mod(\mathcal{T})) \models I_r$.*

Proof. $Sh(\mathcal{C}, J) \models I_r$

$\Leftrightarrow Sh(\mathcal{C}, J) \models \forall x_0, \ldots, \forall x_r(\bigwedge_{i,j \in 0,\ldots,r, i<j}(x_i \wedge x_j = 0) \rightarrow (\bigvee_{i \in 0,\ldots,r} \neg x_i = 1))$.

$\Leftrightarrow \forall X \in Obj(\mathcal{C}), \forall x_0, \ldots, \forall x_r \in \Omega,$

$\forall x_0, \ldots, \forall x_r(\bigwedge_{i,j \in 0,\ldots,r, i<j}(x_i \wedge x_j = 0) \rightarrow (\bigvee_{i \in 0,\ldots,r} \neg x_i = 1))$.

$\Leftrightarrow \forall X \in Obj(\mathcal{C}), \forall x_0, \ldots, \forall x_r \in \Omega_J(X),$

$X \Vdash \forall x_0, \ldots, \forall x_r (\bigwedge_{i,j \in 0,\ldots,r, i<j} (x_i \wedge x_j = 0) \rightarrow (\bigvee_{i \in 0,\ldots,r} \neg x_i = 1)).$

$\Leftrightarrow \forall n \in \omega, \forall X \in E_n^+, Idl(E_n^+(Mod(\mathcal{T}))_{\leq (X(x_0,\ldots,x_r))^{Mod(\mathcal{T})}}) \models I_r.$ (as noted above)

$\Leftrightarrow \forall n \in \omega, \forall X \in E_n^+, DE_n^+(Mod(\mathcal{T}))_{\leq (X(x_0,\ldots,x_r))^{Mod(\mathcal{T})}} \models I_r.$ (by Proposition 5.4)

$\Leftrightarrow \forall n \in \omega, DE_n^+(Mod(\mathcal{T})) \models I_r.$ (by Corollary 5.1) □

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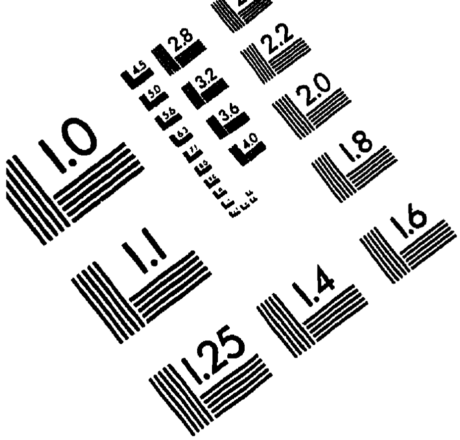

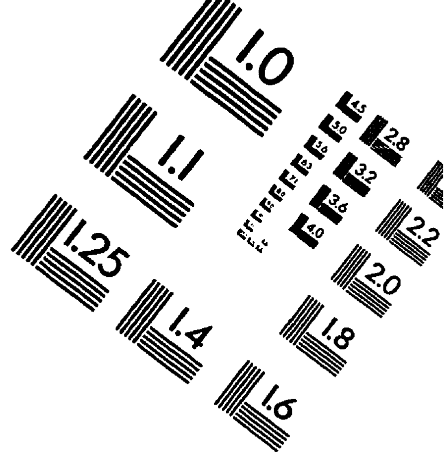

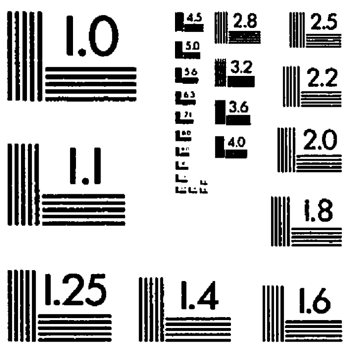

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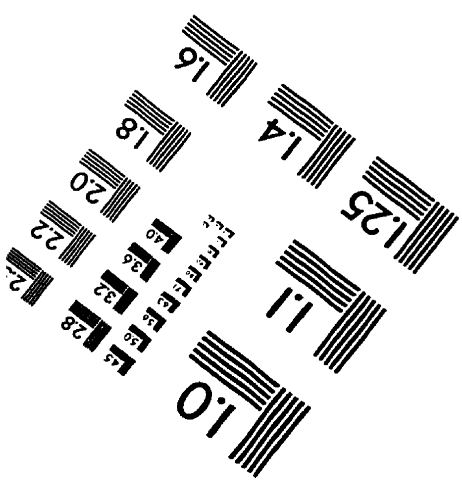

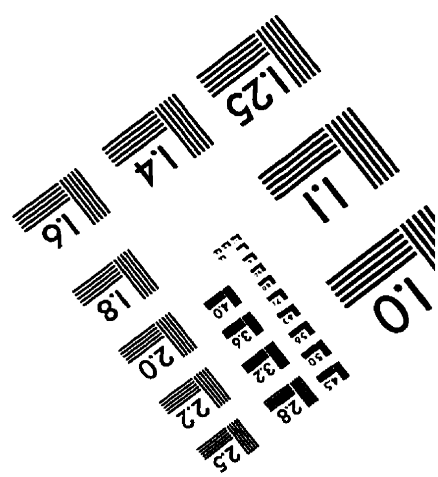

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