Coherent Signalling and Receiver Diversity for Fading Channels

by

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To my father
Abstract

Demands for faster data rates on wireless and cellular channels have led to much current interest in the use of two-dimensional (2-D) M-ary signalling formats and in the use of diversity techniques. Moreover, coherent detection of digital signalings transmitted over wireless fading channels has better power efficiency than conventional differential detectors. However, accurate performance analyses of general coherent two-dimensional signal constellations in fading environments have not been reported, particularly for diversity systems. In this thesis, new analytical expressions for the exact symbol error rates (SER) in slow Ricean, Rayleigh and Nakagami fading with diversity combining are presented for any 2-D signalling format having polygonal decision boundaries. Three types of combining techniques are considered: maximal ratio combining (MRC), equal gain combining (EGC) and selection combining (SC). These numerically efficient SER formulas make it possible for the first time to optimise parameters of various constellations precisely and to determine which constellation has the lowest probability of error. Numerical results for the performances of seventeen coherent 8-ary and 16-ary modulations in additive white Gaussian noise (AWGN) and slowly fading channels with diversity reception are discussed.

Perfect coherent detection is difficult to implement in a wireless environment. In the case of imperfect coherent detection, there is constant or varying error in the channel amplitude estimation or phase estimation or both. The robustness of a 2-D constellation to channel amplitude and phase error is indicated by its amplitude and phase error tolerance, two parameters examined in this thesis. Furthermore, new simple SER expressions for general 2-D constellations with polygonal decision boundaries in the presence of a con-
stant channel estimation error are derived. This study vividly demonstrates the effect of imperfect channel estimation on the error performance of a 2-D constellation and easily distinguishes some signal sets from others for their strong tolerance to channel estimation error although these signal sets all have similar performance in the perfect coherent detection scenario. Practical pilot symbol aided 2-D modulation systems are analysed and simulated to demonstrate the effects of dynamic channel estimation errors on 2-D signalling in fading. The performance studies for both ideal and practical coherent detections provide a good reference for engineers to select a constellation to meet their design requirements.

The probability of error of higher dimensional signalings are investigated for three classes of M-ary orthogonal signals in Rayleigh fading. In particular, new symbol error rates (and bit error rates) for coherent 3-ary and 4-ary orthogonal and transorthogonal signalings, and 6-ary and 8-ary biorthogonal signalings in slow Rayleigh fading are presented and these new expressions are found to be close approximations for the error rates of arbitrary M-ary orthogonal, transorthogonal or biorthogonal signalling.
Acknowledgements

I wish to express my sincere gratitude to Dr. Norman C. Beaulieu and Dr. Paul H. Wittke for their continuous guidance, inspiration, encouragement, technical advice and financial support which have made this thesis possible. They are excellent supervisors for me. Special thanks to Dr. Peter McLane for his encouragement and guidance on my research.

I thank Tao Lu for his technical insights and advice, particularly in the aspect of numerical evaluation methods. I thank Tao Lu, Chris Tan, and Dave Young for their assistance with computer issues. Special thanks to Xiaofei Dong for her help during my off-campus period. Thanks to Chris, Dave, Christine, Zhaohui, Zhihua, Bo, Kareem, Julian, Mike and Marius for their friendship inside and outside of the lab.

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<th>Definition</th>
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<tbody>
<tr>
<td>2-D</td>
<td>Two-dimensional</td>
</tr>
<tr>
<td>AGC</td>
<td>Automatic Gain Control</td>
</tr>
<tr>
<td>AWGN</td>
<td>Additive White Gaussian Noise</td>
</tr>
<tr>
<td>BER</td>
<td>Bit Error Rate</td>
</tr>
<tr>
<td>BPSK</td>
<td>Binary Phase Shift Keying</td>
</tr>
<tr>
<td>CDF</td>
<td>Cumulative Distribution Function</td>
</tr>
<tr>
<td>EGC</td>
<td>Equal Gain Combining</td>
</tr>
<tr>
<td>FDMA</td>
<td>Frequency Division Multiple Access</td>
</tr>
<tr>
<td>FSK</td>
<td>Frequency Shift Keying</td>
</tr>
<tr>
<td>HF</td>
<td>High Frequency</td>
</tr>
<tr>
<td>iid</td>
<td>Independent and identically distributed</td>
</tr>
<tr>
<td>LOS</td>
<td>Line-of-sight</td>
</tr>
<tr>
<td>MPSK</td>
<td>M-ary Phase Shift Keying</td>
</tr>
<tr>
<td>MRC</td>
<td>Maximal Ratio Combining</td>
</tr>
<tr>
<td>ML</td>
<td>Maximum Likelihood</td>
</tr>
<tr>
<td>PDF</td>
<td>Probability Density Function</td>
</tr>
<tr>
<td>PSAM</td>
<td>Pilot Symbol Aided Modulation</td>
</tr>
<tr>
<td>QAM</td>
<td>Quadrature Amplitude Modulation</td>
</tr>
<tr>
<td>RV</td>
<td>Random Variable</td>
</tr>
<tr>
<td>SC</td>
<td>Selection Combining</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
</tr>
<tr>
<td>--------------</td>
<td>---------------------------------------</td>
</tr>
<tr>
<td>SER</td>
<td>Symbol Error Rate</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal-to-Noise Ratio</td>
</tr>
<tr>
<td>TDMA</td>
<td>Time Division Multiple Access</td>
</tr>
<tr>
<td>WSS</td>
<td>Wide Sense Stationary</td>
</tr>
</tbody>
</table>
Symbol Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$,<em>{1}F</em>{1}(\alpha,\beta;z)$</td>
<td>Confluent hypergeometric function</td>
</tr>
<tr>
<td>$,<em>{2}F</em>{1}(\alpha,\beta;\gamma;z)$</td>
<td>Gaussian hypergeometric function</td>
</tr>
<tr>
<td>$A_{i}$</td>
<td>Amplitude of a QAM modulated signal $s_{i}(t)$</td>
</tr>
<tr>
<td>$b$</td>
<td>Distance of $x'$ in unit average energy constellation</td>
</tr>
<tr>
<td>$b_{j}$</td>
<td>$b$ of the $j$th subregion</td>
</tr>
<tr>
<td>$C$</td>
<td>Covariance matrix for amplitude fadings at pilot symbols</td>
</tr>
<tr>
<td>$C_{0}$</td>
<td>Covariance matrix normalised to fading power</td>
</tr>
<tr>
<td>$\text{Cov}(x,y)$</td>
<td>Covariance of random variables $x$ and $y$</td>
</tr>
<tr>
<td>$c$</td>
<td>Speed of light</td>
</tr>
<tr>
<td>$c_{l}, l = 1 \ldots L$</td>
<td>Weighting coefficient of the $l$th diversity branch</td>
</tr>
<tr>
<td>$D$</td>
<td>Decision variable</td>
</tr>
<tr>
<td>$\mathbf{D}$</td>
<td>Decision variable in vector form</td>
</tr>
<tr>
<td>$\text{erfc}(x)$</td>
<td>Complementary error function</td>
</tr>
<tr>
<td>$E[x]$</td>
<td>Statistical expectation of $x$</td>
</tr>
<tr>
<td>$E_{b}$</td>
<td>Average received signal energy per bit</td>
</tr>
<tr>
<td>$E_{p}$</td>
<td>Average energy of pulse shape $p(t)$</td>
</tr>
<tr>
<td>$E_{s}$</td>
<td>Average received signal energy per symbol</td>
</tr>
<tr>
<td>$f(\alpha_{l},\bar{\alpha}_{l})$</td>
<td>Joint probability density function of $\alpha_{l}$ and $\bar{\alpha}_{l}$</td>
</tr>
<tr>
<td>$f(\phi)$</td>
<td>PDF of $\phi$</td>
</tr>
<tr>
<td>${f_{1}(t), f_{2}(t)}$</td>
<td>Orthonormal basis</td>
</tr>
</tbody>
</table>
\( f_c \)  
Carrier frequency

\( f_D \)  
Maximum Doppler frequency or fading bandwidth

\( f_L(x) \)  
PDF of \( X \) for sum of \( L \) diversity branches

\( f_L(\gamma_s) \)  
PDF of SNR \( \gamma_s \) for sum of \( L \) diversity branches

\( f_{1|\gamma_l,\gamma_l}(r, \theta) \)  
Conditional PDF of \( t \) on \( \gamma_l, \hat{\gamma}_l \) in polar form

\( f_X(x) \)  
Probability density function of \( X \)

\( g(t) \)  
Complex fading random process

\( g \)  
RV, a sample of \( g(t) \) at any time \( t \)

\( \hat{g} \)  
Estimate of \( g \)

\( g_{k,t} \)  
Complex fading RV of the \( l \)th symbol in the \( k \)th frame

\( g_l \)  
Complex fading RV of the \( l \)th symbol in the current (0th) frame

\( \hat{g}_l \)  
Estimate of \( g_l \)

\( g_{p,k} \)  
Complex fading on the \( k \)th pilot symbol

\( \hat{g}_{p,k} \)  
Estimate of \( g_{p,k} \)

\( g_R(t) \)  
Real part of the complex fading process \( g(t) \)

\( g_R \)  
RV, a sample of \( g_R(t) \) at any time \( t \)

\( g_I(t) \)  
Imaginary part of the complex fading process \( g(t) \)

\( g_I \)  
RV, a sample of \( g_I(t) \) at any time \( t \)

\( \mathbf{H}_l \)  
Row vector, interpolator coefficients for the \( l \)th symbol in a frame

\( h_{k,l} \)  
Interpolator coefficient for the \( k \)th pilot if current symbol is at the \( l \)th position in a frame

\( h(n) \)  
Interpolator function

\( I_0(x) \)  
Modified Bessel function of the first kind of order zero,

\( I_i \)  
Inphase component of signal \( s_i \)

\( \text{Im}(x) \)  
Imaginary component of \( x \)

\( J \)  
Number of total subregions in a constellation

\( J_0(x) \)  
Bessel function of the first kind of order zero

\( K \)  
Rice factor or
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>Number of pilots used from previous frames</td>
</tr>
<tr>
<td>$K_2$</td>
<td>Number of pilots used from subsequent frames</td>
</tr>
<tr>
<td>$K_l$</td>
<td>Rice $K$ factor of the $l$th diversity branch</td>
</tr>
<tr>
<td>$K_T$</td>
<td>Sum of Rice $K$ factors of $L$ diversity branches</td>
</tr>
<tr>
<td>$k$</td>
<td>Frame index, in Chapter 5</td>
</tr>
<tr>
<td>$L$</td>
<td>Number of diversity branches or Number of symbols in a PSAM frame</td>
</tr>
<tr>
<td>$M$</td>
<td>Number of signals in a modulation format</td>
</tr>
<tr>
<td>$m$</td>
<td>Nakagami-$m$ fading parameter</td>
</tr>
<tr>
<td>$m_l$</td>
<td>Nakagami-$m$ fading parameter of the $l$th branch</td>
</tr>
<tr>
<td>$m_T$</td>
<td>Sum of Nakagami-$m$ fading parameters of $L$ diversity branches</td>
</tr>
<tr>
<td>$m_1$</td>
<td>Non-zero mean of $g_l$ in Ricean fading</td>
</tr>
<tr>
<td>$m_2$</td>
<td>Non-zero mean of $g_Q$ in Ricean fading</td>
</tr>
<tr>
<td>$N$</td>
<td>Noise power</td>
</tr>
<tr>
<td>$N_0$</td>
<td>One-sided power spectral density of white Gaussian noise</td>
</tr>
<tr>
<td>$N_l$</td>
<td>Total noise power of $L$ diversity branches</td>
</tr>
<tr>
<td>$n = (x_1, x_2, \ldots, x_M)$</td>
<td>Additive white Gaussian noise vector</td>
</tr>
<tr>
<td>$n_{ij}$</td>
<td>Normal vector of plane $x_i - x_j = c$, where $c$ is a constant</td>
</tr>
<tr>
<td>$n_{k,l}$</td>
<td>Sampled complex Gaussian noise RV at the $l$th symbol in the $k$th frame</td>
</tr>
<tr>
<td>$n_l$</td>
<td>Sampled complex Gaussian noise RV of the $l$th diversity branch or Sampled complex Gaussian noise RV at the $l$th symbol in the current (0th) frame</td>
</tr>
<tr>
<td>$n_{pk}$</td>
<td>Noise added onto the $k$th pilot</td>
</tr>
<tr>
<td>$n_R$</td>
<td>Real part of the complex noise $n$</td>
</tr>
<tr>
<td>$n_I$</td>
<td>Imaginary part of the complex noise $n$</td>
</tr>
<tr>
<td>$n_w(t)$</td>
<td>White Gaussian noise process</td>
</tr>
<tr>
<td>$P_g$</td>
<td>Probability of an erroneous trilateral region in AWGN</td>
</tr>
</tbody>
</table>
\( P_2 \) \quad \text{BER/SER of BPSK}

\( P_e \) \quad \text{Average symbol error probability of a signal set}

\( P(e|s_1) \) \quad \text{Probability of a symbol error given } s_1

\( P(e,j|s_1) \) \quad \text{Probability of error in subregion } j \text{ given } s_1 \text{ transmitted}

\( P_L \) \quad \text{Probability of an erroneous trilaterai region in fading with}

\( L \)-order diversity combining

\( P_M \) \quad \text{SER of MPSK}

\( P_{MQAM} \) \quad \text{SER of MQAM}

\( P_n \) \quad \text{Noise power}

\( P_{s_i} \) \quad \text{Probability of error given signal } s_i \text{ sent}

\( P(A) \) \quad \text{Probability of event } A

\( p(t) \) \quad \text{Pulse shape}

\( p_k \) \quad \text{The } k\text{th pilot symbol}

\( p_n(x_1,x_2) \) \quad \text{PDF of a two-dimensional noise vector}

\( Q(a,b) \) \quad \text{Marcum-Q function}

\( Q(x) \) \quad \text{Gaussian tail integral}

\( Q_i \) \quad \text{Quadrature component of signal } s_i

\( q \) \quad \text{Ratio of fading amplitude to its estimate}

\( R(\theta) \) \quad \text{Distance in the radius direction defined in eqn. (2.10)}

\( r \) \quad \text{Polar radius of noise vector } n \text{ or combined noise vector } t

\( R_{RR}(\tau) \) \quad \text{Autocorrelation of the real part of a complex fading process}

\( R_{RI}(\tau) \) \quad \text{Autocorrelation of the imaginary part of a complex fading process}

\( R_{RI}(\tau) \) \quad \text{Crosscorrelation of the real and imaginary part of a complex fading process}

\( R_{k,l;i,m} \) \quad \text{The correlation between complex fading samples at the } l\text{th symbol}

\text{in the } k\text{th frame and the } i\text{th symbol in the } m\text{th frame}

\( r_H \) \quad \text{Radius of outer ring}

\( r_L \) \quad \text{Radius of inner ring}

\( r_\Omega \) \quad \text{Ratio of the fading power estimate to the fading power}
Ratio of the fading power estimate at the $l$th symbol position to the fading power

$\hat{r}_\Omega$

Signal vector

$s_i = (s_{i1}, s_{i2}, \ldots, s_{iM})$

Transmitted signal with normalized unit average energy

$s(t)$

The $i$th bandpass signal in an $M$-ary signal set

$s_i(t), i = 1, \ldots, M$

The $i$th complex lowpass signal in an $M$-ary signal set

$s_{ll}(t)$

Lowpass equivalent of $s_i(t)$

$s_{kl, l}$

Lowpass complex sample of $s(t)$ at the $l$th symbol in the $k$th frame

$T$

Sampling period or symbol duration or Decision threshold in Fig. 2.4

$t = (t_1, t_2)$

Combined noise vector in two dimensions

$\mathbf{t}$

Combined complex noise

$T_3$

Transformation matrix of 3-ary orthogonal signal

$T_4$

Transformation matrix of 4-ary orthogonal signal

$TH$

Decision threshold

$V$

Ratio of complex fading to its estimate

$v$

Velocity of a mobile station

$\text{Var}[x]$  

Variance of $x$

$w_j$

A priori probability of subregion $j$

$w(n)$

Window function

$X$

RV, a sample of $x(t)$ at any time $t$

$x$

A value of $X$, or

$\Delta$

Signal magnitude in the absence of fading

$x'$

Distance shown in Fig. 2.3

$x'$

New coordinates for $n$

$y(t)$

Received noiseless signal, including multiplicative fading in a fading channel

$y$

RV, a sample of $y(t)$ at any time $t$
\( y_l \)  
Received noiseless signal from the \( l \)th diversity branch  

\( z = (z_1, z_2, \ldots, z_M) \)  
Received signal vector  

\( z(t) \)  
Received signal  

\( z_{k,l} \)  
Lowpass complex sample of \( z(t) \) at the \( l \)th symbol in the \( k \)th frame  

\( z_l \)  
Received signal from the \( l \)th diversity branch  

\( \alpha(t) \)  
Amplitude of complex fading process \( g(t) \)  

\( \alpha \)  
Amplitude of complex fading process at any time \( t \)  

Constant amplitude of channel gain in AWGN  

\( \hat{\alpha} \)  
Estimate of \( \alpha \)  

\( \alpha_{k,l} \)  
Sample of \( \alpha(t) \) at \( l \)th symbol in the \( k \)th frame  

\( \alpha_l \)  
Fading amplitude of the \( l \)th diversity branch or  

Fading amplitude of the \( l \)th symbol in the current (0th) frame  

\( \hat{\alpha}_l \)  
Estimate of \( \alpha_l \)  

\( \beta \)  
Ring ratio  

\( \beta_{OPT} \)  
Optimum ring ratio  

\( \beta_i, i = 1, 2, \ldots \)  
Orthonormal vector which constitutes a transformation matrix  

\( \eta \)  
Angle shown in Fig. 2.3  

\( \eta_j \)  
Angle \( \eta \) of the \( j \)th subregion  

\( \Gamma(x) \)  
Gamma function  

\( \gamma \)  
Received signal-to-noise ratio  

\( \gamma_b \)  
Received signal-to-noise ratio per bit  

\( \gamma_l \)  
Received signal-to-noise ratio per symbol of the \( l \)th branch  

\( \gamma_s \)  
Received signal-to-noise ratio per symbol  

\( \gamma(\alpha, x) \)  
Incomplete Gamma function  

\( \Lambda \)  
Average signal-to-noise ratio per symbol per branch  

\( \Lambda_l \)  
Average SNR of the \( l \)th diversity branch  

\( \Lambda_T \)  
Total average SNR of \( L \) diversity branches  

\( \mu \)  
Non-centrality parameter of Ricean fading
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_i, i = 1, \ldots, M$</td>
<td>Additive component in ML decision rule</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Average energy in the amplitude of fading</td>
</tr>
<tr>
<td>$\hat{\Omega}$</td>
<td>Estimate of $\Omega$</td>
</tr>
<tr>
<td>$\hat{\Omega}_l$</td>
<td>Estimate of $\Omega$ at the $l$th symbol position</td>
</tr>
<tr>
<td>$\Omega_l$</td>
<td>Average fading energy of the $l$th diversity branch</td>
</tr>
<tr>
<td>$\phi(t)$</td>
<td>Phase of the complex fading random process $g(t)$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>RV, a sample of $\phi(t)$ at any time $t$</td>
</tr>
<tr>
<td>$\hat{\phi}$</td>
<td>Estimate of $\phi$</td>
</tr>
<tr>
<td>$\phi_l$</td>
<td>RV $\phi$ from the $l$th diversity branch</td>
</tr>
<tr>
<td>$\phi_{k,l}$</td>
<td>Sample of phase $\phi(t)$ at the $l$th symbol in the $k$th frame</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Angle shown in Fig. 2.3</td>
</tr>
<tr>
<td>$\psi_j$</td>
<td>Angle $\psi$ from the $j$th subregion</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Normalized correlation coefficient between $\alpha_l$ and $\hat{\alpha}_l$</td>
</tr>
<tr>
<td>$\rho_l$</td>
<td>$\rho$ at the $l$th symbol position</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>Variance of $g_l$ and $g_{\Omega}$</td>
</tr>
<tr>
<td>$\tau_e$</td>
<td>Erroneous decision region</td>
</tr>
<tr>
<td>$\tau_c$</td>
<td>Correct decision region</td>
</tr>
<tr>
<td>$\tau_{s,i}$</td>
<td>The $i$th decision subregion of $\tau_e$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Angle of 2-D noise vector $\mathbf{n}$ or combined noise vector $\mathbf{t}$</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>Starting angle of a subregion</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>Ending angle of a subregion</td>
</tr>
<tr>
<td>$\theta_{j,1}$</td>
<td>Starting angle of the $j$th subregion</td>
</tr>
<tr>
<td>$\theta_{j,2}$</td>
<td>Ending angle of the $j$th subregion</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>Phase difference between the channel fading and its estimate</td>
</tr>
<tr>
<td>$\xi_l$</td>
<td>Phase of a QAM modulated signal $s_i(t)$</td>
</tr>
</tbody>
</table>
Chapter 1

INTRODUCTION

1.1 Background

Wireless communications has been one of the fastest growing segments in the telecommunications industry [2], [3]. The demand for mobile telephone service in the United States itself, for example, has repeatedly exceeded the capacity of systems that have been made available to the public, ever since the first one was introduced in 1946 [4]. Today with 97 million wireless telephone subscribers in the United States and more than 200 million around the world [5], the future looks even brighter with potential for growth. The current mobile network is a combination of digital cellular systems and analog cellular systems, which are generally referred to as the second-generation and first-generation cellular systems, respectively. The various advantages of digital systems over analog systems such as higher immunity to channel impairment and noise as well as increased system capacity, have been major motivations to move to digital transmission. It is to be expected that one day wireless systems will be fully digital. Even though wide deployment of second-generation cellular systems is still in progress, researchers had begun working on the technologies and standards of third-generation systems years before [6]. In third-generation cellular systems, significant increase in the system capacity is expected, and considerable efforts are being made to produce a global standard. In addition to wireless voice service,
markets in paging services, cordless phones, and direct-to-home satellite broadcasting have been expanding.

The trends described above are sufficient to sustain a strong demand for wireless digital communication systems in the future. The major driving force for wireless in the 21st century, however, will likely lie in the increasingly popular Internet [7]. The 90s have witnessed the emergence of the Internet as a source for information access, a way to communicate (email, on-line chat rooms) and an opportunity for business. The impact of the Internet on the traditional telephone industry is far-reaching. For some time, it has been apparent that computing and telecommunications technologies are converging [8]. Today telephone companies not only provide voice but also data dial-up services. Future communication networks, using both wired and wireless interconnection, will seamlessly support a variety of services, including voice, video and data. Such huge information flow can only be realized with high speed transmission. Since present phone lines support data transmission at lower rates, high-speed wireless transmission is being developed as a possible choice for transmission from a central office to the local user.

People want high transmission speeds and considerable effort has been made to raise the transmission speed of networks. The increases in speed of digital wire-line modems during the past twenty five years have come about in the main with a corresponding increase in the number of signals in a two-dimensional (2-D) modulation format. Knowledge of the error performance of a 2-D signal set in additive white Gaussian noise (AWGN) is very useful in selecting a constellation that has the lowest error probability among its peers, or has suboptimum error performance with other advantages valuable to a particular system design, such as reduced implementation complexity. In the latter case, it is then desirable to know the error performance sacrificed in exchange for other gains. Evaluation of the exact symbol error rate (SER) or bit error rate (BER) of an arbitrary 2-D constellation, however, had not been readily available, since the double integral of a bivariate Gaussian noise distribution over an integration region with polygonal boundaries does not have a closed-form formula. The search for an optimum signal set in AWGN was presented in [9]-
using geometry and bounding techniques. The approximate symbol error probability expressions are rather accurate at high signal-to-noise ratios (SNR's), but are not precise for ranges of low SNR. It was not until 1991 that Craig proposed an ingenious solution to this seemingly straightforward yet long-standing elusive problem [12].

In this thesis, we extend Craig's method to applications in wireless fading environments. Mainly, we present new analytical methods and numerical programs that can be used to evaluate the performances of coherent digital communications on wireless fading channels: single channels and also channels employing diversity combining techniques. A literature review of each of these topics will be presented in the corresponding chapter. The theme is to provide new analytical tools and results that can be used by system designers and engineers to determine and compare the performances of two-dimensional coherent modulation schemes. This is essential to the development of high speed wireless data systems. Of the same importance, or maybe more important, is the fact that these tools can be used for a wide range of fading conditions and environments, and are not limited to a specific fading model.

For coherent detection and diversity combining, some channel information, i.e., fading channel amplitude and phase, has to be known. In practice, channel estimation is never perfect, and estimation errors degrade signalling performances. In this thesis, we also investigate the effects of channel estimation error, by considering constant automatic gain control (AGC) errors, as well as dynamic channel estimation error incurred in a pilot symbol aided system. Here AGC is used to scale and rotate the received signal according to the estimated channel amplitude and phase. The AGC error reflects the channel estimation error. We present analytical expressions, numerical and simulation results for error performances of coherent 2-D signalling sets in Rayleigh fading with channel estimation errors.

The efforts to extend Craig's approach to higher dimensional signalling result in new analytical expressions for 3-ary and 4-ary orthogonal signals in Rayleigh fading. Also presented are results for biorthogonal and transorthogonal signals.
This chapter is organised as follows. In Section 1.2, we review in some detail the characteristics of the radio propagation environment. The principles of diversity combining schemes are presented in Section 1.3. These two sections serve to help understand the terminology and concepts used in this thesis. In Section 1.4, we provide an outline and summarise the main contributions of this thesis.

1.2 Multipath Fading Environment

The mobile radio propagation environment places fundamental limitations on the performances of wireless radio systems. There are roughly three independent phenomena that together create a hostile transmission environment: path loss variation with distance, short-term (fast) multipath fading and slow log-normal shadowing. The underlying physical principles behind these three phenomenon are different. Path loss is due to the decay of electro-magnetic wave intensity in the atmosphere. Multipath fading is caused by multipath propagation, while slow shadowing is due to topographical variations along the transmission path. In this thesis, we will focus on multipath fading.

In wireless radio systems, several signals with different amplitudes, phases and delays corresponding to different transmission paths arrive at the receiver. The different signal components add at the receiver constructively or destructively to give the resultant signal. This multipath fading results in rapid variations typically as much as 30 to 40 dB in the envelope of the received signal over a distance corresponding to a fraction of a wavelength. The velocity of the mobile station, $v$, and the carrier frequency, $f_c$, determine the fading rate or fading bandwidth, $f_D$, that is, $f_D = \frac{v}{c} f_c$ where $c$ is the speed of light. Hence, faster motion leads to more rapid fading. Multipath also causes time dispersion, because the multiple replicas of the transmitted signal propagate over different transmission paths and reach the receiver antenna with different time delays. This is called frequency selective fading. On the other hand, the fading is said to be non-selective or flat if time delays in distinct paths are not large enough to result in resolvable replicas of the transmitted signal.
at the receiver antenna. From the frequency domain perspective, all frequency components of the transmitted signal undergo the same attenuation and phase shift through the channel in a frequency-flat fading channel. This thesis is concerned with frequency-flat multipath fading.

In a frequency-flat fading environment, the received signal is simply the transmitted signal multiplied by a complex-valued random process which introduces a fading envelope and a random phase to the transmitted signal. A number of different models have been proposed in the literature to describe the statistical behaviour of the fading envelope of the received signal [13]-[16]. Well known models are the Rayleigh, Ricean and Nakagami distributions. Next we discuss and review the theoretical origins and the characteristics of the three models.

1.2.1 Rayleigh Fading

In Rayleigh fading, the composite received signal consists of a large number of plane waves resulting from scattering at surface elements [15]. Using a central limit theorem, the received complex low-pass signal \( g(t) = \alpha(t) \exp(j\phi(t)) = g_I(t) + j g_Q(t) \) can be modelled as a complex Gaussian random process. In the absence of a line-of-sight (LOS) or specular component, \( g_I(t) \) and \( g_Q(t) \) have zero mean. At any time \( t \), \( g_I \) and \( g_Q \) are Gaussian random variables (RVs) with

\[
E[g_I] = E[g_Q] = 0, \quad \text{Var}[g_I] = \text{Var}[g_Q] = \sigma^2, \quad E[g_I g_Q] = 0, \quad (1.1)
\]

where \( E[x] \) denotes the expected value of \( x \) and \( \text{Var}[x] \) the variance of \( x \). Therefore, the envelope \( \alpha \) of the received signal has a Rayleigh distribution given by

\[
f_\alpha(\alpha) = \frac{\alpha}{\sigma^2} \exp \left( -\frac{\alpha^2}{2\sigma^2} \right), \quad \alpha \geq 0. \quad (1.2)
\]

The Rayleigh fading model agrees well with macrocellular field measurements over the frequency range from 50 to 11,200 MHz at distances of a few tens of wavelengths or greater where the mean signal is sensibly constant [17]. It usually applies to scenarios where there
is no LOS path between the transmitter and receiver antennas. The phase $\phi$ of the received signal is uniformly distributed from 0 to $2\pi$ at any time $t$, and the amplitude and phase are statistically independent. Define a new variable $\gamma_s = \alpha^2/N_0$ proportional to the squared envelope $\alpha^2$, where $N_0$ denotes the one-sided power spectral density of Gaussian noise. Variable $\gamma_s$ denotes received signal-to-noise ratio per symbol. The probability density function (PDF) of $\gamma_s$ is given by

$$f_{\gamma_s}(\gamma_s) = \frac{1}{\Lambda} \exp \left( -\frac{\gamma_s}{\Lambda} \right) u(\gamma_s)$$

(1.3)

where $\Lambda = \sigma^2/N_0$ is the average signal-to-noise ratio per symbol and $u(x)$ is the unit step function.

### 1.2.2 Ricean Fading

If there is a LOS or specular component between the transmitter and receiver, $g_I(t)$ and $g_Q(t)$ have non-zero mean $m_1$ and $m_2$ and the envelope $\alpha$ is a Ricean RV with PDF given by

$$f_{\alpha}(\alpha) = \frac{\alpha}{\sigma^2} \exp \left( -\frac{\alpha^2 + \mu^2}{2\sigma^2} \right) I_0 \left( \frac{\alpha \mu}{\sigma^2} \right), \quad \alpha \geq 0,$$

(1.4)

where $\mu^2 = m_1^2 + m_2^2$ is the non-centrality parameter, and $I_0(x)$ is the zero-order modified Bessel function of the first kind. Ricean fading is often observed in microcellular and satellite applications where a LOS path exists [18], [19]. The Rice $K$ factor is the ratio of the power in the specular and scattered components, that is, $K = \frac{\mu^2}{2\sigma^2}$. For $K = 0$, the channel exhibits Rayleigh fading, and for $K = \infty$, the channel has no fading. It was reported in [18] that a typical value of $K$ for practical microcellular channels is about $K = 7$ dB. Rice factor $K = 12$ dB was reported for a smaller number of cases. Values of Rice factor in outdoor and indoor systems usually range from 0 to 25 [20]. The PDF of SNR variable $\gamma_s$ in a Ricean fading channel can be expressed in terms of the Rice $K$ factor as

$$f_{\gamma_s}(\gamma_s) = \frac{K + 1}{\Lambda} \exp \left[ - \left( K + \frac{K + 1}{\Lambda} \gamma_s \right) \right] I_0 \left( 2\sqrt{\frac{K(K+1)\gamma_s}{\Lambda}} \right) u(\gamma_s)$$

(1.5)
where $\Lambda = (\mu^2 + 2\sigma^2)/N_0$ is the average SNR in Ricean fading.

The phase is no longer uniformly distributed but has a preferred value owing to the presence of a dominant component. The PDF of the phase is given in [21, Eqn. (5.62)].

1.2.3 Nakagami-$m$ Fading

The Nakagami-$m$ distribution was developed by Nakagami in the early 1940's to characterise rapid fading in long distance High-Frequency (HF) channels [14]. It was shown to sometimes have greater flexibility and accuracy in matching some experimental data than either the Rayleigh, Ricean, or log-normal distributions [22]-[24]. The Nakagami-$m$ distribution describes the received envelope amplitude by

$$f_\alpha(\alpha) = \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^m \alpha^{2m-1} \exp \left( -\frac{ma^2}{\Omega} \right), \quad \alpha \geq 0, \quad m \geq \frac{1}{2}$$

(1.6)

where $\Omega = E[\alpha^2]$ is the average power of Nakagami distributed $\alpha$, $\Gamma(x)$ is the gamma function, and the parameter $m$ is defined as the ratio of moments, called the fading parameter,

$$m = \frac{\Omega^2}{E[(\alpha^2 - \Omega)^2]}.$$  

(1.7)

The Nakagami-$m$ distribution is a generalised distribution that can be used to model different fading environments by changing the value of $m$. Rayleigh fading is obtained for $m = 1$, and a one-sided Gaussian RV is described by $m = 0.5$. The nonfading case corresponds to $m = \infty$. For values of $m$ in the range $\frac{1}{2} \leq m \leq 1$, (1.6) models fading conditions more severe than Rayleigh fading. For values of $m > 1$, (1.6) models less severe fading than Rayleigh. Furthermore, the Ricean distribution and log-normal distribution can sometimes be closely approximated by Nakagami fading under certain conditions [14], [22], [23]. A summary of Nakagami-$m$ distribution properties can be found in a review paper by Nakagami [14].

Using a transformation of random variables, the SNR $\gamma_s$ in Nakagami fading has PDF,

$$f_{\gamma_s}(\gamma_s) = \frac{\gamma_s^{m-1}}{m \Lambda} \left( \frac{m}{\Lambda} \right)^m \exp \left( -\frac{m\gamma_s}{\Lambda} \right) \mu(\gamma_s)$$

(1.8)

where $\Lambda = \frac{\Omega}{N_0}$ is the average SNR.
1.3 Principles of Diversity Combining

In order to improve the reliability of transmissions on wireless radio channels, some measures have to be employed to reduce the severity of multipath fading. Diversity techniques have been known to be effective in combating the extreme and rapid signal variations associated with the wireless radio transmission path. Basically, the diversity method requires that a number of transmission paths be available, all carrying the same message but having independent fading statistics. The mean signal strengths of the paths should also be approximately the same. Diversity can be achieved by methods that can be placed into seven categories [25]. In this section, we are concerned with space diversity where the distance between the receiving antennas is made large enough to ensure independent fading. Usually a spatial separation of about a half-wavelength will suffice (typically less than 30cm for frequencies above 500 MHz) [17]. Excellent references on the topic of diversity systems are [17],[25]-[29].

Diversity combining refers to the method by which the signals from the diversity branches are combined. There are several ways of categorising diversity combining methods. Predetection combining refers to diversity combining that takes place before detection, while postdetection combining takes place after detection. For diversity strategies incorporating signal summing, summing after detection can be either equal or inferior to summing before detection because a nonlinear effect is often experienced in a detection process [26]. With ideal coherent detection, there is no performance difference between predetection and postdetection combining [25]. In any case, diversity combining methods include maximal ratio combining (MRC), equal gain combining (EGC) and selection combining (SC).

The received signal from the $l$th diversity branch is represented by

$$z_l = g_l s_i + n_l = y_l + n_l$$

(1.9)

where $g_l = \alpha_l \exp(j\phi_l)$ is a complex channel gain, $s_i$ is the transmitted signal, $y_l = g_l s_i$ is the faded signal, and $n_l$ is additive Gaussian noise.
1.3.1 Maximal Ratio Combining

In this method, the individual branches must be first co-phased and weighted proportionately to their channel gain and then summed. This is equivalent to weighting each branch by the complex conjugate of its channel gain, i.e., \( c_l = g_l^* = \alpha_l \exp(-j\phi_l) \), where \( c_l \) is the weighting coefficient of the \( l \)th branch. It is well known that MRC results in a maximum likelihood (ML) receiver \([25]\) and gives the best possible performance among the diversity combining techniques. Fig. 1.1 shows general block diagrams of a coherent predetection and postdetection \( L \)-branch maximal ratio combiner.

The combined noiseless signal is

\[
y = \sum_{l=1}^{L} c_l y_l = s_i \sum_{l=1}^{L} \alpha_l^2,
\]

(1.10)

The noise powers \( P_n \) in all branches are assumed to be equal. Likewise, the total noise power is the sum of the noise powers in each branch, weighted by the branch gain factors,

\[
N_t = P_n \sum_{l=1}^{L} \alpha_l^2.
\]

(1.11)

The total SNR is

\[
\gamma_s = \frac{y y^*}{2N_t} = \frac{\sum_{l=1}^{L} \alpha_l^2 |s_i|^2}{2P_n} = \sum_{l=1}^{L} \gamma_l,
\]

(1.12)

the sum of the branch SNRs.

When all the diversity branches provide the same average power and they are uncorrelated, the PDF of the total SNR with MRC in Rayleigh fading is given by \([17]\)

\[
f_L(\gamma_s) = \frac{\gamma_s^{L-1} e^{-\gamma_s/\Lambda}}{\Lambda^L (L-1)!},
\]

(1.13)

where \( \Lambda \) is the average signal-to-noise ratio per branch.

In Ricean fading with MRC \([30, pp. 174]\),

\[
f_L(\gamma_s) = \frac{K_T/L + 1}{\Lambda} \left( \frac{K_T/L + 1}{K_T \Lambda} \gamma_s \right)^{L-1} \exp[-(K_T + K_T/L + 1) \gamma_l] \times I_{L-1} \left( 2 \sqrt{\frac{K_L (K_T/L + 1)}{\Lambda} \gamma_l} \right)
\]

(1.14)
Figure 1.1. Block diagram of coherent maximal ratio combining, (a) predetection (b) post-detection.
where $K_T = \sum_{l=1}^{L} K_l = \mu^2/(2\sigma^2)$, $K_l = \mu_l^2/(2\sigma^2)$, $\mu^2 = \sum_{l=1}^{L} \mu_l^2$, $\Lambda = (\mu^2/L + 2\sigma^2)/(2P_n)$ is the mean SNR per branch, and $I_{L-1}(x)$ is the $(L-1)$th-order modified Bessel function of the first kind.

In Nakagami-$m$ fading with MRC [31],

$$f_L(y_s) = \left[ y_s^{m_T-1} / \Gamma(m_T) \right] \left( \frac{m_T}{\Lambda_T} \right)^{m_T} \exp\left( -\frac{m_T y_s}{\Lambda_T} \right)$$

where $m_T = \sum_{l=1}^{L} m_l$ and $\Lambda_T = \sum_{l=1}^{L} \Lambda_l$. $\Lambda_l$ is the mean SNR of the $l$th branch, and the ratio $m_l/\Lambda_l$ is the same for all diversity branches.

### 1.3.2 Equal Gain Combining

Maximal ratio combining requires complete knowledge of channel branch gains. Equal gain combining is similar to MRC because the diversity branches are co-phased, but simpler than MRC as the gains are set equal to a constant value of unity. The block diagram for EGC is the same as Fig. 1.1 except for the weighting coefficients $c_l = \exp(-j\phi_l)$. That is, the channel estimator in EGC only needs to estimate the channel phase but not the channel amplitude. The performance of EGC is not as good as optimal MRC but is comparable to MRC. In practice, coherent postdetection EGC is useful for modulation schemes having equal energy symbols, such as M-ary phase shift keying (MPSK) because only channel phase information is required. For signals of unequal energy, complete channel knowledge is required for coherent detection and therefore postdetection MRC is usually used. Predetection EGC, however, still has merits to be used with unequal energy signals because apart from the $L$ channel phase estimators, predetection EGC only requires one AGC to estimate channel amplitude after the matched filter while predetection MRC would require $L$ AGC's to obtain $L$ channel amplitude knowledge.

The combined noiseless signal is given by

$$y = \sum_{i=1}^{L} \alpha_i s_i = s_i \sum_{i=1}^{L} \alpha_i$$

(1.16)
and the SNR of the combiner output is

\[ \gamma_s = \frac{yy^*}{2P_nL}. \]  

(1.17)

To find the probability distribution function of an EGC output \( y \) which is a sum of Rayleigh RV’s is a difficult task. In Chapter 3 we will adopt Beaulieu’s infinite series result for the distribution of the combiner output signal in Rayleigh, Ricean and Nakagami-\( m \) fading [32]-[34].

### 1.3.3 Selection Combining

Selection diversity is generally the simplest method of all. Its performance suffers some loss compared to MRC and EGC. Ideal selection combining chooses the branch giving the highest SNR at any instant. Fig. 1.2 illustrates the principle of selection combining. In practice, the branch with the largest (S+N) is usually selected, since it is difficult to measure SNR. For radio links using continuous transmission, e.g., frequency division multiple access (FDMA) systems, SC is not very practical, since it requires continuous monitoring of all the diversity branches. If such monitoring is performed, it is probably better to use maximal ratio combining since the implementation is marginally more complex and the performance is better [25]. In time division multiple access (TDMA) systems, however, a form of SC can be implemented where the diversity branch is selected prior to the transmission of a TDMA burst. The selected branch is then used for the transmission of the entire burst [25]. In the following analysis, we assume ideal continuous branch selection. As far as the statistics of the output signal are concerned, it is immaterial where the selection is done. The antenna signals could be sampled, for example, and the best one sent to the receiver.

Assuming all \( L \) diversity branches are independent and identically distributed (iid), the PDF of the signal amplitude at the output of a selection combiner in Ricean fading is given
Select largest SNR branch

Figure 1.2. Block diagram of predetection selection combining.

by [33]

\[ f_L(\alpha) = L[1 - Q(\sqrt{2K}, \alpha \sqrt{\frac{2(1+K)}{\Omega}})] L^{-1} 2\alpha(1+K) \]
\[ \times \exp \left( -K - \frac{1+K}{\Omega} \alpha^2 \right) I_0(2\alpha \sqrt{\frac{K(1+K)}{\Omega}}) \]  \hspace{1cm} (1.18)

where \( \Omega = E[\alpha^2] \) is the total signal power in each Ricean channel, and \( Q(a, b) = \int_0^\infty xe^{-\frac{x^2+b^2}{2}} I_0(ax)dx \) is the Marcum-Q function.

The selection combining Nakagami-\( m \) distributed \( \gamma_s \) is derived for iid diversity branches as

\[ f_L(\gamma_s) = \frac{m^m}{\Gamma(m) L} \gamma_s^{m-1} e^{-\frac{\gamma_s}{\Lambda}} [\gamma(m, \frac{\gamma_s}{\Lambda})]^L-1 \]  \hspace{1cm} (1.19)

where \( \gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt \) is the incomplete gamma function [35, 8.35] and \( \Lambda = E[\gamma_s] \).

For selection combining and Rayleigh fading, the PDF of \( \gamma_s \) assuming iid diversity branches is given by [17]

\[ f_L(\gamma_s) = \frac{L}{\Lambda}(1 - e^{-\gamma_s/\Lambda})^{L-1} e^{-\gamma_s/\Lambda} \]  \hspace{1cm} (1.20)
1.4 Thesis Outline and Contributions

In Chapter 2, we introduce Craig's approach [12] to computing the exact average symbol error rate of coherent two-dimensional constellations with polygonal decision boundaries in AWGN. The precise SER formula is well suited to numerical evaluation with ease and high accuracy, as it is in the form of a single integral with finite integration interval and an elementary function integrand. New exact performances of six commonly known 8-ary and eleven 16-ary signalling formats for a large range of average SNR and peak SNR are obtained and plotted. The SER expression also enables optimisation of parameters such as the ring ratio of a circular constellation to achieve the lowest symbol error rate in AWGN. For example, a star constellation such as that specified in the CCITT V.29 standard can be improved by adjusting the amplitude ratios of the points in the constellation to save about 0.63 dB power in AWGN without sacrificing the phase error tolerance, while maintaining the same error rate. The optimum ring ratios as functions of SNR of six circular signal sets in AWGN are presented for the first time.

In Chapter 3, we present as an extension of Craig's method for AWGN channels a new exact method for computing the average symbol error probability of two-dimensional \( M \)-ary signalling in slow fading. The method is generally applicable to polygonal decision regions. The performances of various coherent 8-ary and 16-ary modulations in slowly fading channels are analysed. Moreover, new expressions for the exact symbol error rates in fading with diversity combining are derived for any two-dimensional signalling format having polygonal decision boundaries. Maximal ratio combining, equal gain combining and selection combining are considered. As an example, new analytical symbol error rate expressions (in closed-form or single integral form) are presented for coherent 16 star-QAM in slowly fading channels. The SER formulae obtained make it possible for the first time to optimise parameters of various constellations precisely and to determine which constellation has the lowest probability of error. New numerical results for optimum ring ratios of various 8-ary and 16-ary circular constellations in fading with and without diversity reception are given. The sensitivity of each constellation to phase error and amplitude error...
is also presented and comparisons are made. The 8-ary and 16-ary signal sets studied in Chapter 2 are examined using the new symbol error probability formulae to determine best signal sets for fading channels. Ideal gain control is assumed in this chapter.

In Chapter 4, we study the effect of automatic gain control error on the performance of 2-D constellations, where there is a constant phase displacement and amplitude error in the estimation of the fading channel. Rayleigh fading is considered. The SER of an arbitrary 2-D constellation with fixed AGC error in Rayleigh fading is found to be expressed by a single integral with finite integration limits, and an expression involving elementary functions as integrand. Numerical results match well with the geometric phase error margins and amplitude error margins tabulated in Chapter 3.

In Chapter 5, we focus on the study of a practical channel estimation method, i.e. a pilot symbol aided modulation (PSAM) scheme used in the coherent detection of two-dimensional signalings. Known pilot symbols are periodically inserted into the data symbols to assist channel estimation and the estimated channel information is used to coherently detect transmitted data symbols. The statistical distribution of the dynamic channel estimation error of PSAM in Rayleigh fading is available in the literature and we apply the probability distribution to the analysis of the SER of the pilot symbol aided 2-D signalling. The analysis, however, results in complicated numerical solutions and therefore, a simulation approach has been used to obtain the error performances of these pilot symbol aided 2-D signalling systems.

In Chapter 6, we investigate the error performance of higher-dimensional modulation formats in Rayleigh fading. Craig's method can not be directly applied to higher-dimensional signalings, and more complicated coordinate transformations are required. New exact expressions for symbol error rates and bit error rates for coherent 3-ary and 4-ary orthogonal and transorthogonal signalings in Rayleigh fading channels are derived. New exact error probability expressions for coherent 6-ary and 8-ary biorthogonal signalings in Rayleigh fading are also presented. The use of these exact expressions as close approximations for the error rates of M-ary orthogonal, biorthogonal and transorthogonal
signalling with arbitrary $M$ is illustrated.

In Chapter 7, we present a summary of the conclusions of the thesis and propose some suggestions for further research.
Chapter 2

Probability of Symbol Error for
Two-dimensional Signallings in AWGN

2.1 Introduction

In search of efficient modulation formats for high speed wire-line data transmission, a considerable amount of work has been done on determining two-dimensional $M$-ary signal constellations which minimise the probability of error in the presence of additive white Gaussian noise under an average power or peak power constraint [9]-[11],[36]. However, an explicit expression for the symbol error rate has been, in general, difficult to obtain for arbitrary decision regions when the number of signals is greater than two. Asymptotic error expressions assuming large signal-to-noise ratio, or some bounding techniques have been used to approximate the symbol error rate [9], [10]. The absence of a simple tractable error probability evaluation technique has prevented a highly precise analytical determination of the performances of those modulations combining both amplitude and phase-shift keying.

Recently, Craig [12] has given a new method for computing the average error probability of two-dimensional, $M$-ary signalling in additive white Gaussian noise. The method is generally applicable to signal sets whose decision region boundaries are polygons surrounding each of the signal points in the constellation, which includes most practical signal
sets. The symbol error probability expression is in the form of a single integral over a finite interval with an integrand containing only elementary functions. Craig's approach is also well elaborated in [37, pp. 137-140].

In this chapter in Section 2.2, we first present the system model for transmission on the AWGN channel. Then in Section 2.3, we introduce Craig's method for the symbol error rate of arbitrary 2-D signal sets in AWGN channels. The SER expressions for widely used modulation formats \( M \)-ary Phase Shift Keying (MPSK) and \( M \)-ary Quadrature Amplitude Modulation (MQAM) are presented as well. Next, as an example, the SER of 16 star-QAM is derived in Section 2.4. Section 2.5 discusses and compares the error performances of six 8-ary and eleven 16-ary constellations in AWGN. Finally, conclusions are given in Section 2.6.

### 2.2 System Model

A block diagram of a baseband equivalent communication system for additive white Gaussian noise channels is illustrated in Fig. 2.1. A more detailed receiver structure is shown in Fig. 2.2. In an AWGN channel, the received signal can be written as

\[
z(t) = \alpha s(t) + n_w(t) = y(t) + n_w(t), \quad 0 \leq t \leq T
\]  

(2.1)

where \( \alpha = \sqrt{E_s} \) denotes the square root of the average received energy per symbol, \( n_w(t) \) is the additive white Gaussian noise with one-sided power spectral density \( N_0 \), and \( s(t) = \{s_i(t)\}, i = 1 \ldots M \) represents the transmitted signal with unit average energy. In combined amplitude and phase modulation, the transmitted bandpass signal is given by

\[
s_i(t) = \sqrt{2p(t)}A_i \cos(\omega_c t + \zeta_i)
\]

\[
= \sqrt{2I_i p(t)} \cos 2\pi f_c t - \sqrt{2Q_i p(t)} \sin 2\pi f_c t, \quad 0 \leq t \leq T
\]

\[
I_i = A_i \cos \zeta_i, \quad Q_i = A_i \sin \zeta_i,
\]  

(2.2)
where \( p(t) \) denotes a pulse shape which has zero intersymbol interference, such as a Nyquist pulse (with zero timing error), and the energy of \( p(t) \) is normalised to one, i.e., \( E_p = 1 \). Parameter \( A_i \) and \( \xi_i \), \( i = 1, \ldots, M \), are the modulated amplitude and phase level of the \( i \)th symbol, and \( I_i \) and \( Q_i \) are the corresponding in-phase and quadrature components of the symbol in an \( M \)-ary modulation scheme. The lowpass equivalent or complex envelope of \( s_i(t) \) is given by

\[
s_{ii}(t) = \sqrt{2} A_i e^{j \xi_i} p(t).
\] (2.3)

The energy of signal \( s_i(t) \) is given by \( A_i^2 E_p = I_i^2 + Q_i^2 \). Hence the average energy of \( s(t) \) over \( M \)-ary signals is \( \sum_{i=1}^{M} P(s_i(t))(I_i^2 + Q_i^2) = 1 \) by assumption, where \( P(s_i(t)) \) is the a priori probability that signal \( s_i(t) \) is sent. The received noiseless signal \( y(t) = \alpha s(t) \) therefore has average energy \( E_s \).

It is well-known that in an \( M \)-ary 2-D signalling system, the waveform of the transmitted signal can be expressed as linear combinations of a set of two orthonormal functions. The noise waveform \( n_w(t) \) and hence received signal \( z(t) \) require a signal space of infinite dimension to represent them. However, as is usual in detection theory, it was shown in [38], [1, pp. 229-232] that projections of the noisy waveform on to the two orthonormal bases
form sufficient statistics for the optimum determination of the received message. Therefore,

\[ z(t) = z_1 f_1(t) + z_2 f_2(t) = (\alpha s_{i1} + x_1) f_1(t) + (\alpha s_{i2} + x_2) f_2(t) \quad (2.4) \]

where \( f_1(t) \) and \( f_2(t) \) are two orthonormal functions given by

\[
\begin{align*}
  f_1(t) &= \sqrt{2} p(t) \cos 2\pi f_c t, \\
  f_2(t) &= \sqrt{2} p(t) \sin 2\pi f_c t,
\end{align*}
\]

and

\[
\begin{align*}
  s_{i1} &= I_i, & s_{i2} &= Q_i, \\
  z_k &= \int_0^T z(t) f_k(t)dt, \\
  x_k &= \int_0^T n_w(t) f_k(t)dt, \quad k = 1, 2.
\end{align*}
\]

This leads to vector representation of waveforms

\[ z = \alpha s_i + n \quad (2.6) \]

where \( s_i = (s_{i1}, s_{i2}) \), \( z = (z_1, z_2) \) and \( n = (x_1, x_2) \). It was shown in [1, p. 232] that the joint probability density function of Gaussian noise \( n = (x_1, x_2) \) is given by

\[ p_n(x_1, x_2) = \frac{1}{\pi N_0} \exp\left(\frac{-x_1^2 + x_2^2}{N_0}\right). \quad (2.7) \]

In polar coordinates, the PDF of Gaussian noise \( n = (r, \theta) \) is expressed as

\[ p_n(r, \theta) = \frac{r}{\pi N_0} \exp\left(-\frac{r^2}{N_0}\right) \quad r \geq 0, \quad 0 \leq \theta \leq 2\pi. \quad (2.8) \]

As shown in Fig. 2.2, two receiver filters matched to the two orthonormal bases are employed. At sampling time \( t = T \), the outputs of the matched filters are \( z_1 \) and \( z_2 \). Assume symbol timing is perfect and the pulse shape \( p(t) \) selected results in zero intersymbol interference. The optimum maximum likelihood detection selects the signal point from a constellation which has the minimum Euclidean distance to the output of the matched filter. Implementation of this decision rule is illustrated in Fig. 2.2.
2.3 Symbol Error Probability of 2-D Signallings in AWGN Channels

Fig. 2.3 illustrates a received noiseless symbol $\alpha s_i$ in 2-D signal space and its decision boundaries. Remember that transmitted symbol $s_i$ is one of the $M$ signals in a signal set which has normalised unit average energy. The square root of the average received energy is given by $\alpha = \sqrt{E_s}$. In general, the decision boundaries define either closed or open regions. If the 2-D Gaussian noise superimposed on $\alpha s_i$ results in the received signal falling outside the appropriate decision boundaries, a symbol error occurs. The correct decision region given $s_i$ sent can be divided into trilaterals (triangles or triangles with a vertex at infinity, see Fig. 2.3(b)). These trilaterals are generated by straight lines originating at the symbol $\alpha s_i$ and terminating at the intersection of two decision boundaries. The erroneous decision region given $s_i$ is therefore composed of disjoint subregions 1, 2, 3 and 4, as illustrated in Fig. 2.3.

Consider shifting the origin of the coordinates to $\alpha s_i$ and a following change from Cartesian coordinates to polar coordinates. Using (2.8), the probability that $z$ falls into
Figure 2.3. Two types of typical decision regions of a signal point: (a) closed region, (b) open region.

Erroneous subregion 1 is given exactly by

\[ P_{g,1} = P_{g,1}(\alpha) = \int_0^\eta \int_{R(\theta)}^\infty p_r(r,\theta) dr d\theta \]
\[ = \frac{1}{2\pi} \int_0^\eta \exp \left[ -\frac{R^2(\theta)}{N_0} \right] d\theta \]  

(2.9)

where

\[ R(\theta) = \frac{x' \sin \psi}{\sin(\pi - \theta - \psi)} = \frac{x' \sin \psi}{\sin(\theta + \psi)} \]  

(2.10)

and \( x' = \sqrt{3}\alpha \) is the distance shown in Fig. 2.3. We use parameter \( \sqrt{3} \) to represent the distance \( x' \) when amplitude \( \alpha \) is one. Hence the average SNR per symbol is given by

\[ \gamma_s = \frac{\alpha^2}{N_0} = \frac{E_s}{N_0}. \]  

(2.11)

Parameters \( \eta \) and \( \psi \) in (2.9) are determined by the geometry of the trilateral corresponding to the subregion, and \( \theta \) is a dummy variable of integration. In summary, the probability of error for subregion 1 can be written either in amplitude \( \alpha \) or SNR \( \gamma_s \) as

\[ P_{g,1} = P_{g,1}(\alpha) = \frac{1}{2\pi} \int_0^\eta \exp \left[ -\frac{b\alpha^2 \sin^2 \psi}{N_0 \sin^2(\theta + \psi)} \right] d\theta, \]  

(2.12)
\[ P_{g,i}(\gamma_s) = \frac{1}{2\pi} \int_0^{\eta} \exp \left[ -\frac{b\gamma_s \sin^2 \psi}{\sin^2(\theta + \psi)} \right] d\theta. \]  

(2.13)

Eqs. (2.12) and (2.13) are well suited to numerical evaluation in that they have finite integration limits and an exponential integrand.

The detection error probability given symbol \( s_i \) is sent, is then the sum of probabilities that the received signal falls into erroneous subregions 1, 2, 3 and 4 in Fig. 2.3, i.e.,

\[ P(e|s_i) = \sum_{j=1}^{4} P_{g,j}(\alpha) = \sum_{j=1}^{4} P_{g,j}(\gamma_s). \]  

(2.14)

The probability of error given any other symbol in the constellation is transmitted, is obtained similarly. The exact average probability of symbol error for \( M \)-ary data in AWGN is the weighted sum of probabilities for all subregions of every possible signal point as given by

\[ P_e = \sum_{i=1}^{M} P(e|s_i)P(s_i) = \sum_{j=1}^{J} \frac{w_j}{2\pi} \int_0^{\eta_j} \exp \left[ -\frac{b_j\alpha^2 \sin^2 \psi_j}{N_0 \sin^2(\theta + \psi_j)} \right] d\theta \]  

(2.15)

where \( P(s_i) = w_j \) is the a priori probability of the symbol \( i \) to which subregion \( j \) corresponds, \( b_j, \psi_j \) and \( \eta_j \) are parameters corresponding to subregion \( j \) as defined in the previous paragraph, and \( J \) is the total number of erroneous subregions for all the signal points in the constellation. Since most constellations have symmetry, the number of subregions with distinct geometry is usually smaller than \( J \). Note that there is a factor of 2 difference between [12, Eqs. (3), (5) and (13)] and the three equations above. It is readily confirmed that [12, Eqs. (3) and (5)] are correct since they give the SER of MPSK which has a symmetric decision region and are twice the probability for one trilateral region as defined by (2.9) or (2.13). In [12, Eq. (13)], however, a factor of 2 is missing (\( \frac{1}{2\pi} \) instead of \( \frac{1}{\pi} \)).

Eq. (2.15) shows that average error probabilities that otherwise must be expressed as a sum of \( M \) double integrals, are now expressed as a sum of single integrals, for an arbitrary 2-D constellation. Also, this simple method gives the probability of error directly, compared to conventional techniques where a more complicated single or double integral is evaluated and the result is subtracted from unity as explained in [37, pp. 130-131]. To
achieve the same accuracy. Fewer significant figures are required in the numerical evaluation of Eq. (2.15) than conventional techniques. For small symbol error probability, the difference is quite significant.

We give probability of error expressions for some widely used signal sets using the above method. The SER of MPSK is given by [12]

$$P_M = \frac{1}{\pi} \int_0^{\pi/\sqrt{2}} \exp \left[ -\gamma \frac{\sin^2(\frac{\pi}{M})}{\sin^2 \theta} \right] d\theta. \quad (2.16)$$

BPSK is a special case of MPSK with $M = 2$,

$$P_2 = \frac{1}{\pi} \int_0^{\pi/2} \exp \left[ -\gamma \frac{\sin^2 \theta}{\sin^2 \theta} \right] d\theta, \quad (2.17)$$

while the BER/SER of BPSK is previously known as [39]

$$P_2 = Q(\sqrt{2}\gamma_b) = \frac{1}{2} \text{erfc}(\sqrt{2}\gamma_b) \quad (2.18)$$

where $Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) dt$ is the Gaussian tail integral and erfc(x) the complementary error function. From here, a new expression for the $Q$-function is obtained as [12]

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp \left( -\frac{x^2}{2\sin^2 \theta} \right) d\theta, \quad x \geq 0. \quad (2.19)$$

This new expression for the $Q$-function is a finite integral and its argument is no longer in its upper or lower limit but in the exponential integrand, which is not only useful for numerical evaluation purposes, but also facilitates some analyses in fading environments. Applications of (2.19) can be found in [40].

The symbol error probability of MQAM in AWGN is given by

$$P_{MQAM} = \frac{\pi}{4} \left( 1 - \frac{1}{\sqrt{M}} \right) \int_0^{\pi/2} \exp \left( -\frac{3\gamma_s}{2(M-1)\sin^2 \theta} \right) d\theta$$

$$- \frac{\pi}{4} \left( 1 - \frac{1}{\sqrt{M}} \right)^2 \int_0^{\pi/2} \exp \left( -\frac{3\gamma_s}{2(M-1)\sin^2 \theta} \right) d\theta \quad (2.20)$$

as opposed to [39]

$$P_{MQAM} = 4 \left( 1 - \frac{1}{\sqrt{M}} \right) Q \left( \sqrt{\frac{3\gamma_s}{M-1}} \right) - 4 \left( 1 - \frac{1}{\sqrt{M}} \right)^2 Q^2 \left( \sqrt{\frac{3\gamma_s}{M-1}} \right). \quad (2.21)$$
Therefore, a new expression for $Q^2(x)$ is obtained as [40]

$$Q^2(x) = \frac{1}{\pi} \int_0^{\pi} \exp \left( -\frac{x^2}{2\sin^2\theta} \right) d\theta, \quad x \geq 0. \quad (2.22)$$

### 2.4 Error Probability of 16 Star-QAM

As an example, we will show in this section how to calculate the exact symbol error probability of 16 star-QAM in AWGN by using the formulae derived in Section 2.3. The signal constellation of 16 star-QAM is illustrated in Fig. 2.4. Decision boundaries are as shown in the figure. Two kinds of decision regions correspond to inner ring signal points with radius $r_L$ and outer ring points with radius $r_H$. The ring ratio is $\beta = r_H/r_L$ and equally-likely transmission of signals is assumed.
From symmetry, the average symbol error probability can be written in terms of the error probability when signal $S_0$ is transmitted, $P_{S_0}$, and the error probability when signal $S_{12}$ is transmitted, $P_{S_{12}}$. These are given by

$$P_{S_0} = \frac{1}{\pi} \sum_{k=1}^{2} \int_{0}^{\eta_k} \exp \left[ -\frac{x_k^2 \sin^2 \psi_k}{N_0 \sin^2(\theta + \psi_k)} \right] d\theta$$

(2.23)

and

$$P_{S_{12}} = \frac{1}{\pi} \sum_{k=3}^{4} \int_{0}^{\eta_k} \exp \left[ -\frac{x_k^2 \sin^2 \psi_k}{N_0 \sin^2(\theta + \psi_k)} \right] d\theta$$

(2.24)

where $x_k, k = 1, \ldots, 4$ are lengths, and $\eta_k$ and $\psi_k$ are angles useful for the analysis and are as shown in Fig. 2.4. The average symbol error probability of 16 star-QAM is given by

$$P_e = \frac{1}{2\pi} \sum_{k=1}^{4} \int_{0}^{\eta_k} \exp \left[ -\frac{x_k^2 \sin^2 \psi_k}{N_0 \sin^2(\theta + \psi_k)} \right] d\theta = \frac{1}{2\pi} \sum_{k=1}^{4} \int_{0}^{\eta_k} \exp \left[ -\frac{b_k \alpha^2 \sin^2 \psi_k}{N_0 \sin^2(\theta + \psi_k)} \right] d\theta$$

(2.25a)

where

$$\eta_1 = \eta_3 = \tan^{-1}[(\sqrt{2} - 1) \frac{\beta + 1}{\beta - 1}],$$

(2.25b)

$$\eta_2 = \pi - \eta_1, \quad \eta_4 = \frac{7\pi}{8} - \eta_1,$$

(2.25c)

$$\psi_1 = \psi_3 = \frac{\pi}{2} - \eta_1, \quad \psi_2 = \frac{\pi}{8}, \quad \psi_4 = \frac{\pi}{8} + \eta_1,$$

(2.25d)

$$b_1 = b_3 = b_4 = \frac{1}{2(\beta^2 + 1)}[(\beta - 1)^2 + (\sqrt{2} - 1)^2(\beta + 1)^2], \quad b_2 = \frac{2}{\beta^2 + 1}$$

(2.25e)

$$\frac{\alpha^2}{N_0} = \gamma_b = 4\gamma_b$$

(2.25f)

where $\gamma_b = E_b/N_0 = \gamma_b/\log_2 M$ is the average received signal-to-noise ratio per bit. Result (2.25) is exact and, to the best of the author's knowledge, new. Note that the error probability which is a function of the decision boundaries will depend on the parameter $T$ (indicated in Fig. 2.4) and on the ring ratio $\beta$. The optimum parameter $T$ is readily determined to be at the midpoint of the radius between a signal on the inner ring and a signal on the outer ring by [1, Section 4.2]. The optimum ring ratio, $\beta_{OPT}$, is a function of the SNR. The exact error rate expressions derived here are well suited to numerically optimising the ring ratio.
2.5 Two-Dimensional 8-ary and 16-ary Signalling

In this section, we apply the formulae of Section 2.3 to the performance analysis and comparison of seventeen conventional 8-ary and 16-ary modulation formats in AWGN. The performances of these constellations in slow fading will be studied in subsequent chapters. A MATLAB program has been written to accept any two dimensional constellation as input and to then calculate the exact symbol error probability automatically by separating the decision regions into the appropriate trilaterals. This greatly facilitates the analysis. The main MATLAB source codes and the use of the program are presented in detail in Appendix A. In [12], Craig presented new SER results versus peak $E_b/N_0$ for three 16-ary constellations: rotated (8,8), (4,12) and CCITT V.29. Except for simple modulation formats such as 8PSK, 16PSK and 16 rectangular-QAM whose SER’s are known and the ones studied in [12], the precise symbol error rates of many 2-D signalling constellations in AWGN have not been reported before. Here we give new precise SER’s for eleven constellations. Also presented for the first time are the optimum ring ratios for the circular constellations in AWGN.

2.5.1 Eight-ary modulation formats

The 8-ary signal sets considered are 8PSK, the 8-ary max-density set proposed in [10] and [11], rectangular, (4,4), triangular, and (1,7), as shown in Fig. 2.5. The constellation plots in Fig. 2.5 are not scaled to have identical power but are presented to show the particular structures. The last four constellations mentioned above were studied in both [9] and [10]. The exact symbol error probability curves of these 8-ary constellations as a function of average SNR per bit are plotted for AWGN in Fig. 2.6 using the formulae derived in Section 2.3. To the best of the authors’ knowledge, except for the SER of 8PSK, these error probability results are new.

Before we proceed to compare constellations, the parameter of set (4,4) needs to be optimised. The performance of circular constellations obviously depends on the ratio of the radii of the outer and inner rings, as demonstrated by (2.25) for 16 star-QAM. An optimum
Figure 2.5. Signal constellations of 8-ary signal sets: (a) 8PSK (b) Rectangular (c) (4,4) (d) Triangular (e) (1,7) (f) Max-density.
Figure 2.6. Average symbol error probabilities of 8-ary signal sets versus average SNR per bit in AWGN channels.
ratio exists which yields the lowest error probability for a given SNR. At asymptotically large SNR, the optimum ring ratio can be determined by constellation geometry. In general, the asymptotic optimum ring ratio is the ratio that results in the minimum distance between inner ring signal points being equal to the minimum distance between an inner ring point and an outer ring point. Any other ring ratio will give a smaller minimum Euclidean distance for the dominant (at large SNR) error event when the average signal power remains constant and hence larger probability of error. The optimum ring ratio of (4,4) in AWGN is plotted in Fig. 2.10 on Page 35 together with circular 16-ary constellations which will be discussed shortly in the next subsection.

Fig. 2.6 gives the SER’s of the 8-ary signal formats as a function of average SNR per bit in AWGN. It clearly shows that the order of performance from worst to best is 8PSK, rectangular, (4,4), triangular, the max-density set and (1,7) for the range of SER from $10^{-1}$ to $10^{-4}$ in AWGN. Similar performance ordering was reported in [9, Fig. 7] without considering the max-density set. It is observed from both Fig. 2.6 and [9, Fig. 7] that constellation (1,7) outperforms the triangular constellation in the range of small to medium SNR. When the SNR is very large and correspondingly the SER is smaller than $10^{-8}$, however, the triangular constellation does give lower symbol error probability than (1,7) in AWGN as expected. The SER curves of the triangular set and (1,7) have a crossover point at $\frac{E_b}{N_0} = 14.5$dB, and the max-density set crosses (1,7) at $\frac{E_b}{N_0} = 10.6$dB. Asymptotically, the max-density set has the lowest SER followed by the triangular set and (1,7). This is expected since the max-density set has the densest packing and therefore optimum performance in AWGN at large SNR. References [9, Fig. 7] and [10, Fig. 8] demonstrated similar relative performance for medium to large SNRs using the approximate SER expressions derived there, though each of them considered fewer signal sets and a smaller SNR range. Table 2.1 gives the SNR per bit required to achieve a SER= $10^{-6}$ for the five 8-ary constellations in AWGN. From the exact SERs obtained, we have a clarified understanding of the performances of these 8-ary signalling formats for all SNR’s.

Fig. 2.7 depicts the SER of 8-ary signal sets versus the peak SNR per bit in AWGN. The
2.5.2 Sixteen-ary modulation formats

The 16-ary constellations considered are illustrated in Figs. 2.8-2.9. Among them, the hexagonal constellation, rotated (8,8), (4,12), triangular, rectangular, (1,5,10) and (5,11) constellation, were studied for AWGN channels in [9], using approximate symbol error probability formulae. What we have termed the V.29 constellation was specified in CCITT recommendation V.29 for 9600 bits per second transmission over wire-line channels [41, p. 243]. Similar star constellations such as 16 star-QAM are now of interest for fading applications [42]. Modulation format (4,4,4,4) is of the same form as the V.29 constellation but the radii of the four rings, $r_i$, $i = 1..4$, are optimised for large SNR and the AWGN channel. That is, $\frac{\sigma}{r_1} = 1.932$, $\frac{\sigma}{r_2} = 2.414$ and $\frac{\sigma}{r_3} = 3.346$. We present the precise symbol error probabilities of these signalling formats for AWGN in this subsection. We give also the SER of the 16-ary max-density signalling format proposed in [10] and [11]. This constellation gives the best error performance among 16-ary signal sets in AWGN for large average SNR.

The optimum ring ratios of (4,4), (5,11) (4,12), 16 star-QAM, rotated (8,8) and (1,5,10) are plotted in Fig. 2.10 as a function of SNR per bit in AWGN. They approach asymptotic values at large SNR. A straightforward analysis of 16 star-QAM leads to $\beta_{OPT} = \ldots$
Figure 2.7. Average symbol error probabilities of 8-ary signal sets versus peak SNR per bit in AWGN channels.
Figure 2.8. Signal constellations of 16-ary signal sets: (a) 16 star-QAM (b) Hexagonal (c) V.29 (d) Rotated (8,8) (e) (4,12) (f) (4,4,4,4).
Figure 2.9. Signal constellations of 16-ary signal sets: (g) Triangular (h) Rectangular (i) (1,5,10) (j) (5,11) (k) Max-density.
TABLE 2.2
ASYMPTOTICALLY OPTIMUM AWGN RING RATIOS AND RING RATIOS USED BY THOMAS ET AL

<table>
<thead>
<tr>
<th>Constel.</th>
<th>$\beta_{OPT}$</th>
<th>ring ratios in [9]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4,4)</td>
<td>1.93185</td>
<td>1.932</td>
</tr>
<tr>
<td>(5,11)</td>
<td>2.175</td>
<td>2.0164</td>
</tr>
<tr>
<td>(4,12)</td>
<td>2.732</td>
<td>2.7298</td>
</tr>
<tr>
<td>(8,8)</td>
<td>1.5867</td>
<td>1.6628</td>
</tr>
<tr>
<td>(1,5,10)</td>
<td>1.902</td>
<td>2</td>
</tr>
</tbody>
</table>

$\sqrt{2 - \sqrt{2}} + 1 = 1.76537$, which agrees with the result of a numerical search for the optimum ring ratio in AWGN. The ring ratios presented in Thomas et al.'s paper [9] are close to the asymptotically optimum ring ratios found in [43]. The ring ratio differences lead to very slight improvement of about 0.02 dB per bit at a SER of $10^{-6}$. At a SER of $10^{-2}$, the maximum power per bit saved is around 0.14 dB when the optimum ring ratios (not asymptotically optimum) are used in contrast to the ring ratios presented in [9].

The exact SERs of 16-ary constellations in AWGN are plotted in Figs. 2.11-2.14, as a function of average and peak SNR.

Table 2.3 provides the SNR required per bit for the various constellations to attain a target symbol error probability of $10^{-6}$ in the AWGN channel. Fig. 9 in [10] shows similar relative performance among the rotated (8,8), (1,5,10), rectangular and the 16-ary optimum constellations. One interesting comparison is between V.29 and (4,4,4,4). It is observed that (4,4,4,4) with the optimised ring ratios gives 0.63 dB improvement per bit over V.29, without a sacrifice in peak power performance, amplitude error tolerance and phase error tolerance, as will be discussed in the next chapter. This is a significant difference.
Figure 2.10. Optimum ring ratios of circular 8-ary and 16-ary signal sets in AWGN.
Figure 2.11. Average symbol error probabilities of 16-ary signal sets (Set 1) versus average SNR in AWGN channels.
Figure 2.12. Average symbol error probabilities of 16-ary signal sets (Set 2) versus average SNR in AWGN channels.
Figure 2.13. Average symbol error probabilities of 16-ary signal sets (Set 1) versus peak SNR in AWGN channels.
Figure 2.14. Average symbol error probabilities of 16-ary signal sets (Set 2) versus peak SNR in AWGN channels.
TABLE 2.3
SNR PER BIT REQUIRED TO ACHIEVE SER=10^{-6} IN AWGN

<table>
<thead>
<tr>
<th>$\frac{E_b}{N_0}$ (dB)</th>
<th>Star-QAM</th>
<th>Hexagonal</th>
<th>V.29</th>
<th>(8,8)</th>
<th>(4,12)</th>
<th>(4,4,4,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>16.235</td>
<td>15.96</td>
<td>15.83</td>
<td>15.69</td>
<td>15.435</td>
<td>15.2</td>
</tr>
<tr>
<td>Triangular</td>
<td>Rectangular</td>
<td>(1,5,10)</td>
<td>(5,11)</td>
<td>Max-density</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{E_b}{N_0}$ (dB)</td>
<td>14.97</td>
<td>14.895</td>
<td>14.865</td>
<td>14.805</td>
<td>14.42</td>
<td></td>
</tr>
</tbody>
</table>

2.6 Conclusions

In this chapter, we have elaborated Craig’s method for calculating the exact symbol error probability of arbitrary $M$-ary two-dimensional constellations in AWGN. The results are applicable to general 2-D signal sets with polygonal decision boundaries. The analytical SER expressions do not require intensive numerical computations to achieve high accuracy. The error performances of various 8-ary and 16-ary signalling formats have been readily obtained as functions of all ranges of average SNR and peak SNR. With this powerful tool we have been able to optimise parameters of 2-D signal sets such as ring ratios of circular constellations. One significant example is the CCITT V.29 constellation. By optimising its ring ratios, a power savings of about 0.63 dB per bit can be achieved with no extra implementation complexity required. Among 16-ary constellations, the max-density constellation and the (5,11) constellation stand out as achieving low error rate in AWGN channels.
Chapter 3

Two Dimensional Signalling
Constellations for Fading Channels

3.1 Introduction

Demands for faster data rates on wireless and cellular channels have led to much current interest in the use of two-dimensional $M$-ary signalling formats and in the use of diversity techniques. The error probability analysis of constant envelope modulation $M$-ary phase shift keying (MPSK) has been carried out for various channel and system models [44]-[49]. In [50] and [51], the performances of non-constant envelope coherent modulations such as rectangular $M$-ary quadrature amplitude modulation (MQAM) over AWGN and slowly fading channels have been studied taking into account amplitude and phase reference errors. In [52], the performances of MQAM in Rayleigh fading with maximum ratio diversity combining and selection diversity combining have been derived in closed-form. The binary digital modulation performance in Rayleigh fading with equal gain combining was studied in [53] and [54]. Recently, a number of papers have been published on the performance analysis of MQAM in generalised fading channels with diversity combining [55]-[57]. However, accurate performance analyses of general coherent two-dimensional signal constellations in fading environments have not been reported, particularly for di-
versity systems. Yet, current fast digital signal processors make implementation of more complex modulation formats with diversity combining economical.

In this chapter, Craig's method for determining the average error probability for the AWGN case is extended to give new results for determining average probabilities of symbol error for two-dimensional $M$-ary signalling in slow fading. The results are used to determine the exact average symbol error probability of a range of signal constellations that are candidates for higher rate transmission on fading channels with and without diversity. They are the 2-D 8-ary and 16-ary signal constellations discussed in Chapter 2. Since multipath fading degrades significantly the performances of higher order modulation formats, current digital cellular and land mobile applications rarely use modulation formats with more than 16 signalling symbols. Our analyses and techniques, however, are applicable to any $M$-ary 2-D constellation with polygonal decision boundaries. Ideal gain control is assumed. The contributions of this chapter include: the derivation of precise expressions for symbol error probabilities of the constellations in slow Rayleigh, Ricean and Nakagami fading with maximal ratio diversity combining, equal gain diversity combining, and selection diversity combining; the determination of the optimum parameters, such as ring ratio of circular constellations; assessments of the amplitude and phase error sensitivities of the constellations considered; and comparisons of all the constellations with regard to the former performance criteria.

This chapter is organised as follows. In Section 3.2, we discuss in detail the system model used in this chapter. In Section 3.3, the symbol error rates of two-dimensional, $M$-ary signalling formats in slow Ricean, Rayleigh and Nakagami fading with maximal ratio combining (MRC), equal gain combining (EGC) and selection combining (SC) are derived. In Section 3.5, performance comparisons of well-known 8-ary and 16-ary constellations are carried out to determine optimum designs. Phase jitter vulnerability and amplitude jitter vulnerability of each constellation are also analysed. For circular signalling formats, the optimum ring ratios at large SNR are determined and tabulated. Finally, conclusions are given in Section 3.6.
Figure 3.1. Block diagram of a communication system employing coherent detection in flat fading channels.

### 3.2 System Model

The equivalent system block diagram of coherent detection in a flat fading channel is illustrated in Fig. 3.1. In contrast to an AWGN channel, the received complex signal in a frequency flat fading channel can be represented by

\[
z(t) = g(t)s_i(t) + n_w(t), \quad 0 \leq t \leq T
\]  

(3.1)

where \( g(t) = \alpha(t) \exp(j\phi(t)) \) is the multiplicative fading process as defined in Section 1.2.1, \( s_i(t) \) is the information bearing signal with unit average energy and \( n_w(t) \) the white Gaussian noise, as defined in Section 2.2.

If the channel varies slowly with regard to the symbol duration, i.e., \( f_DT \ll 1 \), it can be assumed that \( g(t) \) remains approximately constant over the symbol duration. Under this condition, the explicit dependency of \( g(t) \) on time can be removed so that the received signal becomes

\[
z(t) = gs_i(t) + n_w(t) = \alpha e^{j\phi} s_i(t) + n_w(t), \quad 0 \leq t \leq T
\]  

(3.2)

where the fading gain \( g \) is a complex random variable. If the fading gain is Gaussian with zero (non-zero) mean, the magnitude \( \alpha \) will be Rayleigh (Ricean) distributed and the phase
\( \phi \) is uniformly distributed over \([-\pi, \pi]\) (non-uniformly distributed [21, p. 135]). In some scenarios, the magnitude \( \alpha \) is modelled as a Nakagami-\( m \) random variable. As discussed in Chapter 2, the outputs of the matched filters can be expressed in vector form, where each component is the projection of the received signal onto one of two orthonormal basis functions.

\[
\mathbf{z} = g \mathbf{s}_i + \mathbf{n}
\]

where vectors \( \mathbf{z}, \mathbf{s}_i, \mathbf{n} \) are two-dimensional. The received signal-to-noise ratio can be expressed as \( \gamma_s = \alpha^2 / N_0 \), and the average SNR is given by \( \Lambda = E[\gamma_s] \).

In the case of ideal coherent detection, the amplitude and phase of the fading gain \( g \) at each symbol duration can be tracked perfectly. There are two ways to use this information to coherently detect the received signal. First, the decision variable is obtained as \( D = \frac{\mathbf{z}}{g} \), which is implemented by the block named CGC (complex gain control) device in Fig. 3.1, and the decision boundaries are fixed for signals \( \mathbf{s}_i \); second, the decision variable is \( D = \mathbf{z} \), while the constellation and the decision boundaries (slicer) on the constellation rotate and scale according to \( \phi \) and \( \alpha \). Either way will give the same result. In this chapter, we will follow the second method, where constellations rotate and scale proportionally to \( g \). In the later chapters, analyses follow the first method. The randomly distributed phase introduced by the channel is fully compensated at detection. Because the noise vector \( \mathbf{n} \) is circular symmetric, i.e., the phase of the noise is uniform between \([-\pi, \pi]\), and the channel fading is independent of the additive noise, the perfectly known fading phase has no effect on the detection. The changing magnitude \( \alpha \), however, is equivalent to varying the received SNR from symbol to symbol and its effect on the detection performance can be taken into account by ensemble averaging across its distribution.

### 3.3 Symbol Error Probability in Fading Diversity Systems

In this section, we extend Craig's method to fading diversity systems. As shown in Chapter 2, the SER (2.15) is the weighted sum of basic probability expressions given by (2.12)
or (2.13). It will be convenient to use the form (2.13)

\[ P_\delta(\gamma_s) = \frac{1}{2\pi} \int_0^\pi \exp \left[ -\frac{b\gamma_s \sin^2 \psi}{\sin^2(\theta + \psi)} \right] d\theta. \tag{3.4} \]

Note that (3.4) is valid for AWGN (i.e., no fading) with \( \gamma_s = \frac{E_s}{N_0} \) a constant. In slowly fading channels, the SNR \( \gamma_s \) and \( E_s \) are random variables and the probability expression (3.4) has to be averaged over the SNR distribution to obtain the probability of error. Ideal gain control is assumed so that the decision boundaries at the receiver vary proportionally with the fading. Then the error probability \( P_L \) for reception with \( L \)-order diversity using (3.4) is given by

\[ P_L = \int_0^\infty f_L(\gamma_s)P_\delta(\gamma_s)d\gamma_s = \int_0^\infty f_\alpha(\alpha)P_\delta(\alpha)d\alpha \tag{3.5} \]

where \( f_L(\gamma_s) \) and \( f_\alpha(\alpha) \) are the probability density functions of the SNR and magnitude \( \alpha \) in fading with \( L \)-order diversity, respectively. Three types of combining techniques for Ricean, Rayleigh and Nakagami-\( m \) fading will be discussed.

### 3.3.1 Maximal Ratio Diversity Combining

At the output of a maximal ratio combiner, the total instantaneous SNR \( \gamma_s \) is the sum of the instantaneous SNR of all the branches, i.e., \( \gamma_s = \sum_{l=1}^L \gamma_l \). When these \( L \) diversity branches are statistically identical and independent, the PDF's \( f_L(\gamma_s) \) of the SNR at an MRC output in Ricean, Nakagami and Rayleigh fading are given in [30],[31],[17], respectively. The symbol error probability in these fading channels can then be obtained by substituting (3.4) and those PDF's into (3.5). In the case of independent and distinct branches, which have different distributions or the same distributions but distinct parameters, it is no longer possible to express the PDF of the total SNR of MRC output in a tractable closed-form. As shown in [55], the joint PDF of the instantaneous SNR sequence \( \gamma_l, l = 1..L \) is given by

\[ f_{\gamma_1,\gamma_2,\ldots,\gamma_L}(\gamma_1,\gamma_2,\ldots,\gamma_L) = \prod_{l=1}^L f_\gamma(\gamma_l), \tag{3.6} \]
since the branches are statistically independent. The probability of error with MRC reception is now given by

$$P_L = \int_0^\infty \cdots \int_0^\infty P_s(\gamma_s) \prod_{l=1}^L f_{\gamma_l}(\gamma_l) d\gamma_1 d\gamma_2 \cdots d\gamma_L.$$  \hspace{1cm} (3.7)

Notice that $P_s(\gamma_s)$ in MRC can be expressed in a product form

$$P_s(\gamma_s) = \frac{1}{2\pi} \int_0^\infty \prod_{l=1}^L \exp \left[ -\frac{b \gamma_l \sin^2 \psi}{\sin^2 (\theta + \psi)} \right] d\theta,$$  \hspace{1cm} (3.8)

and substituting (3.8) into (3.7), we have

$$P_L = \int_0^\infty \cdots \int_0^\infty \frac{1}{2\pi} \int_0^\infty \prod_{l=1}^L \exp \left[ -\frac{b \gamma_l \sin^2 \psi}{\sin^2 (\theta + \psi)} \right] f_{\gamma_l}(\gamma_l) d\theta d\gamma_1 d\gamma_2 \cdots d\gamma_L.$$  \hspace{1cm} (3.9)

The integrand in (3.9) is continuous over the semi-infinite rectangular integration region, hence the order of integration can be interchanged [58, pp. 191-194]. Therefore,

$$P_L = \frac{1}{2\pi} \int_0^\infty \prod_{l=1}^L \int_0^\infty f(\gamma_l) \exp \left[ -\frac{b \gamma_l \sin^2 \psi}{\sin^2 (\theta + \psi)} \right] d\gamma_l d\theta.$$  \hspace{1cm} (3.10)

The SNR distribution for Ricean fading was introduced in Chapter 1 and is rewritten here [59]

$$f(\gamma_l) = \frac{K_l + 1}{\Lambda_l} \exp[-(K_l + \frac{K_l + 1}{\Lambda_l})\gamma_l] I_0 \left(2 \sqrt{\frac{K_l(K_l + 1)}{\Lambda_l}} \gamma_l\right)$$  \hspace{1cm} (3.11)

where $K_l$ is the Rice factor, $\Lambda_l = E[\gamma_l] = \frac{E_{s,l}}{N_0}$ is the average SNR per symbol, $E_{s,l}$ is the ensemble average of the signal energy per symbol of the $l$th branch and $I_0(x)$ is the zeroth-order modified Bessel function of the first kind [35, 8.431]. Therefore combining (3.10) and (3.11) gives

$$P_L = \frac{1}{2\pi} \int_0^\infty \prod_{l=1}^L \left[ \frac{\sin^2 (\theta + \psi)}{\sin^2 (\theta + \psi) + \frac{\Lambda_l b \sin^2 \psi}{K_l + 1}} \right] \exp \left[ -\frac{K_l \Lambda_l b \sin^2 \psi}{L(K_l + 1) \sin^2 (\theta + \psi) + \Lambda_l b \sin^2 \psi} \right] d\theta$$  \hspace{1cm} (3.12)
where [35, 6.631.4] has been used.

The Nakagami-\(m\) distributed \(\gamma_l\) of the \(l\)th diversity channel is given by [14, Eq. (11)]

\[
f(\gamma_l) = \left[\gamma_l^{m_l-1} / \Gamma(m_l)\right] \left(\frac{m_l}{\Lambda_l}\right)^{m_l} \exp\left(-\frac{m_l\gamma_l}{\Lambda_l}\right)
\]

(3.13)

where \(\Gamma(x)\) is the gamma function, and \(m_l \geq \frac{1}{2}\) is the fading parameter of the \(l\)th branch. Combining (3.10) and (3.13) leads to

\[
P_L = \frac{1}{2\pi} \int_0^n \prod_{l=1}^L \left[1 + \frac{\Lambda_l}{m_l \sin^2(\theta + \psi)} \right]^{-m_l} d\theta
\]

(3.14)

where [35, 3.462.1] has been used.

For maximal ratio combining with Rayleigh fading, let \(K_l = 0\) in (3.12) or \(m_l = 1\) in (3.14) and we obtain

\[
P_L = \frac{1}{2\pi} \int_0^n \prod_{l=1}^L \frac{\sin^2(\theta + \psi)}{\sin^2(\theta + \psi) + \Lambda_l b \sin^2 \psi} d\theta.
\]

(3.15)

The results here can also be easily generalised to more complicated scenarios, for example, any mixture of Ricean/Nakagami-Rayleigh channels, by employing appropriate \(f(\gamma_l)\) in (3.10). Single channel reception \((L = 1)\) and iid diversity channels are both special cases of (3.12), (3.14) and (3.15). For single channel Ricean, Nakagami-\(m\) and Rayleigh fading, we have respectively

\[
P_{L=1} = \frac{1}{2\pi} \exp(-K) \int_0^n w(\theta) \exp[Kw(\theta)] d\theta
\]

(3.16)

where

\[
w(\theta) = \frac{\sin^2(\theta + \psi)}{\sin^2(\theta + \psi) + \Lambda_l b \sin^2 \psi},
\]

(3.17)

\[
P_{L=1} = \frac{1}{2\pi} \int_0^n \left[1 + \frac{1}{m \sin^2(\theta + \psi)}\right]^m d\theta
\]

(3.18)

and

\[
P_{L=1} = \frac{n}{2\pi} - \frac{1}{2\pi} \sqrt{\frac{p}{p + 1}} \left[\tan^{-1}\left(\sqrt{\frac{p + 1}{p}} \tan(\eta + \psi)\right) - \tan^{-1}\left(\sqrt{\frac{p + 1}{p}} \tan \psi\right)\right]
\]

(3.19)

where \(p = \Lambda b \sin^2 \psi\) and [35, 2.562.1] has been used.
3.3.2 Equal Gain Diversity Combining

The performance of EGC is almost as good as that of MRC though EGC does not require estimation of the fading channel amplitude of each diversity branch. More details on EGC can be found in Chapter 1. Beaulieu [32] has given a convergent infinite series for the complementary probability distribution function (CDF) of a sum of independent random variables, which can be used to obtain the CDF and PDF of the SNR at the output of L-branch EGC in Ricean [33], Rayleigh [32] and Nakagami [34] fading. This series is used here to determine the SER for EGC.

Let \( \alpha_l, l = 1, \ldots, L, \) be independent random variables describing the faded symbol amplitudes normalised so that \( E[\alpha_l^2] = E_x \) on the \( L \) channels, and let \( f_{\alpha_l}(\alpha_l) \) be the PDF of \( \alpha_l \). The amplitude of an equal gain combiner output is then \( \alpha = \sum_{l=1}^{L} \alpha_l \). The PDF of \( \alpha \) can be written as [34]

\[
f_{\alpha}(\alpha) = \frac{2}{T} \sum_{n=1, odd}^{\infty} \{e^{-in\omega\alpha}\Phi(n\omega) + e^{in\omega\alpha}\Phi(-n\omega)\} \tag{3.20}
\]

where

\[
\omega = \frac{2\pi}{T},
\]

\[
\Phi(n\omega) = \prod_{i=1}^{L} \Phi_i(n\omega), \tag{3.21}
\]

\[
\Phi_i(n\omega) = E[e^{in\omega\alpha_l}] = E[\cos(n\omega\alpha_l)] + jE[\sin(n\omega\alpha_l)], \tag{3.22}
\]

and polar expressions for \( \Phi(n\omega) \) and \( \Phi_i(n\omega) \) are

\[
\Phi(n\omega) := A_n e^{j\tau_n}, \quad \Phi_i(n\omega) := A_{ln} e^{j\tau_l}. \tag{3.23}
\]

Hence,

\[
A_n = \prod_{l=1}^{L} A_{ln}, \quad \tau_n = \sum_{l=1}^{L} \tau_l. \tag{3.24}
\]
Define

\[ \Phi_{R_l} := E[\cos(n\omega \alpha_l)], \quad \Phi_{I_l} := E[\sin(n\omega \alpha_l)] \]  

(3.25)

and then,

\[ A_{ln} = \sqrt{\Phi_{R_l}^2 + \Phi_{I_l}^2}, \quad \tau_l = \tan^{-1}\left( \frac{\Phi_{I_l}}{\Phi_{R_l}} \right) \]  

(3.26)

where \( \tau_l \) must be chosen to be in the correct quadrant. It can be shown that [34]

\[ \Phi(-n\omega) = A_n e^{-j\tau_n}. \]  

(3.27)

Using the above definitions, the series for the PDF of \( \alpha \) can be rewritten as

\[ f_\alpha(\alpha) = \frac{2}{T} \sum_{n=1, \text{odd}}^{\infty} \{A_n e^{j\tau_n} e^{-j(n\omega \alpha)} + A_n e^{-j\tau_n} e^{j(n\omega \alpha)}\} \]

\[ = \frac{4}{T} \sum_{n=1, \text{odd}}^{\infty} A_n \cos(n\omega \alpha - \tau_n) \]  

(3.28)

where \( A_n \) and \( \tau_n \) depend on the specific channel probability distribution of each diversity branch. Regarding the selection of \( T \), there is a tradeoff between accuracy and the number of terms needed [34]. Since \( T \) is the period of the square wave used to calculate the cdf and pdf of \( \alpha \) [34], it has the same units as amplitude \( \alpha \), i.e., volts in this case. A larger value of \( T \) results in greater accuracy and is necessary to compute small probabilities. In our computations, \( T \) was chosen to lie between 500 and 1500. Note that \( \alpha \) in (3.28) is a random variable representing the sum of the amplitude of the \( L \) branches. Therefore, the resulting instantaneous SNR for equal gain diversity combining is given by

\[ \gamma_s = \frac{\alpha^2}{N_0L}. \]  

(3.29)

Hence, the error probability expression \( P_L \) in (3.5) is obtained as

\[ P_L = \frac{2}{\pi T} \int_0^n \sum_{n=1, \text{odd}}^{\infty} A_n \cos \tau_n \int_0^{\infty} \cos(n\omega \alpha)e^{-\alpha^2} d\alpha d\theta \]

\[ + \frac{2}{\pi T} \int_0^n \sum_{n=1, \text{odd}}^{\infty} A_n \sin \tau_n \int_0^{\infty} \sin(n\omega \alpha)e^{-\alpha^2} d\alpha d\theta \]  

(3.30)
where

\[ c = c(\theta) = \frac{b \sin^2 \psi}{N_0 L \sin^2(\theta + \psi)}. \]  

(3.31)

The change of integration order and the term-by-term integration of the infinite series in (3.30) is guaranteed by [58, pp. 191-194] and [58, Theorem 15, p. 423]. Define

\[ Q_n(c(\theta)) := \int_0^\infty \cos(n\omega \alpha)e^{-\alpha^2}d\alpha = \frac{\sqrt{\pi}}{2\sqrt{c}} \exp\left(-\frac{n^2\omega^2}{4c}\right) \]  

(3.32)

\[ P_n(c(\theta)) := \int_0^\infty \sin(n\omega \alpha)e^{-\alpha^2}d\alpha = \frac{n\omega}{2c} \exp\left(-\frac{n^2\omega^2}{4c}\right)_{1}F_1\left(\frac{1}{2}; \frac{3}{2}; -\frac{n^2\omega^2}{4c}\right) \]  

(3.33)

where [35, 3.952.7], [35, 3.952.8] and \( {}_{1}F_1(0; \frac{1}{2}; z) = 1 \) have been used, and \( {}_{1}F_1(\cdot; \cdot; \cdot) \) is the confluent hypergeometric function [35, 9.210]. The confluent hypergeometric function \( {}_{1}F_1\left(\frac{1}{2}; \frac{3}{2}; z\right) \) is easily computed from its series definition

\[ {}_{1}F_1\left(\frac{1}{2}; \frac{3}{2}; z\right) = \sum_{n=0}^{\infty} \frac{z^n}{(2n+1)n!}. \]  

(3.34)

Eq. (3.30) finally takes the form of

\[ P_L = \frac{2}{\pi R^2} \sum_{\eta=1, \text{odd}}^{\infty} \int_0^\eta A_n(Q_n(\theta)\cos \tau_n + P_n(\theta)\sin \tau_n)d\theta \]  

(3.35)

where \( A_n \) and \( \tau_n \) are computed from (3.24) and (3.26) once \( \Phi_{R_i} \) and \( \Phi_{I_i} \) are specified.

The expressions of \( \Phi_{R_i} \) and \( \Phi_{I_i} \) for Ricean distributed \( \alpha_i \) are written as [33]

\[ \Phi_{R_i} = e^{-K_i} \sum_{j=0}^{\infty} \frac{K_i^j}{j!} {}_{1}F_1\left(j + 1; \frac{1}{2}; -\frac{n^2\omega^2\Omega_i}{4(1 + K_i)}\right) \]  

(3.36)

\[ \Phi_{I_i} = \frac{n\omega e^{-K_i}}{\sqrt{(1 + K_i)\Omega_i}} \sum_{j=0}^{\infty} \frac{\Gamma(j + \frac{3}{2})}{(j!)^2} K_i^j {}_{1}F_1\left(j + \frac{3}{2}; \frac{3}{2}; -\frac{n^2\omega^2\Omega_i}{4(1 + K_i)}\right) \]  

(3.37)
where $K_i$ is the Rice factor and $\Omega_i = E[\alpha_i^2]$ is the total signal energy from the $l$th Ricean channel. A recursive algorithm for computing $\text{I}_1 F_l(j; m; -z)$ is given in [34] and for computing $\text{I}_1 F_l(j + \frac{3}{2}; \frac{3}{2}; -z)$ in [33], where $j$ is an integer. The average SNR per symbol of each branch is defined as

$$\Lambda_i := \frac{E[\alpha_i^2]}{N_0} = \frac{\Omega_i}{N_0}.$$  \hspace{1cm} (3.38)

For Nakagami distributed $\alpha_l$ [34],

$$\Phi_{R_i} = \text{I}_1 F_l(m_l; \frac{1}{2}; -\frac{n^2 \omega^2 \Omega_l}{4 m_l}),$$  \hspace{1cm} (3.39)

$$\Phi_{l} = n\omega \sqrt{\frac{\Omega_l}{m_l}} \frac{\Gamma(m_l + \frac{1}{2})}{\Gamma(m_l)} \text{I}_1 F_l(m_l + \frac{3}{2}; \frac{3}{2}; -\frac{n^2 \omega^2 \Omega_l}{4 m_l})$$  \hspace{1cm} (3.40)

where $m_l$ is the Nakagami fading parameter of the $l$th received channel.

For Rayleigh distributed $\alpha_l$, letting $m_l$ in (29) equal 1, we have

$$\Phi_{R_i} = \text{I}_1 F_l(1; \frac{1}{2}; -\frac{n^2 \omega^2 \Omega_l}{4}),$$  \hspace{1cm} (3.41)

$$\Phi_{l} = \frac{\sqrt{\pi \Omega_l}}{2} n\omega \exp \left( -\frac{n^2 \omega^2 \Omega_l}{4} \right)$$  \hspace{1cm} (3.42)

where $\Omega_l$ is the average signal energy of the $l$th Rayleigh fading channel.

### 3.3.3 Selection Diversity Combining

Selection combining is a simple method of implementing diversity although its performance is not as good as that of MRC and EGC. The PDF of the signal amplitude at the output of a selection combiner when all branches are assumed to be independent and identically Ricean distributed is given by [33]

$$f_L(\alpha) = L[1 - Q(\sqrt{2K} \alpha \sqrt{\frac{\Omega}{2(1+K)}})] L - 1 2 \alpha (1+K) \frac{\Omega}{2 \Omega} \exp \left( -K - \frac{1+K}{\Omega} \alpha^2 \right)$$

$$\times I_0(2\alpha \sqrt{\frac{K(1+K)}{\Omega}})$$  \hspace{1cm} (3.43)
where $\Omega = E[\alpha^2]$ is the total signal energy in each Ricean channel and $Q(a, b) = \int_b^\infty xe^{-\frac{x^2+1}{2}}$ $I_0(ax)dx$ is the Marcum-Q function. Combining (3.43) and (3.5) results in

$$P_L = \frac{1}{2\pi} \int_0^\pi d\theta \int_0^\infty f_L(\alpha) \exp \left[ -\frac{b\alpha^2 \sin^2 \psi}{\Lambda_0 \sin^2 (\theta + \psi)} \right] d\alpha.$$  \hspace{1cm} (3.44)

For $L = 2$, it is possible to obtain a closed-form expression involving the Marcum-Q function for the inner integral by using [60, (34) and (40)], but the result is cumbersome. The two-dimensional integrand of (3.44) is a smooth bivariate positive function and (3.44) can be easily and efficiently evaluated by numerical methods for any order $L$. With the change of variable $t = e^{-\alpha}$, the domain of the integrand is transformed from $[0, \infty)$ into $(0, 1]$. Then Gaussian quadrature can be applied to compute the two-dimensional integral with ease and high accuracy.

The selection combining Nakagami-$m$ distributed $\gamma_s$ is derived for iid diversity branches as

$$f_L(\gamma_s) = L(\frac{m}{L})^m \gamma_s^{m-1} e^{-\frac{\gamma_s}{\Lambda} L_1 \left[ \gamma(m, \frac{m}{L}) \right] L^{-1}}$$  \hspace{1cm} (3.45)

where $\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt$ is the incomplete gamma function [35, 8.35] and $\Lambda = E[\gamma_s]$.

The probability of error for SC Nakagami fading is given by

$$P_L = \frac{1}{2\pi} \int_0^\pi d\theta \int_0^\infty f_L(\gamma_s) \exp \left[ -\frac{b\gamma_s \sin^2 \psi}{\sin^2 (\theta + \psi)} \right] d\gamma_s d\theta.$$  \hspace{1cm} (3.46)

which has to be evaluated numerically. When the parameter $m$ is an integer, the incomplete gamma function can be expressed in an analytical form as given in [35, 8.352.1]. Then (3.45) simplifies to

$$f_L(\gamma_s) = L(\frac{m}{L})^m \gamma_s^{m-1} e^{-\frac{\gamma_s}{\Lambda} \left[ 1 - e^{-\frac{\gamma_s}{\Lambda}} \sum_{k=0}^{m-1} \frac{\gamma_s^k}{k! \left( \frac{m}{L} \right)^k} \right] L^{-1}}.$$  \hspace{1cm} (3.47)

It is possible to obtain error probability expressions from (3.46) and (3.47) for small $L$. For example, when $L = 2$,

$$P_{L=2} = \frac{1}{\pi} \int_0^\pi \left\{ \left[ \frac{m \sin^2 (\theta + \psi)}{m \sin^2 (\theta + \psi) + \Lambda b \sin^2 \psi} \right]^m - \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \left( \frac{1}{2m \sin^2 (\theta + \psi) + \Lambda b \sin^2 \psi} \right)^{m+k} \right\} d\theta$$  \hspace{1cm} (3.48)
and when \( L = 3 \),

\[
P_{L=3} = \frac{3}{2\pi} \int_0^{\pi} \left\{ \left[ \frac{m \sin^2(\theta + \psi)}{m \sin^2(\theta + \psi) + \Lambda b \sin^2 \psi} \right]^m \right. \\
- \sum_{k=0}^{m-1} \frac{2(m+k-1)!}{(m-1)! k!} \left( \frac{m \sin^2(\theta + \psi)}{2m \sin^2(\theta + \psi) + \Lambda b \sin^2 \psi} \right)^{m+k} \\
+ \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \frac{(m+k+j-1)!}{(m-1)! k! j!} \left( \frac{m \sin^2(\theta + \psi)}{3m \sin^2(\theta + \psi) + \Lambda b \sin^2 \psi} \right)^{m+j+k} \left\} \right. d\theta,
\]

(3.49)

where [35, 3.381.4] has been used.

For selection combining and Rayleigh fading, the PDF of \( \gamma_s \) assuming iid diversity branches is given by [17]

\[
f_L(\gamma_s) = \frac{L}{\Lambda} (1 - e^{-\gamma_s/\Lambda})^{L-1} e^{-\gamma_s/\Lambda}.
\]

(3.50)

Substituting (3.4) and (3.50) into (3.5) yields

\[
P_L = \frac{L}{2\pi} \sum_{k=0}^{L-1} \frac{(-1)^k}{k+1} \left( \begin{array}{c} L-1 \\ k \end{array} \right) \left\{ \eta - \sqrt{\frac{p}{p+1}} \tan^{-1} \left( \sqrt{\frac{p+1}{p}} \tan(\eta + \psi) \right) \\
- \tan^{-1} \left( \sqrt{\frac{p+1}{p}} \tan \psi \right) \right\}
\]

(3.51)

where \( p = \Lambda b \sin^2 \psi/(k+1) \) and [35, 2.562.1] has been used. Eqn. (3.51) is a new closed-form expression for the symbol error probability of an arbitrary polygonal 2-D signal set in Rayleigh fading with selection combining.
3.4 Sixteen Star-QAM in Fading with Single Channel Reception

As an example of the application of the theory developed in this chapter, the symbol error rates of 16 star-QAM in AWGN, Ricean, Nakagami-\(m\) and Rayleigh fading without diversity reception are evaluated from eqns. (2.25), (2.25f), (3.16), (3.17), (3.18) and (3.19) and are shown in Fig. 3.2. As in AWGN, the error probability which is a function of the decision boundaries will depend on the parameter \(T\), indicated in Fig. 2.4 and on the ring ratio \(\beta\). The optimum parameter \(T\) in AWGN is known to be at the midpoint of the radius between a signal on the inner ring and a signal on the outer ring and is optimum for all values of SNR. Hence, the same optimum value of \(T\) obtains for the fading signal case.

Optimum ring ratios of 16 star-QAM for AWGN, Ricean, Nakagami-\(m\) and Rayleigh fading environments are determined by numerical search and are plotted as functions of the SNR in Figs. 3.3 and 3.4. These results, derived using the precise error probabilities given above are new. Observe that the optimum ratios approach asymptotic values as the SNR increases. This is expected since the probability of an error event in slow fading with perfect phase and envelope tracking is dominated by the minimum distance error event at large values of SNR. Also note that the asymptotic values for Ricean fading and Rayleigh fading are the same whereas the asymptotic values for Nakagami fading depend on the fading severity parameter, \(m\). This is to be expected since the tail of the Ricean signal amplitude distribution decays at the same rate as the tail of the Rayleigh signal amplitude distribution whereas the tail of the Nakagami signal amplitude distribution does not. Table 3.1 summarises the asymptotically optimum ring ratios of 16 star-QAM for various fading channels. Note that the asymptotically optimum ring ratio for Ricean channels is independent of the Ricean \(K\) factor, whereas \(\beta_{OPT}\) for Nakagami channels depends on \(m\). This is because of the previously mentioned behaviour of the tails of the signal amplitude distributions.

Since the transmitter operates with a fixed ring ratio, we have chosen the asymptotic
Figure 3.2. Average symbol error probabilities of coherent 16 star-QAM (with optimum ring ratios) and 16 rectangular-QAM in fading channels.
Figure 3.3. Optimum ring ratios of coherent 16 star-QAM in AWGN and Ricean fading as functions of average SNR.
Figure 3.4. Optimum ring ratios of coherent 16 star-QAM in Nakagami fading.
TABLE 3.1
ASYMPTOTICALLY OPTIMUM RING RATIOS FOR 16 STAR-QAM

<table>
<thead>
<tr>
<th>Channel</th>
<th>AWGN</th>
<th>Ricean</th>
<th>Nakagami (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Rayleigh</td>
<td>0.6</td>
</tr>
<tr>
<td>$\beta_{OPT}$</td>
<td>1.7654</td>
<td>1.9512</td>
<td>2.0117</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>1.9512</td>
<td>1.8785</td>
<td>1.8459</td>
</tr>
</tbody>
</table>

ring ratio values for the evaluation of SER in Fig. 3.2; also, the receiver is assumed to have perfect gain control. Also shown in Fig. 3.2 are the SER’s of coherent 16 rectangular-QAM. No closed-form expression exists for the fading SER except for the Rayleigh fading case [25] and these results have also been obtained using the new method described in Section 3.3. The difference in SNR required for a target error rate, for instance, SER= $10^{-4}$, in AWGN is 1.26 dB in favour of 16 rectangular-QAM [37, pp. 627-631]. The difference is much smaller (typically less than 0.4 dB) for fading environments except for light fading corresponding to moderate to large values of $m$ (say $m \geq 2$) for Nakagami fading and large values of $K$ (say $K > 5$dB) for Ricean fading.

3.5 Two-dimensional 8-ary and 16-ary Signalling

In this section, we apply the formulae of Section 3.3 to the performance analysis and comparison of seventeen conventional 8-ary and 16-ary modulation formats presented in Chapter 2 in slow fading with diversity reception. The computer program for calculating the SER of any 2-D constellation in AWGN is modified to include extensive combinations of fading and diversity combining scenarios. Our results for slowly fading channels are new. Here presented for the first time are the optimum ring ratios for the circular constellations in fading diversity systems. They are shown to be quite different from the values for AWGN.
The amplitude and phase error sensitivities of each constellation are also discussed. The discussions in this section assume that all the diversity branches are statistically independent and identically distributed.

### 3.5.1 Eight-ary modulation formats

The 8-ary modulation formats discussed in this subsection were introduced in Chapter 2 and their constellations were shown in Fig. 2.5. As before, the ring ratios of circular constellations need to be optimised before comparing the performances of these signal sets. The optimum ring ratios of constellation (4,4) for asymptotically large SNR are summarized in Table 3.2 for various channel models. It is observed that the asymptotically optimum ring ratios vary with diversity order and channel models, but do not depend on the combining type. That is, the asymptotically optimum values of ring ratio for a circular constellation employing MRC, EGC, or SC techniques are identical. This behavior is explained as follows. It is well-known that the asymptotic dependence of the error rate on the SNR approaches a straight line on a log-log plot and, importantly, the slope is independent of the type of diversity combining though the magnitude of the slope increases as the diversity order increases. See, for example, Fig. 3.10 on Page 67 of this thesis. This fact is the essence of the statement that "a channel error rate of \( P \) becomes essentially \( P^L \) with \( L \)-fold diversity" [61, p. 79] though such statements are approximately rather than rigorously true (In fact \( P_e(\text{diversity}) = f(SNR, L)^P \) where \( f(SNR, L) \) depends on the combining type). Since the asymptotic dependence of the error rate on the logarithmic SNR has the same slope for different combining types, one has \( P_e(\text{MRC}) = h(SNR, \beta) \), \( P_e(\text{SC}) = h(SNR + \Delta_{SC}, \beta) \) and \( P_e(\text{EGC}) = h(SNR + \Delta_{EGC}, \beta) \) as SNR goes to infinity, where \( \Delta_{SC} \) and \( \Delta_{EGC} \) are the SNR penalties in dB's of SC and EGC compared to MRC respectively, and \( \beta \) is the ring ratio used. Therefore, the asymptotic optimum ring ratio \( \beta_{OPT} \) that minimises \( P_e(\text{MRC}) \) at infinite SNR also minimises \( P_e(\text{SC}) \) and \( P_e(\text{EGC}) \) at infinite SNR. However, the optimum ring ratios corresponding to small or medium ranges of SNR are not the same for MRC, EGC and SC. This is seen graphically in Fig. 3.12 on Page 70. In the results involving the
(4,4) constellation in Figs. 3.5-3.9 the asymptotically optimum ring ratio is used.

For each signal constellation, the combinations of channel model, diversity combining type and diversity order result in a large number of cases. In this thesis, the symbol error probabilities of only a few cases are plotted for discussion purposes.

The SER curves for various fading channels in Figs. 3.5-3.9 demonstrate that fading substantially reduces the performance differences among the six 8-ary signalling formats. Diversity combining reduces the probability of error significantly, but also enlarges the performance differences among the six signalling formats, as expected, since diversity diminishes the effects of fading and makes the received signal plus noise more like that in the AWGN channel. Moreover, it is observed that diversity improvements are more significant for more severe fading environments, e.g., Rayleigh vs. Ricean, Nakagami-$m$ where $m = 0.7$ vs. $m = 2$. Also note that the 8-ary max-density set no longer produces the lowest probability of error but is second to the (1,7) constellation in fading channels. In general, the optimum signalling format in AWGN is not necessarily optimum in fading. The 8PSK curve in Fig. 3.6 for $L=1$ agrees with [50, Fig. 3(a)]. Fig. 3.10 depicts the symbol error rate of the (4,4) constellation in Rayleigh fading with three combining techniques, MRC, EGC, and SC. The error rate curves demonstrate the well-known relationships between the various forms of combining. Maximal ratio combining requires approximately 1.5 dB less
power than selection combining at diversity order \( L = 2 \) and 2.6 dB less at \( L = 3 \), to achieve the same error rate. Equal gain combining gives performance close to that of MRC. Although not shown here, it is observed in all our results that the combining type does not change the performance ordering of these constellations.

### 3.5.1.1 Amplitude and phase error sensitivity

The use of signal constellations and decision regions to calculate the error performance assumes coherent detection. To implement coherent detection in both AWGN and fading channels, some form of carrier amplitude and phase tracking is required. As a measure of the robustness of a constellation to errors in carrier phase recovery, a phase tolerance angle is defined as the minimum angular displacement \( \delta \) that will cause a symbol error in the absence of noise. This is illustrated in Fig. 3.11(a). The amplitude error tolerance is defined as the minimum relative radius difference that will cause a symbol error in the absence of noise, as shown in Fig. 3.11(b).

To indicate the robustness of the constellation to error in carrier recovery, the amplitude tolerance (in dB) and phase tolerance angle are calculated for each constellation. A signalling format yielding a low probability of symbol error may be less immune to amplitude and phase error. Amplitude tolerance margins of the six 8-ary signal sets are summarized in Table 3.3, and their phase tolerance margins are given in Table 3.4. Note that both the amplitude error tolerance and the phase error tolerance angle of constellation (4,4) depend on its ring ratio. The amplitude error tolerance of (4,4) is \( \min(|20 \log \frac{\beta^2-1}{2\beta^2-\sqrt{2}\beta}|, 20 \log \frac{\beta^2-1}{\sqrt{2}\beta-2}) \). For the range of ring ratios asymptotically optimum for various channel models, the angle is approximately 30° and the amplitude tolerance is between 4.72dB and 4.77dB. In view of the amplitude and phase error tolerance and the error performance, the triangular signal set is a rather good candidate for fading channels. Constellation (4,4) has the greatest phase error tolerance. Since the error performance of constellation (4,4) is quite comparable to that of the (1,7) constellation in fading channels, both are good candidates as efficient modulation formats for cellular and land mobile systems.
Figure 3.5. Average symbol error probabilities of 8-ary signal sets in Ricean fading channels with maximal ratio combining and $K = 5\text{dB}$. 
Figure 3.6. Average symbol error probabilities of 8-ary signal sets in Rayleigh fading channels with maximal ratio combining.
Figure 3.7. Average symbol error probabilities of 8-ary signal sets in Nakagami fading channels with maximal ratio combining and $m = 0.7$. 
Figure 3.8. Average symbol error probabilities of 8-ary signal sets in Nakagami fading channels with maximal ratio combining and $m = 2$. 
Figure 3.9. Average symbol error probabilities of 8-ary signal sets in Rayleigh fading channels with selection combining.
Figure 3.10. Average symbol error probabilities of the (4,4) constellation in Rayleigh fading channels with maximal ratio combining, equal gain combining and selection combining.
Figure 3.11. Illustration of (a) phase error margin and (b) amplitude error margin.

### TABLE 3.3
AMPLITUDE ERROR TOLERANCE OF SIX 8-POINT CONSTELLATIONS

<table>
<thead>
<tr>
<th></th>
<th>8PSK</th>
<th>Triangular</th>
<th>(1,7)</th>
<th>Rectangular</th>
<th>(4,4)</th>
<th>Max-density</th>
</tr>
</thead>
<tbody>
<tr>
<td>dB</td>
<td>∞</td>
<td>7.96</td>
<td>6.02</td>
<td>6</td>
<td>4.72 to 4.77</td>
<td>4.17</td>
</tr>
</tbody>
</table>

### TABLE 3.4
PHASE ERROR TOLERANCE OF SIX 8-POINT CONSTELLATIONS

<table>
<thead>
<tr>
<th></th>
<th>(4,4)</th>
<th>Triangular</th>
<th>(1,7)</th>
<th>Max-density</th>
<th>Rectangular</th>
<th>8PSK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle</td>
<td>~ 30°</td>
<td>26.9°</td>
<td>25.7°</td>
<td>24.8°</td>
<td>24.295°</td>
<td>22.5°</td>
</tr>
</tbody>
</table>
3.5.2 Sixteen-ary modulation formats

In this subsection, we present numerical results for the eleven 16-ary signalling sets studied in Chapter 2 for fading diversity environments, by employing the formulas derived in Section 3.3. We start with optimising the ring ratios of circular signal sets 16 star-QAM, rotated (8,8), (4,12), (5,11) and (1.5,10). Fig 3.12 depicts the optimum ring ratios of rotated (8,8) constellation in Rayleigh fading with three diversity combining techniques. Different combining techniques have distinct optimum ring ratio values at low SNR, however, the asymptotic optimum ring ratios converge to the same value at very large SNR. This has been explained in the previous subsection. The asymptotically optimum ring ratios of 16-ary circular constellations in various fading channels are summarised in Tables 3.5-3.9. It is observed from these tables that the asymptotically optimum ring ratios depend on the constellation geometry, the channel type and the diversity order.

**TABLE 3.5**

<table>
<thead>
<tr>
<th>Diversity Order</th>
<th>ASYMPTOTICALLY OPTIMUM RING RATIOS FOR 16 STAR-QAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>L = 1</td>
<td>AWGN 1.765, Ricean Rayleigh 1.951, Nakagami (m) 1.993, 1.879, 1.846</td>
</tr>
<tr>
<td>L = 2</td>
<td>1.879, 1.914, 1.828, 1.808</td>
</tr>
<tr>
<td>L = 3</td>
<td>1.846, 1.874, 1.808, 1.795</td>
</tr>
<tr>
<td>L = 4</td>
<td>1.828, 1.851, 1.798, 1.787</td>
</tr>
</tbody>
</table>

The error rate curves in Figs. 3.13-3.22 are arranged in the order of highest to lowest probability of error at large SNR, although the order may change at lower SNR. Again, the performance differences between constellations are greatly diminished in fading channels, while diversity combining increases the differences to some extent. It is observed that the order of constellation performance varies with channel model and diversity order, but again is invariant to diversity combining type. The best constellation can be chosen from the SER.
Figure 3.12. Optimum ring ratios of rotated (8,8) constellation in Rayleigh fading with diversity combining.
### TABLE 3.6
ASYMPTOTICALLY OPTIMUM RING RATIOS FOR ROTATED (8,8) CONSTELLATION

<table>
<thead>
<tr>
<th>Diversity Order</th>
<th>AWGN</th>
<th>Ricean</th>
<th>Nakagami (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Rayleigh</td>
<td>0.7</td>
</tr>
<tr>
<td>L = 1</td>
<td>1.587</td>
<td>1.867</td>
<td>1.918</td>
</tr>
<tr>
<td>L = 2</td>
<td>1.772</td>
<td></td>
<td>1.820</td>
</tr>
<tr>
<td>L = 3</td>
<td>1.724</td>
<td></td>
<td>1.766</td>
</tr>
<tr>
<td>L = 4</td>
<td>1.696</td>
<td></td>
<td>1.732</td>
</tr>
</tbody>
</table>

### TABLE 3.7
ASYMPTOTICALLY OPTIMUM RING RATIOS FOR (4,12) CONSTELLATION

<table>
<thead>
<tr>
<th>Diversity Order</th>
<th>AWGN</th>
<th>Ricean</th>
<th>Nakagami (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Rayleigh</td>
<td>0.7</td>
</tr>
<tr>
<td>L = 1</td>
<td>2.732</td>
<td>2.559</td>
<td>2.588</td>
</tr>
<tr>
<td>L = 2</td>
<td>2.509</td>
<td></td>
<td>2.533</td>
</tr>
<tr>
<td>L = 3</td>
<td>2.488</td>
<td></td>
<td>2.506</td>
</tr>
<tr>
<td>L = 4</td>
<td>2.480</td>
<td></td>
<td>2.491</td>
</tr>
</tbody>
</table>

### TABLE 3.8
ASYMPTOTICALLY OPTIMUM RING RATIOS FOR (5,11) CONSTELLATION

<table>
<thead>
<tr>
<th>Diversity Order</th>
<th>AWGN</th>
<th>Ricean</th>
<th>Nakagami (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Rayleigh</td>
<td>0.7</td>
</tr>
<tr>
<td>L = 1</td>
<td>2.175</td>
<td>2.324</td>
<td>2.360</td>
</tr>
<tr>
<td>L = 2</td>
<td>2.258</td>
<td></td>
<td>2.291</td>
</tr>
<tr>
<td>L = 3</td>
<td>2.224</td>
<td></td>
<td>2.253</td>
</tr>
<tr>
<td>L = 4</td>
<td>2.205</td>
<td></td>
<td>2.230</td>
</tr>
</tbody>
</table>
TABLE 3.9
ASYMPTOTICALLY OPTIMUM RING RATIOS FOR (1,5,10) CONSTELLATION

<table>
<thead>
<tr>
<th>Diversity Order</th>
<th>AWGN</th>
<th>Ricean Rayleigh</th>
<th>Nakagami (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L = 1</td>
<td>1.902</td>
<td>2.069</td>
<td>2.100 2.015 1.990</td>
</tr>
<tr>
<td>L = 2</td>
<td>2.015</td>
<td>2.041 1.975 1.957</td>
<td></td>
</tr>
<tr>
<td>L = 3</td>
<td>1.990</td>
<td>2.012 1.957 1.942</td>
<td></td>
</tr>
<tr>
<td>L = 4</td>
<td>1.975</td>
<td>1.994 1.946 1.934</td>
<td></td>
</tr>
</tbody>
</table>

Curves for different fading channels, although some results are not plotted in this thesis due to space limitations. These results show that the 16-ary max-density constellation is outperformed slightly by (1,5,10) in Ricean fading with small K-factor, in Rayleigh fading, and in Nakagami fading with a small m parameter (for example, m = 0.7), without diversity reception. The 16 rectangular-QAM results in Ricean and Rayleigh fading agree with those in [50]-[52]. Fig. 3.23 illustrates the same relationships among different combining types as does Fig. 3.10.
Figure 3.13. Average symbol error probabilities of 16-ary signal sets (Set 1) in Ricean fading channels with maximal ratio combining and $K = 5\text{dB}$. 
Figure 3.14. Average symbol error probabilities of 16-ary signal sets (Set 2) in Ricean fading channels with maximal ratio combining and $K = 5\text{dB}$. 

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The tolerances of each 16-ary constellation to amplitude and phase error are given in Table 3.11 and Table 3.12, respectively. The amplitude error tolerances of circular constellations depend on their ring ratios and are given in Table 3.10.

In Table 3.11, the amplitude error tolerances for circular constellations are obtained using the range of asymptotic optimum ring ratios for AWGN and fading. It is to be noted that the phase tolerance angle of rotated (8,8) increases as its ring ratio gets larger and the value given in Table 3.12 is for the asymptotically optimum ring ratio in AWGN. The maximum (for any ring ratio) of rotated (8,8)’s phase error tolerance angle is 22.5°.

The values of amplitude error tolerance indicate that these 16-ary constellations are
Figure 3.15. Average symbol error probabilities of 16-ary signal sets (Set 1) in Rayleigh fading channels with maximal ratio combining.
Figure 3.16. Average symbol error probabilities of 16-ary signal sets (Set 2) in Rayleigh fading channels with maximal ratio combining.
Figure 3.17. Average symbol error probabilities of 16-ary signal sets (Set 1) in Rayleigh fading channels with selection combining.
Figure 3.18. Average symbol error probabilities of 16-ary signal sets (Set 2) in Rayleigh fading channels with selection combining.
Figure 3.19. Average symbol error probabilities of 16-ary signal sets (Set 1) in Nakagami fading channels with maximal ratio combining and $m = 0.7$. 

Nakagami, $m=0.7$ Set 1

- Hexagonal (a)
- 16 star-QAM (b)
- Triangular (c)
- (4,12) (d)
- Rotated (8,8) (e)
- Rectangular (f)
- V.29 (g)
Figure 3.20. Average symbol error probabilities of 16-ary signal sets (Set 2) in Nakagami fading channels with maximal ratio combining and $m = 0.7$. 
Figure 3.21. Average symbol error probabilities of 16-ary signal sets (Set 1) in Nakagami fading channels with maximal ratio combining and $m = 2$. 

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Figure 3.22. Average symbol error probabilities of 16-ary signal sets (Set 2) in Nakagami fading channels with maximal ratio combining and $m = 2$. 
Figure 3.23. Average symbol error probabilities of the 16-ary max-density constellation in Rayleigh fading channels with maximal ratio combining, equal gain combining and selection combining.
more sensitive to errors where the estimated amplitude is larger than the real channel amplitude. As expected, 16 rectangular-QAM has better tolerance than 16 star-QAM to amplitude error but less tolerance to phase error.

As shown in Tables 3.11 and 3.12, (4,4,4,4) has a slightly larger amplitude error tolerance and phase tolerance angle than the V.29 constellation. This confirms that constellation V.29 can be optimised in terms of its ring ratio parameter, leading to the (4,4,4,4) constellation, without sacrificing the amplitude and phase error margin.

### 3.6 Conclusions

Precise analytical expressions for the symbol error probability of multi-point 2-D constellations in fading with diversity reception have been derived. These make an accurate optimisation of constellation parameters feasible at arbitrary signal-to-noise ratios and permit accurate comparisons of the performances of various signal sets. Asymptotically optimum ring ratios for circular constellations have been found to vary with the channel model and the number of diversity branches but are invariant to the combining type.

Among 8-ary constellations, (1,7), max-density, triangular and (4,4) constellations have the best error performance in fading with (4,4) having a slightly higher error rate. The amplitude error tolerance is greatest for 8PSK, triangular and (1,7), while the carrier phase error tolerance is greatest for (4,4), triangular and (1,7). Among 16-ary constellations, the max-density constellation and the (1,5,10) constellation stand out as achieving low error

---

**TABLE 3.12**

<table>
<thead>
<tr>
<th></th>
<th>(4,4,4,4)</th>
<th>Star-QAM</th>
<th>V.29</th>
<th>(8,8)</th>
<th>(1,5,10)</th>
<th>Rectangular</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle</td>
<td>22.9°</td>
<td>22.5°</td>
<td>22.5°</td>
<td>~ 21.66°</td>
<td>18°</td>
<td>16.86°</td>
</tr>
<tr>
<td>Max-density</td>
<td>(5,11)</td>
<td>Triangular</td>
<td>(4,12)</td>
<td>Hexagonal</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Angle</td>
<td>16.42°</td>
<td>16.36°</td>
<td>15.52°</td>
<td>15°</td>
<td>13.5°</td>
<td></td>
</tr>
</tbody>
</table>
rate in fading channels, and (4.4.4.4). V.29 and 16 star-QAM rank as the best three for
carrier phase error tolerance and the worst three for amplitude error tolerance.

As shown in Chapter 2, the ring ratios of the V.29 constellation can be optimised to save
0.63 dB power in AWGN. In this chapter, it is proved that this optimisation improves both
the amplitude and phase error tolerance slightly. Furthermore, diversity techniques while
significantly improving system performance as expected, change the performance ordering
of the constellations. Finally, the relative performances of the signalling formats and their
error margin analysis in various fading channels with and without diversity reception pro-
vide a good reference for practising engineers to select a constellation to meet their design
requirements.
Chapter 4

Effect of Constant Channel Estimation Errors on the Performance of Coherent 2-D Signallings

4.1 Introduction

The analyses in previous chapters all assume perfectly coherent detection. Coherent detection of a digital signal transmitted over a flat fading channel has better power efficiency and it eliminates the error floor exhibited by the conventional differential phase shift keying (DPSK) receivers in fast fading. However, a receiver which employs coherent detection must have complete knowledge of the channel amplitude and phase. In general, the receiver acquires this knowledge from signals received previously over the channel. The result of this estimation process is a noisy amplitude and phase reference which is then used by the receiver for the detection of the incoming signals. The complex gain control (CGC) scales the received signal according to the complex channel gain (both amplitude and phase) so that the decision regions correspond to the transmitted signal constellation. This process is also called fading compensation. Error in fading channel estimation will result in improper scaling and rotation of the received signal. In the worst case, a symbol error can occur even
in the absence of noise. Chapter 3 presented error margins which give the largest tolerable displacement in amplitude and phase of various 8-ary and 16-ary signalling constellations before a symbol error occurs.

In this and the next chapter, we investigate the effects of channel estimation error on the symbol error rates of coherent 2-D signalings in slow Rayleigh fading. As the first step, we consider in this chapter a scenario where there is a fixed estimation error in channel amplitude, or phase, or both. The different amplitude and phase error tolerances of the 8-ary and 16-ary constellations presented in the previous chapter, suggest that these constellations will respond differently to channel estimation errors. In Section 4.2, we discuss the system model used in this chapter. Next, the SER of any 2-D constellation in Rayleigh fading with fixed channel estimation error is derived in Section 4.3. Numerical results for error performance of several 8-ary and 16-ary signaling sets are discussed in Section 4.4. Finally, conclusions are presented in Section 4.5.

### 4.2 System Model

Fig. 4.1 is a block diagram of a coherent receiver with channel estimation and fading compensation. The received signal is given by (3.1) and the flat Rayleigh fading is assumed to be slow enough that it is almost constant over one symbol duration. After the matched filter, we use vector representation for the received signal as

\[ z = g_s i + n \]  

(4.1)

where \( s_i \) is one of the \( M \) signals which has unit average energy, and \( n \) is Gaussian noise with zero mean and variance \( N_0/2 \) in each of its components. Both \( s_i \) and \( n \) are two-dimensional. The fading \( g = \alpha \exp(j\phi) \) is a zero-mean complex Gaussian random variable with its amplitude \( \alpha \) being Rayleigh distributed.

An estimate of the channel fading \( g \) is denoted by \( \hat{g} \). We define \( V = \frac{\hat{g}}{g} = q \exp(j\varphi) \), where \( q = \alpha/\hat{\alpha} \) is the ratio of channel amplitude to the estimated channel amplitude, and \( \varphi = \phi - \hat{\phi} \) is the difference between channel phase and estimated channel phase. In this
Figure 4.1. Block diagram of a coherent receiver with channel estimation in a fading channel.

chapter, \( V \) is a fixed complex number, as we study the effects of a fixed amplitude error and phase error. The decision variable given \( s_1 \) was sent, at the decision threshold is given by

\[
\mathbf{D} = \frac{\mathbf{z}}{\mathbf{g}} = \mathbf{s}_1 + \mathbf{t},
\]

\[
\mathbf{t} = (V - 1)\mathbf{s}_1 + \frac{V\mathbf{n}}{g}.
\] (4.2)

In this way, the detection constellation corresponds to the transmitted constellation where \( s_1 \) is transmitted. If \( \mathbf{D} \) falls into the decision region of \( s_1 \), a correct detection follows. From (4.2), the decision variable can be considered to be the transmitted signal \( s_1 \) translated by a combined noise term \( t = (t_1, t_2) = (r\cos\theta, r\sin\theta) \).

If the sum of \( t \) and \( s_1 \) falls outside the decision region of \( s_1 \), a symbol error occurs. The symbol error probability given \( s_1 \) is given by

\[
P(e|s_1) = \int_{t \in \mathcal{E}_e} \int f_{t|s_1}(t_1, t_2) dt_1 dt_2
\] (4.3)

where \( \mathcal{E}_e \) is the erroneous decision region, and \( f_{t|s_1}(t_1, t_2) \) is the PDF of the combined noise \( t \) conditioned on \( s_1 \). The average probability of a symbol error is therefore given by

\[
P_e = \sum_{i=1}^{M} P(e|s_i)P(s_i)
\] (4.4)

where \( P(s_i) = \frac{1}{M} \) for equally-likely transmission of \( M \) signals.
4.3 Probability of Error in Rayleigh fading with Constant Estimation Error

As in the case of ideal coherent two-dimensional signaling, we can divide the correct decision region of $s_1$ into small disjoint trilaterals, and the erroneous region is correspondingly separated into disjoint subregions. Because $\frac{V_n}{g}$ in (4.2) has circular symmetric phase, we can rotate the constellation so that $s_1$ is on the in-phase axis, without affecting the analysis. This facilitates the analysis and hence we always assume $s_1 = (s_1, 0)$. In addition, translate the origin to $s_1$, with the new in-phase axis still in the direction from the old origin to $s_1$. Typical decision regions of $s_1$ are shown in the new coordinates in Fig. 4.2. The probability of a symbol error given $s_1$ is the summation over $j$ of the probability $P(e, j|s_1)$ that $t$ falls into the $j$th erroneous subregion, that is, [62]

$$P(e|s_1) = \sum_{j=1}^{4} P(e, j|s_1).$$  (4.5)

For the first subregion, and $t = (r\cos\theta, r\sin\theta)$ in polar coordinates,

$$P(e, 1|s_1) = \int_{\theta_1}^{\theta_2} \int_{R(\theta)}^{\infty} f_{t|s_1}(r, \theta) dr d\theta,$$

$$R(\theta) = \frac{x'\sin\psi}{\sin(\theta - \theta_1 + \psi)}$$  (4.6)

where angles $\theta_1, \theta_2, \psi$ and distance $x'$ are shown in Fig. 4.2, $f_{t|s_1}(r, \theta)$ is the probability density function of $t$ in polar coordinates given $s_1$ and can be obtained as

$$f_{t|s_1}(r, \theta) = \int f_{t|s_1,g}(r, \theta) f_g(g) dg$$  (4.7)

where $f_g(g)$ is the PDF of the complex channel fading $g$ and $f_{t|s_1,g}(r, \theta)$ is the PDF of $t$ conditional on signal $s_1$ and channel fading $g$. Combined noise $t$ conditioned on $s_1$ and $g$ is Gaussian, with mean $(V - 1)s_1$ and variance $\frac{N_0 |V|^2}{2} = \frac{N_0 g^2}{\alpha^2}$ in each of its components. That is,

$$f_{t|s_1,g}(r, \theta) = f_{t|s_1,\alpha}(r, \theta) = \frac{\alpha^2 r}{\pi N_0 q^2} \exp\left(-\frac{\alpha^2 |t - (V - 1)s_1|^2}{q^2 N_0}\right).$$  (4.8)
Figure 4.2. Decision region of a signal point in the new coordinates: (a) closed region, (b) open region.

It is obvious that (4.8) depends only on the amplitude of the channel fading $g$, and is independent of the phase of the fading. In Rayleigh fading, the amplitude $\alpha$ of the channel fading process $g$ has PDF

$$f_\alpha(\alpha) = \frac{\alpha}{\sigma^2} \exp\left(-\frac{\alpha^2}{2\sigma^2}\right), \quad 0 < \alpha < \infty. \quad (4.9)$$

Therefore, $f_\beta(g)$ in (4.7) can be replaced with (4.9) and the integration is over $\alpha$ instead of $g$. The PDF of $t$ conditional on $s_1$ is given by

$$f_{t|s_1}(r, \theta) = \int_0^\infty f_{t|s_1, \alpha}(r, \theta) f_\alpha(\alpha) d\alpha$$

$$= \int_0^\infty \frac{r}{\pi N_0 q^2 \sigma^2} \alpha^3 \exp\left[-\left(\frac{1}{2\sigma^2} + \frac{|t-(V-1)s_1|^2}{q^2 N_0}\right) \alpha^2\right] d\alpha$$

$$= \frac{1}{\pi} \frac{r}{\sqrt{\gamma}} \frac{\sqrt{\gamma}}{q} (r^2 + 2r(\mu \cos \theta + \nu \sin \theta) + u^2 + v^2) \quad (4.10)$$

where $\gamma = \frac{2\sigma^2}{N_0}$ is the signal-to-noise ratio per symbol and

$$u = (1 - q \cos \varphi)s_1, \quad v = -qs_1 \sin \varphi. \quad (4.11)$$
The probability of subregion 1 error is obtained by substituting (4.10) into (4.6),

\[
P(e, 1|s_1) = \int_{\theta_1}^{\theta_2} \int_{R(\theta)}^{\infty} \frac{1}{\pi} \frac{r}{\left[ \frac{q}{\sqrt{\gamma}} + \frac{q^2}{\sqrt{\gamma}}(r^2 + 2r(u \cos \theta + v \sin \theta) + u^2 + v^2) \right]^2} dr d\theta
\]

\[
= \int_{\theta_1}^{\theta_2} \frac{q^2}{\pi} \int_{R(\theta)}^{\infty} \frac{r}{(a + br + cr^2)^2} dr d\theta
\]

(4.12a)

where

\[
a = \sqrt{\gamma} s_1^2(q^2 + 1 - 2q \cos \varphi) + \frac{q^2}{\sqrt{\gamma}},
\]

(4.12b)

\[
b = 2\sqrt{\gamma} s_1(\cos \theta - q \cos(\theta - \varphi)),
\]

(4.12c)

\[
c = \sqrt{\gamma} s_1,
\]

(4.12d)

\[
\Delta = 4ac - b^2 = 4q^2 + 4\gamma s_1^2(\sin \theta - q \sin(\theta - \varphi))^2.
\]

(4.12e)

Solving the inner integral with respect to \( r \) in (4.12), we obtain

\[
P(e, 1|s_1) = \int_{\theta_1}^{\theta_2} \frac{q^2}{\pi} W d\theta,
\]

\[
W = \frac{2a + bR(\theta)}{\Delta(a + bR(\theta) + cR^2(\theta))} - \frac{\pi b}{\Delta^{3/2}} + \frac{2b}{\Delta^{3/2}} \tan^{-1}\left( \frac{b + 2cR(\theta)}{\sqrt{\Delta}} \right)
\]

(4.13)

where [35, 2.175.2 and 2.172] have been used. Eqn. (4.13) is new. Note that due to the presence of estimation errors, the phase distribution of the combined noise term \( t = (r \cos \theta, r \sin \theta) \) is no longer uniform between \([0, 2\pi]\). Therefore the integration interval in (4.13) is dependent on the absolute angles of the two sides of a decision subregion with respect to the in-phase (real) axis, not just dependent on the angle difference as in the ideal coherent case (integration interval \([0, \eta]\) where \( \eta = \theta_2 - \theta_1 \)). For other subregions, parameters \( \theta_1, \theta_2, \chi', \psi \) will be different and correspond to the geometry of a particular subregion.

4.4 Two-dimensional 8-ary and 16-ary Signallings

In this section, we evaluate the error performance or performance degradation of coherent 8-ary and 16-ary signal sets in the presence of channel estimation errors in slowly fading
Rayleigh channels. The new formula given by (4.13) is easy to compute as it is a single integral with finite limits and with elementary functions as integrand. One thing worth mentioning is that the \( \tan^{-1}(x) \) function in MATLAB does not achieve high accuracy when \( x \) is large. It is therefore more accurate to use \( \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right) \) for large \( x \). In the presence of channel estimation error, it is also necessary to calculate \( \theta_1 \) and \( \theta_2 \) of each subregion in the MATLAB program.

Fig. 4.3 shows the SER of 16 rectangular-QAM and 16 star-QAM in Rayleigh fading as a function of channel amplitude estimation error. Fig. 4.4 shows the SNR penalty caused by amplitude estimation error for both rectangular and star-QAM when the symbol error rate is fixed at \( 10^{-3} \). The asymptotically optimum ring ratio of 16 star-QAM in Rayleigh fading is used in the calculation of its SER. As discussed in Chapter 3 and apparent in Figs. 4.3 and 4.4, 16 star-QAM has a smaller amplitude error margin than 16 rectangular-QAM. The penalty in the error performance due to amplitude error is substantial, especially for large SNR. A negative amplitude error in dB means the estimated channel magnitude is bigger than the real channel amplitude, while a positive amplitude error in dB indicates a smaller estimated channel amplitude. It is observed in these two figures that the SER performance is not symmetric about 0 dB amplitude error (no estimation error). A bigger amplitude estimation than the real value results in worse performance than an equally deviated but smaller estimation does. This is because the geometry of a signal point's decision region is usually asymmetric in the radial direction about the signal point. When the amplitude estimation error is close to the constellation's amplitude error tolerance, the detection quality is very poor and increasing the SNR does not remedy the poor performance.

The symbol error probability of 16 rectangular-QAM and star-QAM as a function of channel phase estimation error is shown in Fig. 4.5. Fig. 4.6 demonstrates the penalties in SNR (dB) caused by phase estimation error to attain symbol error rate at \( 10^{-3} \), for both rectangular and star-QAM. Sixteen star-QAM has larger phase error tolerance than 16 rectangular-QAM, as expected. In the case of perfect coherent detection as shown in Fig. 4.5, rectangular-QAM is slightly better than star-QAM in terms of error performance.
Figure 4.3. Symbol error probability of 16 rectangular-QAM and 16 star-QAM in Rayleigh fading in the presence of channel amplitude estimation error.
Figure 4.4. The SNR penalty due to channel amplitude estimation error of 16 rectangular-QAM and 16 star-QAM in Rayleigh fading for SER = 10^{-3}.
When the phase estimation error exceeds \( \pm 0.1 \) radian (\( \pm 5.73^\circ \)), star-QAM begins to surpass rectangular-QAM in error performance. It is also observed that the symbol error probabilities are symmetric about zero phase estimation error in sharp contrast to the case of amplitude error.

Fig. 4.7 is a SER plot of 16 rectangular-QAM in the presence of various values of combined channel amplitude and phase estimation error. Observation of the results in Fig. 4.7 and comparison with the results in Figs. 4.3 and 4.5 indicates that combined amplitude and phase error has a more severe effect on the error performance than either amplitude error or phase error alone, as expected. An interesting observation from the third upper curve in Fig. 4.7 is that with the gradual increase of channel amplitude and phase estimation error, the symbol error performance deteriorates to a point where the error rate curve no longer decreases as SNR increases, but slightly increases for high and increasing SNR. This phenomenon can be explained as follows. If the error in the channel estimation causes the received noiseless signal to move very close to the decision boundary, the noise superimposed on the received noiseless signal at a particular power level will act as "signal" and bring the total received signal into the correct decision region. Therefore, the lowest error probability achieves at a certain SNR value in the presence of large channel estimation errors.

Fig. 4.8 shows the analytical and simulated symbol error performance of 16 rectangular-QAM in the presence of channel estimation errors at \( q = -2 \) dB, \( \phi = 10^\circ \) and \( q = 2 \) dB, \( \phi = 10^\circ \). The analytical results are shown to agree with simulation results very well. Again, it is observed that underestimating the channel amplitude is less harmful than overestimating the amplitude for 16 rectangular-QAM.

Fig. 4.9 presents the SER curves of eleven 16-ary signal sets as a function of SNR in the presence of channel amplitude error \( q = -2 \) dB and phase estimation error \( \phi = 10^\circ \). Since the effect of a particular channel estimation error on each constellation is unique, the relative performance of these signal sets is significantly different than that in the presence of different channel estimation errors. The SER of 16-ary signal sets in Rayleigh fading

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Figure 4.5. Symbol error probability of 16 rectangular-QAM and 16 star-QAM in Rayleigh fading in the presence of channel phase estimation error.
Figure 4.6. The SNR penalty due to channel phase estimation error of 16 rectangular-QAM and 16 star-QAM in Rayleigh fading for SER = $10^{-3}$. 
Figure 4.7. Symbol error probability of 16 rectangular-QAM in Rayleigh fading in the presence of channel amplitude and phase error.
Figure 4.8. Symbol error probability of 16 rectangular-QAM in Rayleigh fading in the presence of channel amplitude and phase error.
for the case of perfect channel estimation is plotted in Fig. 3.15. The performance differences among these signal sets in this case are almost negligible. However, in the presence of channel estimation error shown in Fig. 4.9, the differences are enlarged enormously. Signal sets hexagonal, rectangular, V.29, (4,4,4,4) and max-density do not have useful performances at all in Fig. 4.9, while (4,12) and (5, 11) are clear winners. It is observed that circular constellations with two rings are generally more robust to combined amplitude and phase estimation error.

Figs. 4.10 and 4.11 present the SER curves of eleven 16-ary signal sets as a function of SNR in the presence of channel amplitude error \( q = 2 \) dB and phase estimation error \( \varphi = 10^\circ \). Except for the hexagonal signal set, the other constellations all have error curves that decrease with an increase in SNR. Sixteen rectangular-QAM has rather poor performance compared to the best four signal sets (1,5,10), (5, 11), 16 star-QAM and (4,4,4,4) in this scenario.

Fig. 4.12 shows the symbol error probability of six 8-ary sets in Rayleigh fading with channel estimation error \( q = -3 \) dB and \( \varphi = 15^\circ \). Again, the figure shows that a circular type of constellation, e.g. 8PSK, is less adversely affected by the channel amplitude and phase estimation error. Rectangular and max-density sets produce poor performances with this level of channel estimation error. Fig. 4.13 shows the symbol error probability of the 8-ary sets in Rayleigh fading with channel estimation error of \( q = 3 \) dB and \( \varphi = 15^\circ \). Underestimating channel amplitude results in better performance than equally overestimating channel amplitude. The (1,7) set outperforms other 8-ary constellations in both Figs. 4.12 and 4.13.

### 4.5 Conclusions

In this chapter we have analysed the symbol error probability of arbitrary polygonal two-dimensional constellations in slow Rayleigh fading channels with arbitrary constant channel amplitude and phase estimation error. The error rate expression is in the form of a single
Figure 4.9. Symbol error probabilities of 16-ary signal sets in Rayleigh fading with channel amplitude error $q = -2\text{dB}$ and phase error $\varphi = 10^\circ$. 
Figure 4.10. Symbol error probabilities of 16-ary signal sets in Rayleigh fading with channel amplitude error $q = 2$ dB and phase error $\varphi = 10^\circ$. 
Figure 4.11. Symbol error probabilities of 16-ary signal sets in Rayleigh fading with channel amplitude error $q = 2\, \text{dB}$ and phase error $\varphi = 10^\circ$. 
Figure 4.12. Symbol error probabilities of 8-ary signal sets in Rayleigh fading with channel amplitude error $q = -3$dB and phase error $\varphi = 15^\circ$. 
Figure 4.13. Symbol error probabilities of 8-ary signal sets in Rayleigh fading with channel amplitude error $q = 3\text{dB}$ and phase error $\varphi = 15^\circ$. 
integral with finite integration interval and elementary functions as integrand, which is a general easy-to-compute formula applicable to any 2-D constellation with polygonal decision regions. The numerical results for various 8-ary and 16-ary signal sets have demonstrated that imperfect channel estimation degrades the error performance of a constellation significantly, and that some signal sets have much better amplitude and/or phase estimation error immunity than others. In general, a circular constellation with one or two rings is more robust to channel estimation errors than other constellation structures. Although the previous chapter has demonstrated that in the worst fading case (Rayleigh fading), the performances of various coherent 2-D modulation formats are not that much different from one another, the study in this chapter clearly eliminates some candidate signal sets such as the 16-ary hexagonal, the rectangular-QAM, V.29, (4,4,4,4), the 16-ary max-density, the 8-ary rectangular and the 8-ary max-density set because of their poor performances in the presence of channel estimation error, as shown in Figs. 4.9, 4.10 and 4.12.
Chapter 5

Pilot Symbol Aided Modulation for Two-Dimensional Signallings

5.1 Introduction

The effect of constant channel estimation errors on the performance of coherent 2-D signal constellations in Rayleigh fading has been examined in Chapter 4. Substantial degradations in the symbol error probability experienced by a 2-D signalling due to the presence of channel estimation errors have been observed and some constellations have shown greater immunity to such errors than other constellations. Practical channel estimation techniques in general result in estimations correlated to the complex Gaussian fading process and have particular distributions. Usually channel fading estimates are assumed to be also Gaussian distributed and hence the estimation error and the channel fading are joint complex Gaussian processes [44], [51]. This is true for minimum mean square error (MMSE) estimation and pilot symbol aided modulation (PSAM) schemes. The Gaussian nature of the fading estimates enables analytical study of the performance of these channel estimation techniques.

Pilot symbol aided modulation schemes use known symbols called pilot symbols to estimate channel fading and apply the estimate to the detection of other symbols. Among
the various fading estimation techniques. PSAM proves to be an effective choice. It was first proposed by Lodge and Moher [63], [64], and Sampei and Sunaga [65]. Cavers [66] conducted theoretical analysis of pilot symbol assisted modulation for Rayleigh fading channels and derived the optimum interpolation filter to minimise the variance of the estimation error. In [66], the BER of BPSK and QPSK, and also a union bound for the SER of 16 rectangular-QAM in a PSAM system were presented. A suboptimum interpolator, the sinc interpolator, was proposed by Kim et al. in [67], while Gaussian interpolation was used in [68]. In [69], the BER performance of $M$-QAM with PSAM in a Rayleigh flat fading channel was evaluated by numerical computation and verified by simulation. The sinc interpolator was also adopted in this analysis. Studies of pilot symbol aided modulation in various wireless environments can also be found in [70]-[75].

In this chapter, we investigate the performance of 2-D signalling using pilot symbol aided modulation in slow Rayleigh fading. The chapter is organised as follows. In Section 5.2, we outline the system model and describe the PSAM system. The symbol error probability of pilot symbol aided 2-D signalling in slow Rayleigh fading is derived in Section 5.3, followed by some simulation results given in Section 5.4. The chapter is concluded by Section 5.5.

5.2 System Model

5.2.1 Pilot Symbol Aided Modulation

A block diagram of a pilot symbol aided system is given in Fig. 5.1. In pilot symbol aided systems, known symbols called pilot symbols are inserted into the transmitted data sequence periodically. A frame is defined as a sequence of symbols beginning from the pilot symbol to the symbol right before the next pilot symbol, as shown in Fig. 5.2. Assume there are $L$ symbols in a frame, and that each symbol has duration $T$. Consider a Rayleigh fading channel. The channel fading at the $l$th symbol interval in the $k$th frame is denoted by the zero-mean complex Gaussian random variable $g_{k,l} = \alpha_{k,l}e^{j\theta_{k,l}}$, where amplitude $\alpha_{k,l}$ is
Rayleigh distributed and phase $\phi_{k,l}$ is uniformly distributed between $[-\pi, \pi)$. The lowpass equivalent representation of the received signal sampled at the $l$th symbol position in the $k$th frame is given by

$$z_{k,l} = g_{k,l}s_{k,l} + n_{k,l}$$  \hspace*{2.5cm} (5.1)$$

where $s_{k,l} = \{s_i\}, i = 1, \ldots, M$ and $s_i = A_i e^{j\varphi_i}$ is the sampled complex envelope of the transmitted signal, and $n_{k,l} = n_R + jn_I$ is the zero-mean Gaussian noise RV with variance $N_0/2$ in both of its real and imaginary parts.

Let $k = 0$ represent the current frame, and omit $k = 0$ in the subscript for notational
simplicity. Channel fading $g_l$ (equivalent to $g_{0,l}$) is estimated from $K$ pilot symbols, that is, the previous $K_1 = [(K - 1)/2]$ pilot symbols, the current pilot and the subsequent $K_2 = [K/2]$ pilots as

$$
\hat{g}_l = \hat{a}_l e^{j\hat{\phi}_l} = \sum_{k=-K_1}^{K_2} h_{k,l} \hat{g}_{pk}
$$

(5.2)

where $[\cdot]$ is the floor function. $\hat{g}_{pk}$ is the estimated channel fading at the $k$th pilot symbol interval and the $h_{k,l}$ is the weighting coefficient for $\hat{g}_{pk}$ and is dependent on the current symbol position $l$. The estimated channel fading for the $k$th pilot symbol is given by

$$
\hat{g}_{pk} = \frac{z_{k,0}}{\rho_k} = g_{pk} + \frac{n_{pk}}{\rho_k}
$$

(5.3)

where $z_{k,0}$ is the received signal at the first symbol interval in the $k$th frame and $\rho_k$ is the known pilot symbol which is the first symbol in the $k$th frame. The channel fading $g_{pk} = g_{k,0}$ and the additive noise $n_{pk} = n_{k,0}$ are independent complex Gaussian variables: therefore $\hat{g}_{pk}$ and $\hat{g}_l$ are Gaussian as well. The current channel fading $g_l$ and its estimate $\hat{g}_l$ are correlated complex Gaussian variables and their joint amplitude probability density is bivariate Rayleigh given by [76]

$$
f(\alpha_l, \hat{\alpha}_l) = \frac{4\alpha_l \hat{\alpha}_l}{(1-\rho)\tilde{\Omega}\tilde{\Omega}} I_0 \left( \frac{2\sqrt{\rho} \alpha_l \hat{\alpha}_l}{(1-\rho)\sqrt{\Omega}\tilde{\Omega}} \right) \exp \left[ -\frac{1}{1-\rho} \left( \frac{\alpha_l^2}{\Omega} + \frac{\hat{\alpha}_l^2}{\tilde{\Omega}} \right) \right], 0 \leq \rho < 1
$$

(5.4)

where $\rho$ is the average of $\rho_l$ over $l = 1, \ldots, L$ and $\rho_l = \frac{\text{Cov}(\alpha_l^2, \hat{\alpha}_l^2)}{\sqrt{\text{Var}(\alpha_l^2)}\sqrt{\text{Var}(\hat{\alpha}_l^2)}}$ is the correlation coefficient between $\alpha_l^2$ and $\hat{\alpha}_l^2$, $\tilde{\Omega} = E[\alpha_l^2]$, $\tilde{\Omega}$ is the average of $\tilde{\Omega}_l$ over $l = 1, \ldots, L$ and $\tilde{\Omega}_l = E[\hat{\alpha}_l^2]$. The PDF of the phase difference between $g_l$ and $\hat{g}_l$ is given by [77]

$$
f(\varphi) = \frac{1-\rho}{4\pi^2} \left[ (1-q^2)^{1/2} + q'(\pi - \cos^{-1} q') \right] (2\pi - |\varphi|), -2\pi \leq \varphi \leq 2\pi
$$

(5.5)

where $\varphi = \phi_l - \hat{\phi}_l$ and $q' = \sqrt{\rho} \cos \varphi$.

### 5.2.2 Interpolation methods

Three interpolation methods for determining $h_{k,l}$ have been reported. The Wiener interpolator proposed by Cavers [66] achieves the optimal performance, but it requires a priori
information of the autocorrelation function of the channel gain, Doppler frequency and SNR to obtain filter coefficients and a large number of computations. Sampei et al. [68] used a simple Gaussian interpolator that works well when pilot symbols are frequently inserted compared with the Nyquist rate of the fading. However, as the period of pilot symbols increases, the performance of the Gaussian interpolator degrades rapidly compared with the Wiener filter [66]. The sinc interpolator proposed by Kim et al. [67] has near optimum performance and is simple to implement. The sinc interpolator is adopted in this chapter.

The pilot symbol frequency is given by $\frac{1}{L_d}$. In order to use the fading samples obtained at pilot symbol intervals to fully represent the fading process without aliasing effects, it is necessary to satisfy the sampling theorem which requires $L \leq \frac{1}{2L_d}$. This gives an upper bound on pilot symbol spacing. In practice, the Doppler spectrum is not known in the receiver. Therefore, lowpass interpolation filters are designed to approximate a brickwall magnitude response up to frequency $\frac{1}{2L}$ and have a linear phase response. This leads to the sinc interpolator given by

$$h(n) = \text{sinc} \left( \frac{n}{L} \right), \quad -K_1L \leq n \leq K_2L. \quad (5.6)$$

To smooth the abrupt truncation of rectangular windowing (in the time domain), a Hamming window can be applied to the sinc interpolator so that

$$h(n) = w(n) * \text{sinc} \left( \frac{n}{L} \right), \quad -K_1L \leq n \leq K_2L \quad (5.7)$$

where

$$w(n) = 0.54 - 0.46 \cos \left( \frac{2\pi n}{KL - 1} + \frac{2\pi \left\lfloor \frac{KL}{2} \right\rfloor}{KL - 1} \right), \quad -K_1L \leq n \leq K_2L. \quad (5.8)$$

The estimated fading is hence obtained as

$$\hat{g}_l = \sum_{k=-K_1}^{K_2} h(l-kL) \hat{g}_{pk} = \sum_{k=-K_1}^{K_2} h_{k,l} \hat{g}_{pk} \quad (5.9)$$

where $1 \leq l \leq L - 1$. The interpolation coefficients with a rectangular window are given by

$$h_{k,l} = \text{sinc} \left( \frac{l}{L} - k \right), \quad -K_1 \leq k \leq K_2 \quad (5.10)$$
while with the Hamming window,

\[ h_{k,l} = \text{sinc} \left( \frac{l}{L} - k \right) \left[ 0.54 - 0.46 \cos \left( \frac{2\pi (l - kL)}{KL - 1} + \frac{2\pi |K_2|}{KL - 1} \right) \right], \quad -K_1 \leq k \leq K_2. \]  

(5.11)

### 5.2.3 Derivation of \( \rho \) and \( r_\Omega \)

The joint distribution of \( \alpha_1 \) and \( \hat{\alpha}_1 \) depends on parameters \( \rho \), \( \Omega \) and \( \hat{\Omega} \), and the phase difference distribution \( \phi \) depends on \( \rho \), as shown in (5.4) and (5.5). In this subsection, we derive the correlation coefficient \( \rho \) and the ratio \( r_\Omega = \hat{\Omega}/\Omega \), which will be used in the subsequent symbol error probability analysis. The method presented here follows [69].

The complex Rayleigh fading process is expressed as \( g(t) = g_R(t) + jg_I(t) \), where \( g_R(t) \) and \( g_I(t) \) are zero mean Gaussian random processes and are mutually independent. The autocorrelation and cross-correlations are given by

\[
\begin{align*}
R_{gR}(\tau) &= E[g_R(t)g_R(t+\tau)] = R(\tau) = \frac{\Omega}{2} J_0(2\pi f_D \tau) \\
R_{gI}(\tau) &= E[g_I(t)g_I(t+\tau)] = R(\tau) = \frac{\Omega}{2} J_0(2\pi f_D \tau) \\
R_{RI}(\tau) &= E[g_R(t)g_I(t+\tau)] = 0.
\end{align*}
\]  

(5.12)

In slow fading, the fading samples in one symbol interval are assumed to be approximately constant. Therefore, \( g_{k,l} = g_{R,kl} + jg_{I,kl} \) is the fading sample in the \( l \)th symbol position in the \( k \)th frame. The correlation between complex fading \( g_{k,l} \) and \( g_{i,m} \) is defined as [17]

\[
R_{k,l;i,m} = \frac{1}{2} E[(g_{R,kl} + jg_{I,kl})(g_{R,im} - jg_{I,im})] \\
= \frac{1}{2} \left( E[g_{R,kl}g_{R,im}] + E[g_{I,kl}g_{I,im}] \right) \\
= R(|(i-k)L+(m-l)|T) = \frac{\Omega}{2} J_0(2\pi f_D |(i-k)L+(m-l)|T). \tag{5.13}
\]

Define a covariance matrix \( C \) for fading at pilot symbols as

\[
C_{ki} = \frac{1}{2} \text{Cov}(g_{p_k}, g_{p_i}^*), \\
= \frac{1}{2} E[g_{p_k} g_{p_i}^*] \\
= R(|k-i|LT), \quad -K_1 \leq i, k \leq K_2. \tag{5.14}
\]

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The Rayleigh fading estimate $\hat{g}_l$ in the current frame $k = 0$ (drop subscript $k = 0$) is given by (5.2) and (5.3) as

$$\hat{g}_l = \sum_{k=-K_l}^{K_l} h_{k,l} \hat{g}_{pk}$$

$$= \sum_{k=-K_l}^{K_l} h_{k,l}(g_{pk} + \frac{n_{pk}}{p_k}).$$  \hspace{1cm} (5.15)

The average power $\hat{\Omega}_l$ of the zero mean fading estimate $\hat{g}_l$ is given by

$$\hat{\Omega}_l = E[\hat{g}_l \hat{g}_l^*] = E[\hat{\alpha}_l^2]$$

$$= E[\sum_{k=-K_l}^{K_l} h_{k,l}g_{pk}^2] + E[\sum_{k=-K_l}^{K_l} h_{k,l} \frac{n_{pk}}{p_k}^2]$$

$$= \sum_{k=-K_l}^{K_l} \sum_{i=-K_l}^{K_l} h_{k,l}h_{i,l}2C_{kl} + \sum_{k=-K_l}^{K_l} \frac{N_0 h_{k,l}^2}{|p_k|^2}. \hspace{1cm} (5.16)$$

Define a row vector $H_l = [h_{-K_l,l}, \ldots, h_{K_l,l}]$, and hence

$$\hat{\Omega}_l = 2H_l^C H_l^T + \frac{N_0}{|p_k|^2} |H_l|^2.$$  \hspace{1cm} (5.17)

Since the pilot symbols are selected from the signal set $\{s_i\}, 1 \leq i \leq M$ with unit average energy, we can take $|p_k|^2 = 1$. The ratio of the estimated fading average power to the fading average power is therefore given by

$$r'_\Omega = \frac{\hat{\Omega}_l}{\Omega} = H_l C_0 H_l^T + \frac{|H_l|^2}{\Lambda}$$  \hspace{1cm} (5.18)

where $\Lambda = \frac{\Omega}{N_0}$ and $C_0 = \frac{2}{\Omega} \mathbf{C}$. It is seen from (5.18) that $r'_\Omega$ depends on the symbol position $l$ within a frame. In the later symbol error probability calculation, we use the average of $r'_\Omega$ over $L - 1$ data positions within a frame, i.e.,

$$r_\Omega = \frac{1}{L-1} \sum_{l=1}^{L-1} r'_\Omega.$$  \hspace{1cm} (5.19)
To obtain $p$, first we have to derive

\[
\text{Cov}(\alpha_l^2, \hat{\alpha}_l^2) = E[\alpha_l^2\hat{\alpha}_l^2] - E[\alpha_l^2]E[\hat{\alpha}_l^2] = E[g_l^*g_l^*] = \Omega_l
\]

Since the real and imaginary parts in fading $g_l = g_{R,l} + jg_{I,l}$ are zero mean Gaussian and mutually independent, and by using

\[
\]

where $a, b, c, d$ are jointly normal with zero mean, we have

\[
E[g_l^*g_l^*] = E[(g_{R,l}^2 + g_{I,l}^2)(g_{R,pl}^2 + g_{I,pl}^2)] = 2E[g_{R,l}^2]E[g_{R,pl}^2] + 4E[g_{R,l}g_{R,pl}]E[g_{R,I}g_{R,pl}].
\]

Therefore,

\[
\text{Cov}(\alpha_l^2, \hat{\alpha}_l^2) = 2\Omega_lH_l\mathbf{CH}_l^T + 4\left(\sum_{k=-K_1}^{K_2} h_{k,l}E[g_{R,l}g_{R,pl}]\right)^2 + \frac{|H_l|^2\Omega_l^2}{\Lambda} - \Omega_l^2.
\]

The variance of $\alpha_l^2$ and $\hat{\alpha}_l^2$ is given by

\[
\text{Var}(\alpha_l^2) = E[\alpha_l^4] - E^2[\alpha_l^2] = \Omega_l^2
\]

\[
\]
The correlation coefficient \( \rho_l \) is therefore written as
\[
\rho_l = \frac{\text{Cov}(\alpha_l^2, \delta_l^2)}{\sqrt{\text{Var}(\alpha_l^2) \text{Var}(\delta_l^2)}} = \frac{4(\sum_{k=-K}^{K} h_{k,l} R(|kL-l|T)^2)}{\Omega_l \hat{\Omega}_l} = \frac{\Lambda(\sum_{k=-K}^{K} h_{k,l} R(|kL-l|T)^2)}{\Lambda H_l C_0 H_l^T + |H_l|^2}.
\]
(5.25)

The correlation coefficient \( \rho \) is obtained by averaging \( \rho_l \) over \( L - 1 \) data symbol positions.

### 5.3 Symbol Error Probability of PSAM

The decision variable for \( l \)th symbol in the current frame \((k = 0 \text{ and is omitted in the subscript}) \) given \( s_1 \), is given by
\[
D = \frac{z_l}{\hat{g}_l} = \frac{g_l}{\hat{g}_l} s_1 + \frac{n_l}{\hat{g}_l} = s_1 + t,
\]
\[
t = \frac{g_l}{\hat{g}_l} s_1 - s_1 + \frac{n_l}{\hat{g}_l} = (V - 1) s_1 + \frac{n_l}{\hat{g}_l}
\]
(5.26)

where \( V = \frac{g_l}{\hat{g}_l} = \frac{\alpha_l}{\hat{\alpha}_l} e^{j\phi} \). As discussed in the previous sections, fading \( g_l \) and its estimate \( \hat{g}_l \) are jointly Gaussian and independent of the noise Gaussian variable \( n_l \). Therefore, conditioned on \( s_1, g_l \) and \( \hat{g}_l \), the PDF of \( t = re^{j\phi} \) is Gaussian, given by
\[
\begin{align*}
  f_{t|g_l,\hat{g}_l}(r, \phi) &= \frac{r^2 \hat{\alpha}_l^2}{\pi N_0} \exp \left[ -\frac{\hat{\alpha}_l^2 |re^{j\phi} - (V - 1)s_1|^2}{N_0} \right] \quad \text{(5.27)}
\end{align*}
\]

The probability that the received signal falls into erroneous subregion \( j \) is given by
\[
P(e, j|s_1) = \int_{\theta_{j,1}}^{\theta_{j,2}} \int_{R(\theta)}^{\infty} \int_{-2\pi}^{2\pi} f_{t|g_l,\hat{g}_l}(r, \phi) f(\alpha_l, \hat{\alpha}_l) f(\phi) d\phi d\alpha_l d\hat{\alpha}_l dr d\theta
\]
\[
\times \int_{\theta_{j,1}}^{\theta_{j,2}} \int_{R(\theta)}^{\infty} \int_{-2\pi}^{2\pi} f_{t|g_l,\hat{g}_l}(r, \phi) f(\alpha_l, \hat{\alpha}_l) f(\phi) d\phi d\alpha_l d\hat{\alpha}_l dr d\theta
\]
(5.28)

where angles \( \theta_{j,1} \) and \( \theta_{j,2} \) are \( \theta_1 \) and \( \theta_2 \) as shown in Fig. 4.2 corresponding to the \( j \)th decision region, and
\[
R(\theta) = \frac{b_j \sin \psi_j}{\sin(\theta - \theta_{j,1} + \psi_j)}
\]
(5.29)

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as explained in Chapters 2 and 3. Drop subscript $l$ in (5.28) to simplify notation. To solve (5.28), perform integration over $r$ first,

$$
M_1 = \int_{R(\theta)}^\infty f_{\rho \xi, \phi}(r, \theta)dr
= \frac{1}{2\pi} \exp \left( -\frac{(\alpha R - \rho)^2 + \nu^2}{N_0} \right) + \frac{\mu}{\sqrt{\pi N_0}} Q\left( (\alpha R - \rho) \sqrt{\frac{2}{N_0}} \exp\left( -\frac{\nu^2}{N_0} \right) \right) \tag{5.30}
$$

where

$$
u = s_1(\alpha \cos(\theta - \varphi) - \alpha \cos \theta), \quad \nu = s_1(\alpha \sin(\theta - \varphi) - \alpha \sin \theta). \tag{5.31}
$$

Next, perform the integration over $\alpha$ and $\hat{\alpha}$ in (5.28).

$$
M_2 = \int_0^\infty \int_0^\infty M_1 f(\alpha, \hat{\alpha}) d\alpha d\hat{\alpha}
= M_{21} + M_{22} \tag{5.32}
$$

where

$$
M_{21} = \int_0^\infty \int_0^\infty \frac{1}{2\pi} \exp \left( -\frac{(\alpha R - \rho)^2 + \nu^2}{N_0} \right) f(\alpha, \hat{\alpha}) d\alpha d\hat{\alpha},
\quad M_{22} = \int_0^\infty \int_0^\infty \frac{\mu}{\sqrt{\pi N_0}} Q\left( (\alpha R - \rho) \sqrt{\frac{2}{N_0}} \exp\left( -\frac{\nu^2}{N_0} \right) \right) f(\alpha, \hat{\alpha}) d\alpha d\hat{\alpha}. \tag{5.33}
$$

Combining the integral expression for $I_0(z)$.

$$
I_0(z) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-z\sin \lambda} d\lambda \tag{5.34}
$$

with $M_{21}$ and $M_{22}$ in (5.33) and changing the integration order, we have

$$
M_{21} = \frac{2}{\pi^2(1-\rho)\Omega^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\infty \int_0^\infty \alpha \hat{\alpha} e^{-\left( A^2 \alpha^2 + B^2 \hat{\alpha}^2 + 2C \alpha \hat{\alpha} \right)} d\alpha d\hat{\alpha} d\lambda,
\quad M_{22} = \frac{4}{\pi(1-\rho)\Omega^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\infty \int_0^\infty s_1(\alpha \cos(\theta - \varphi) - \alpha \cos \theta) \frac{\mu}{\sqrt{\pi N_0}} Q\left( (\alpha R - \rho) \sqrt{\frac{2}{N_0}} \exp\left( -\frac{\nu^2}{N_0} \right) \right) d\alpha d\hat{\alpha} d\lambda \tag{5.35}
$$

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where

\[
A^2 = \frac{1}{(1-\rho)\Omega} + \frac{s_1^2}{N_0},
\]

\[
B^2 = \frac{1}{(1-\rho)\hat{\Omega}} + \frac{R^2 + 2s_1R\cos\theta + s_1^2}{N_0},
\]

\[
C = \frac{\sqrt{\rho}\sin\lambda - \frac{s_1R\cos(\theta - \varphi) + s_1^2\cos\varphi}{(1-\rho)\sqrt{\Omega\hat{\Omega}}}}{N_0},
\]

\[
D^2 = \frac{s_1^2}{N_0}\sin^2(\theta - \varphi) + \frac{1}{(1-\rho)\Omega},
\]

\[
E^2 = \frac{s_1^2}{N_0}\sin^2\theta + \frac{1}{(1-\rho)\Omega},
\]

\[
F = \frac{\sqrt{\rho}\sin\lambda - \frac{s_1^2\sin\theta\sin(\theta - \varphi)}{(1-\rho)\sqrt{\Omega\hat{\Omega}}}}{N_0}
\]

(5.36)

We make a change of variables \( x = \frac{\cos\omega}{A} \) and \( \alpha = \frac{\sin\omega}{B} \) to \( M_{21} \),

\[
M_{21} = \frac{2}{\pi^2(1-\rho)\Omega\hat{\Omega}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \sin\omega \cos\omega A^2B^2 e^{-(1+C\sin\omega)\frac{s_1^2}{AB}} \sin 2\omega d\omega d\lambda
\]

\[
= \frac{1}{\pi^2(1-\rho)\Omega\hat{\Omega}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2(AB + C\sin 2\omega)^2 \sin\omega d\omega d\lambda
\]

\[
= \frac{1}{2\pi^2(1-\rho)\Omega\hat{\Omega}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\omega' \sin 2\omega' d\omega' d\lambda
\]

\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2\pi^2(1-\rho)\Omega\hat{\Omega}(A^2B^2 - C^2)} \left[ 1 - \frac{2C}{\sqrt{A^2B^2 - C^2}} \tan^{-1} \left( \frac{AB - C}{AB + C} \right) \right] d\lambda
\]

(5.37)

where [35, 2.551.1] has been used.

Similarly, make a change of variables \( x = \frac{\cos\omega}{D} \) and \( \alpha = \frac{\sin\omega}{E} \) to \( M_{22} \) giving,

\[
M_{22} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} U \int_{0}^{\infty} x^4 Q(mx) e^{-px^2} dx d\omega d\lambda
\]

(5.38)

where

\[
U = \frac{2\sin 2\omega s_1 (E \cos(\theta - \varphi) \cos\omega) - D \cos\theta \sin\omega)}{\pi\sqrt{nN_0(1-\rho)\Omega\hat{\Omega}D^3E^3}},
\]

\[
m = \sqrt{\frac{2}{N_0}} \left[ \frac{(R + s_1 \cos\theta) \sin\omega - s_1 \cos(\theta - \varphi) \cos\omega}{E} \right],
\]

\[
p = 1 + \frac{F \sin 2\omega}{DE}
\]

(5.39)
By using integration by parts and the integral identity [35, 6.285.1], the integral

\[ I = \int_0^\infty x^4 Q(mx) e^{-px^2} dx \]

\[ = \begin{cases} 
\frac{3\tan^{-1}\left(\frac{\sqrt{2}}{6}\right)}{8\sqrt{\pi}p^2} - \frac{10mp+3m^3}{4\sqrt{2\pi}p^2(2p+m)^2} & m \geq 0 \\
\frac{3(\pi - \tan^{-1}\left(\frac{\sqrt{2}}{6}\right))}{8\sqrt{\pi}p^2} - \frac{10mp+3m^3}{4\sqrt{2\pi}p^2(2p+m)^2} & m < 0
\end{cases} \]  \quad (5.40)

Therefore,

\[ M_{22} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} U_1 d\omega d\lambda. \]  \quad (5.41)

Recall that the received average signal-to-noise ratio is \( \Lambda = \frac{\Omega}{N_0} \) and the fading power ratio is \( r_\Omega = \hat{\Omega}/\Omega \). The probability of error in (5.28) is then written as

\[
P(\epsilon, j|s_1) = \int_{-2\pi}^{2\pi} \int_{\theta_{j,1}}^{\theta_{j,2}} M_{21} f(\phi) d\theta d\phi + \int_{-2\pi}^{2\pi} \int_{\theta_{j,1}}^{\theta_{j,2}} M_{22} f(\phi) d\theta d\phi
\]

\[ = \int_{-2\pi}^{2\pi} \int_{\theta_{j,1}}^{\theta_{j,2}} G(\lambda, \theta, \phi) f(\phi) d\lambda d\theta d\phi
\]

\[ + \int_{-2\pi}^{2\pi} \int_{\theta_{j,1}}^{\theta_{j,2}} H(\omega, \lambda, \theta, \phi) f(\phi) d\lambda d\omega d\theta d\phi \]  \quad (5.42)

where

\[
G(\lambda, \theta, \phi) = \frac{1}{2\pi^2(1-\rho)(a^2+b^2-c^2)} \left[ 1 - \frac{2c}{\sqrt{a^2b^2-c^2}} \tan^{-1}\left(\frac{ab-c}{ab+c}\right) \right],
\]

\[
a^2 = \frac{1}{1-\rho} + \Lambda s_1^2,
\]

\[
b^2 = \frac{1}{1-\rho} + \Lambda r_\Omega (R^2 + 2s_1 R \cos \theta + s_1^2),
\]

\[
c = \frac{\sqrt{\rho} \sin \lambda}{1-\rho} - \Lambda \sqrt{r_\Omega} [s_1 R \cos(\theta - \phi) + s_1^2 \cos \phi] \]  \quad (5.43)
and

\[
H(\omega, \lambda, \theta, \varphi) = \begin{cases} 
\frac{3\tan^{-1}(\sqrt{\frac{p}{\omega}})}{2p^2} - \frac{10mp+3m^3}{\sqrt{2}p^2(2p+m)^2} & m \geq 0 \\
\frac{3(\pi-\tan^{-1}(\sqrt{\frac{p}{\omega}}))}{2p^2} - \frac{10mp+3m^3}{\sqrt{2}p^2(2p+m)^2} & m < 0,
\end{cases}
\]

\[
k = \frac{\sin 2\omega \sqrt{\Lambda s_1 (g \cos(\theta - \varphi) \cos \omega - h \sqrt{r_\omega} \cos \theta \sin \omega)}}{2\pi^2(1 - \rho) h^3 g^3},
\]

\[
p = 1 + f \frac{\sin 2\omega}{gh},
\]

\[
m = \frac{\sqrt{2}\Lambda}{g} \left[ \frac{\sqrt{r_\omega} (R + s_1 \cos \theta) \sin \omega}{h} - \frac{s_1 \cos(\theta - \varphi) \cos \omega}{h} \right],
\]

\[
h^2 = \frac{1}{1 - \rho} + \Lambda s_1^2 \sin^2(\theta - \varphi),
\]

\[
g^2 = \frac{1}{1 - \rho} + \Lambda r_\omega s_1^2 \sin^2 \theta,
\]

\[
f = \frac{\sqrt{\rho} \sin \lambda}{1 - \rho} - \Lambda \sqrt{r_\omega} s_1^2 \sin \theta \sin(\theta - \varphi).
\]

(5.44)

The symbol error probability of an arbitrary 2-D signalling is thus obtained by adding the probability (5.42) accounting for all the erroneous subregions \((j = 1, \ldots, J)\) and then repeating for all possible transmitted symbols \((s_1, \ldots, s_M)\).

### 5.4 Simulation

The symbol error probability analysis for an arbitrary 2-D constellation with PSAM presented in the previous section involves one three-fold integral and one four-fold integral. This makes the numerical evaluation of (5.42) highly computationally intensive. In this case, it becomes more reasonable to use a simulation technique to obtain the error performance of pilot symbol aided 2-D modulation. In this section, the simulation results are presented and discussed for 16 rectangular-QAM, 16 star-QAM and constellation (5, 11) as examples. Constellation (5, 11) was shown to have a rather robust performance in the presence of constant channel estimation error in Chapter 4.

The simulations are implemented using Cadence’s Signal Processing Worksystem (SPW)
4.5. The SPW design block diagram for the BER of a pilot symbol aided 16 rectangular-QAM system is illustrated in Fig. 5.3. In order to get results with high confidence, the total number of bits for one simulation run is $4 \times 10^5$ for small to intermediate SNR values, that is, $\text{SNR} < 30 \text{ dB}$, and for large SNRs, the simulation length is between $2 \times 10^7$ and $9 \times 10^7$ bits. The SPW only requires the user to specify one noise seed for a system simulation, and automatically provides independent seeds for every random source in the simulation. We always run one simulation three times with three different noise seeds, and use the average error probability as the final result. This applies to BER and SER simulations for all constellations, 16 rectangular-QAM, 16 star-QAM and (5,11).

The Rayleigh flat fading channel is simulated by Jakes' model using 16 oscillators [25]. The fading is slow enough to assume that it is constant over one symbol interval. The autocorrelation of the fading samples then follows the Bessel function $J_0(\cdot)$ as given in (5.13). The Jakes' simulator using 16 oscillators generates fading samples with good correlation property only when the correlation lag (normalised time delay) is smaller than $f_D \tau = 7$. In this section, all the simulations use pilot spans within the effective correlation distance of Jakes' simulator.

In the previous chapters, the symbol error rates of the 2-D signalings were discussed. Here, both the symbol and bit error rate of pilot symbol aided 16 rectangular-QAM, 16 star-QAM and (5, 11) signaling are simulated. As a benchmark test, the symbol error rate of 16 rectangular-QAM with frame length $L = 7$, fading rate $f_D T = 0.01$, and $E_b/N_0 = 20 \text{ dB}$ was simulated as a function of interpolation order $K$ and compared with Fig. 3 in [67]. Similarly, the bit error rate of the 16 rectangular-QAM with frame length $L = 15$, interpolation order $K = 30$ and fading rate $f_D T = 0.03$ as a function of $E_b/N_0$ was simulated and compared with Fig. 11 in [69]. The simulations show consistent results with both references (within 0.4 dB in $E_b/N_0$).

In order to get the bit error rate, a symbol to bit mapping has to be performed. As is well known, Gray encoding is the optimum mapping method. Figs. 5.4-5.6 show the bit encoding of 16 rectangular-QAM, 16 star-QAM and (5, 11) signalling, respectively. The
Coherent Detection of Pilot Aided 16 QAM

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Richar K factor</td>
<td>0.0</td>
</tr>
<tr>
<td>Normalized Doppler BW (Hz)</td>
<td>0.01</td>
</tr>
<tr>
<td>Bit rate k</td>
<td>40</td>
</tr>
<tr>
<td>Symbols begin to count</td>
<td>-17300</td>
</tr>
<tr>
<td>Symbols in Count</td>
<td>15</td>
</tr>
<tr>
<td>k</td>
<td>30</td>
</tr>
<tr>
<td>Sampling frequency (Hz)</td>
<td>10.0</td>
</tr>
<tr>
<td>SNR of symbol error</td>
<td>17.5</td>
</tr>
<tr>
<td>No of symbol errors to count</td>
<td>200000</td>
</tr>
<tr>
<td>Symbols in Count</td>
<td>-400000</td>
</tr>
</tbody>
</table>

Figure 5.3: SPW block diagram of pilot symbol aided 16 rectangular-QAM.
16 star-QAM and (5, 11) constellations are encoded in a manner such that symbols of the minimum distance neighbours differ in the least number of bits, and the same principle is applied to the second nearest neighbours, third nearest neighbours, etc., all under the condition of the asymptotically optimum ring ratio chosen for the Rayleigh fading without diversity.

Figs. 5.7 and 5.8 depict the BER of 16 rectangular-QAM, 16 star-QAM and (5, 11) with PSAM in Rayleigh fading for $f_DT = 0.01$ and $f_DT = 0.03$, respectively. The frame length is set at $L = 15$, and the interpolation order $K = 30$. It can be seen that PSAM 16 star-QAM has the best BER performance, followed by 16 rectangular-QAM. Signalling (5, 11) has slightly higher BER than 16 rectangular-QAM. Note that the bit encoding of
Figure 5.6. Quasi-Gray encoding of (5, 11) with ring ratio $\beta = 2.324$.

(5, 11) is less ideal than 16 rectangular-QAM because nearest neighbouring signals cannot always be encoded to have only one different bit (for example, points 0000 and 0011 on the outer ring), and this may lead to some BER performance degradation. For a fading rate of $f_D T = 0.01$, 16 star-QAM saves about 0.488 dB in power over 16 rectangular-QAM at BER $= 10^{-3}$; while for a faster fading rate $f_D T = 0.03$, 16 star-QAM requires about 0.546 dB less power than its rectangular counterpart to achieve BER $= 10^{-3}$. The performance gap between 16 star-QAM and 16 rectangular-QAM is enlarged with increasing fading rate.

Figs. 5.9 and 5.10 depict the SER of 16 rectangular-QAM, 16 star-QAM and (5, 11) in a PSAM system with $f_D T = 0.01$ and $f_D T = 0.03$, respectively. The frame length is set at $L = 15$, and the interpolation order $K = 30$. It can be seen that the SER of (5, 11) is slightly lower than that of 16 rectangular-QAM and that the performance difference is more pronounced at a higher fading rate, by comparing Fig. 5.10 and Fig. 5.9. Observe that 16 star-QAM offers about 0.576 dB power savings over 16 rectangular-QAM at a SER of $2 \times 10^{-3}$. Compared with perfect coherent detection, the channel estimation error in PSAM 16 star-QAM leads to 2.37 dB-2.86 dB power loss, 3.42 dB-3.84 dB degradation in average SNR for 16 rectangular-QAM and about 3.42 dB degradation in average SNR for (5, 11), for the parameters used in Fig. 5.9 and Fig. 5.10.

The choice of the two parameters, frame length $L$ and interpolation order $K$, depends
Figure 5.7. Average bit error probabilities of PSAM 16-ary signal sets in a Rayleigh fading channel with $f_D T = 0.01, L = 15, K = 30$. 
Figure 5.8. Average bit error probabilities of PSAM 16-ary signal sets in a Rayleigh fading channel with $f_{DT} = 0.03, L = 15, K = 30$. 
Figure 5.9. Average symbol error probabilities of PSAM 16-ary signal sets in a Rayleigh fading channel with $f_0 T = 0.01, L = 15, K = 30$. 

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Figure 5.10. Average symbol error probabilities of PSAM 16-ary signal sets in a Rayleigh fading channel with $f_o T = 0.03, L = 15, K = 30$. 
on a number of factors. The larger the frame length $L$, the less power loss incurred from
the pilots inserted in the data stream. As stated in Section 5.2.2. $L$ is upper-bounded by the
sampling theorem at $L \leq \frac{1}{2f_D^T}$. For $f_D T = 0.03$, $L$ should be no larger than 16; while for
$f_D T = 0.01$, we have $L \leq 50$. In the PSAM simulations in this Chapter, $L$ is always set at
15.

The interpolation order $K$ plays an important role in determining the symbol buffer
size that the system must have in order to estimate the fading and hence the delay brought
about in data detection. In general, without sacrificing error performance the smaller the
$K$ the better, assuming high SNR. Figs. 5.11 and 5.12 show the BER performances of 16
rectangular-QAM and 16 star-QAM as a function of the interpolation order $K$ with $L = 15$.
It can be seen that at $f_D T = 0.03$, $K = 30$ is a good choice for $E_b/N_0 = 17.5$dB, while
at $f_D T = 0.01$, $K = 4$ is sufficient for $E_b/N_0 = 17.5$dB and $K = 6$ for $E_b/N_0 = 27.5$dB.
Therefore, choosing parameter $K$ depends on the channel fading rate, the signal-to-noise
ratio and frame size $L$. It does not seem to depend on the signal constellation used. In fact,
by calculating $r_\Omega$ and $p$ from (5.18) and (5.25) as functions of $E_b/N_0$, $L$ and $K$, we are able
to decide appropriate values of $K$. A good choice of $K$ would result in $r_\Omega$ and $p$ as close to
1 as possible.

5.5 Conclusions

The effect of dynamic channel estimation errors in a pilot symbol aided modulation scheme
with coherent detection of 2-D signalling has been studied in this Chapter. The theoretical
analyses of the SER for an arbitrary pilot aided 2-D signalling system lead to a result
with high computational complexity and therefore a computer simulation method was used
to give some meaningful results. It has been shown that although the effect of dynamic
channel estimation errors is less dramatic than fixed, constant amplitude and phase errors
as discussed in Chapter 4, 16 star-QAM does provide better error performance than 16
rectangular-QAM in a PSAM system, and it suffers much less from channel estimation
Figure 5.11. Average BER of PSAM 16 rectangular-QAM as a function of $K$ in Rayleigh fading with $L = 15$. 
Figure 5.12. Average BER of PSAM 16 star-QAM as a function of $K$ in Rayleigh fading with $L = 15$. 
errors than 16 rectangular-QAM does. Constellation (5, 11) also demonstrated robustness to fading estimation errors, and Figs. 5.9 and 5.10 show the trend that (5, 11) would probably perform better in a fast fading environment where channel estimation errors are large. However, its less ideal bit encoding scheme required by its constellation structure leads to some BER degradation.
Chapter 6

Probability of Error Expressions for Classes of Orthogonal Signals in Rayleigh Fading

6.1 Introduction

One potential extension of Craig's approach to the symbol error probability of arbitrary two-dimensional constellations in additive white Gaussian noise lies in the error performance analysis of higher dimensional modulation schemes. The precise probability of error of a coherent arbitrary $M$-dimensional signal set in AWGN is an open problem, and upper and lower bounds have been presented to approximate it [78]. We attempt to adopt the basic idea of Craig's approach, that is, translating the origin to a signal point, separating decision regions into smaller subregions and applying new coordinate systems, to the analysis of the SER of an $M$-dimensional signal set in AWGN and Rayleigh fading. Although a general formulation of the error probability of any $M$-dimensional signal set with a given decision region is not possible, we acquire analytical expressions for the symbol and bit error probabilities of three classes of $M$-ary orthogonal signalings in AWGN and Rayleigh fading for small values of $M$. 
The $M$-ary orthogonal signalling, including orthogonal signalling, biorthogonal signalling and transorthogonal signalling are important benchmark modulation schemes that approach Shannon-limit performance in additive white Gaussian noise when $M$ tends to infinity [39],[37]. Recent performance analyses of coherent $M$-ary orthogonal signalling in fading have been reported in [79]-[81]. In [79], the performances of $M$-ary orthogonal signalling in Rayleigh and Rician fading at infinitely large signal-to-noise ratio were indicated by an asymptotical ($\text{SNR} \to \infty$) parameter, while the asymptotic ($M \to \infty$) performances of $M$-ary orthogonal signalling in Rayleigh and Nakagami fading were examined in [80] and [81], respectively. To the best of our knowledge, no results have been published on the performances of coherent $M$-ary biorthogonal and transorthogonal signalling in slow fading.

In this chapter, we derive exact analytical expressions for the symbol and bit error probabilities for 3-ary and 4-ary orthogonal and transorthogonal signalling, and 6-ary and 8-ary biorthogonal signalling in Rayleigh fading and AWGN. There are no other closed-form expressions for the Rayleigh fading cases. Our solutions for the AWGN case are advantageous numerically to previous solutions. It is also indicated that these results can be used as close approximations to the symbol error and bit error probabilities of other $M$-ary orthogonal, biorthogonal and transorthogonal signalling schemes.

### 6.2 Three-ary and Four-ary Orthogonal Signalling

Consider a set of $M$ equal-energy orthogonal signals \( \{s_i(t)\}, i = 1, \ldots, M \) transmitted over a channel disturbed only by AWGN. In signal space, the noiseless received signals are represented by vectors \( s_i, i = 1, \ldots, M \)

\[
\begin{align*}
        & s_1 = (\sqrt{E_s}, 0, \ldots, 0) \\
        & s_2 = (0, \sqrt{E_s}, \ldots, 0) \\
        & \vdots \\
        & s_M = (0, 0, \ldots, 0, \sqrt{E_s}) \\
\end{align*}
\]

(6.1)
where $E_s$ is the average energy per symbol of the $M$ orthogonal signals. In the case of equally-likely signals, the average symbol error probability is the same as the conditional symbol error probability given signal $s_i$. Without loss of generality, we assume signal $s_1$ is transmitted. The output, $z = (z_1, z_2, \ldots, z_M)$, of an optimum receiver with matched filtering can be expressed as

$$z = s_1 + n$$  \hspace{1cm} (6.2)

where $n = (x_1, x_2, \ldots, x_M)$ is an $M$-dimensional Gaussian noise vector with zero mean and a covariance matrix of $diag(N_0/2, N_0/2, \ldots, N_0/2)$, and $N_0$ is the one-sided spectral density of the noise [1]. Therefore, $z$ given $s_1$ sent is an $M$-dimensional Gaussian vector with probability density function

$$f_z(z) = \frac{1}{(\pi N_0)^{M/2}} e^{-\frac{|z-s_1|^2}{N_0}}.$$  \hspace{1cm} (6.3)

The probability of a symbol error is the probability that vector $z$ falls outside the decision region of $s_1$ due to the presence of noise. That is,

$$P_e(M) = \int_{z \in \tau_e} \frac{1}{(\pi N_0)^{M/2}} e^{-\frac{|z-s_1|^2}{N_0}} dz = 1 - \int_{z \notin \tau_e} \frac{1}{(\pi N_0)^{M/2}} e^{-\frac{|z-s_1|^2}{N_0}} dz$$  \hspace{1cm} (6.4)

where $\tau_e$ and $\tau_c$ are respectively the erroneous and correct decision region of $s_1$, and one is the complement of the other ($\tau_e = \overline{\tau_e}$) in $M$-dimensional space. The erroneous decision region $\tau_e$ can be expressed as

$$\tau_e := \{z : z_2 - z_1 > 0, z_3 - z_1 > 0, \ldots, z_M - z_1 > 0 | s_1 \text{ transmitted}\}.$$  \hspace{1cm} (6.5)

It can be shown that decision region $\tau_e$ consists of $M - 1$ disjoint symmetrical regions, one of which is given by

$$\tau_{s,i} := \{z : z_i - z_1 > 0, z_i - z_2 > 0, \ldots, z_i - z_{i-1} > 0, z_i - z_{i+1} > 0$$

$$\ldots, z_i - z_M > 0\}, \ i = 2, \ldots, M.$$  \hspace{1cm} (6.6)

The region (6.6) is the received signal region where the received symbol will be incorrectly detected as $s_i (i \neq 1)$ instead of $s_1$, and $\tau_e = \bigcup_{i=2}^{M} \tau_{s,i}$. 

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Consider shifting the origin in the signal space to \( s_1 = (\sqrt{E_s}, 0, \ldots, 0) \). Note that the noise vector \( n = (x_1, x_2, \ldots, x_M) = z - s_1 = (z_1 - \sqrt{E_s}, z_2, \ldots, z_M) \). Then, the probability of a symbol error is given by

\[
P_e(M) = \sum_{i=2}^{M} \int_{n \in \tau_{s,i}} \frac{1}{(\pi N_0)^{M/2}} e^{-\frac{\|n\|^2}{2N_0}} dn = \int_{n \in \tau_{s,2}} \frac{M-1}{(\pi N_0)^{M/2}} e^{-\frac{\|n\|^2}{2N_0}} dn
\]  

(6.7)

where the region \( \tau_{s,i} \) in the new coordinates is determined by

\[
\tau_{s,i} := \{ n : x_i - x_1 > \sqrt{E_s}, x_i - x_2 > 0, \ldots, x_i - x_{i-1} > 0, x_i - x_{i+1} > 0 \\
\ldots, x_i - x_M > 0 \}, \quad i = 2, \ldots, M.
\]  

(6.8)

The second equality in (6.7) results from the symmetry of the \( \tau_{s,i} \)'s.

The probability of error (6.7) is a function of the SNR \( \gamma = E_s/N_0 \). In Rayleigh fading channels, the SNR \( \gamma \) is a random variable with PDF

\[
f_\gamma(\gamma) = \frac{1}{\Lambda} e^{-\frac{\gamma}{\Lambda}}
\]  

(6.9)

where \( \Lambda = E[\gamma] \) is the average SNR per symbol. Assuming perfect gain and phase tracking in the receiver, the probability of error for coherent \( M \)-ary orthogonal signals in slowly fading channels is written as

\[
P_{f,c}(M) = \int_{0}^{\infty} f_\gamma(\gamma) P_e(M, \gamma) d\gamma.
\]  

(6.10)

For \( M(=2^k) \) equally likely orthogonal signals, the equivalent bit error probability is related to the symbol error probability by [39]

\[
P_b(M) = \frac{2^{k-1}}{2k-1} P_e(M)
\]  

(6.11)

where \( k \) is an integer.

### 6.2.1 Two-ary Orthogonal Signalling

The symbol/bit error probability of 2-ary orthogonal signalling in AWGN is well-known and is given here for illustration and for comparisons with the 3-ary and 4-ary cases.
Using the alternative definite integral form for the Gaussian probability integral, 
\[ Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\chi^2}{2}\right) d\chi, \]
given in Chapter 2 by (2.19), one has
\[ P(2) = \frac{1}{\pi} \int_0^\pi \frac{\gamma}{\sin^2 \theta} d\theta. \]  
(6.12)
Combining \( P_{e}(M, \gamma) \) in (6.10) with (6.12) leads to
\[ P_f(2) = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\Lambda}{2 + \Lambda}} \]  
(6.13)
as expected [39, eqn. (14-4-15)].

### 6.2.2 Three-ary Orthogonal Signalling

The decision region of \( s_1 \) is illustrated in Fig. 6.1(a), formed by plane AOC and BOC. The three-dimensional (3-D) integration region \( \tau_{s,2} \) in (6.8) is defined by two 2-D planes as
\[ \tau_{s,2} : \begin{cases} 
 x_2 - x_1 > \sqrt{E_s} & \text{normal vector } \mathbf{n}_{21} = \frac{1}{\sqrt{2}} (-1, 1, 0)^T \\
 x_2 - x_3 > 0 & \text{normal vector } \mathbf{n}_{23} = \frac{1}{\sqrt{2}} (0, 1, -1)^T 
\end{cases} \]  
(6.14)
and the symbol error probability of 3-ary orthogonal signalling in AWGN is given by
\[ P_{e}(3) = \frac{2}{(\pi N_0)^{3/2}} \int_{(x_1, x_2, x_3) \in \tau_{s,2}} e^{-\frac{x_1^2 + x_2^2 + x_3^2}{2N_0}} dx_1 dx_2 dx_3. \]  
(6.15)
To solve this 3-D integration, we use a transformation of coordinates. Previously, Craig presented a clever application of transformation of rectangular coordinates to polar coordinates that resulted in new efficient analytical error rate expressions for two-dimensional signalling with polygonal decision boundaries in AWGN [12],[37]. This method was extended to fading in [62],[55]. Motivated by this idea, we perform similar though more general transformations in higher order dimensions. We are unaware of other work that has extended Craig's idea to higher dimensions. The first unit vector of the transformation matrix is chosen as perpendicular to all \( \mathbf{n}_{ij}, i \neq j \). That is, \( \beta_1^T = \frac{1}{\sqrt{2}} (1, 1, 1) \). In Fig. 6.1(a), it is the orthonormal vector perpendicular to the plane where signals \( s_1, s_2 \) and \( s_3 \) lie, and parallel to line OD. The second orthonormal vector is selected as \( \beta_2^T = \frac{1}{\sqrt{6}} (-2, 1, 1) \) and,
Figure 6.1. Decision region for $s_1$ in 3-ary orthogonal signaling, (a) 3-D signal set, (b) 2-D projection after rotation of coordinates.

hence, $\beta_3^T = \frac{1}{\sqrt{2}} (0, 1, -1)$. The new coordinates $\mathbf{x}' = (x'_1, x'_2, x'_3)$ are related to the old coordinates $\mathbf{n} = (x_1, x_2, x_3)$ by

$$\mathbf{x}' = \mathbf{n} \mathbf{T}_3,$$

where $\mathbf{T}_3$ is a transformation matrix given by

$$\mathbf{T}_3 = (\beta_3, \beta_2, \beta_1) = \begin{pmatrix} 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \quad (6.16)$$

It is easily shown that $|\mathbf{x}'| = |\mathbf{n}|$. This transformation of coordinates is a rotation without any scaling in 3-D space. The projection of the decision region on a plane perpendicular to axis $x'_3$ is shown in Fig. 6.1(b), where the shaded region corresponds to the integration region $\tau_{s,2}$. In the new coordinates, $\tau_{s,2}$ is given as

$$\tau_{s,2} : \begin{cases} 
\sqrt[3]{x_1} + \sqrt{3}x'_2 > \sqrt{2E_s} \\
x'_1 > 0 \\
-\infty < x'_3 < \infty,
\end{cases} \quad (6.17)$$
and hence the integration over \( x_3' \) is decoupled from the integration over \( x_1' \) and \( x_2' \). Furthermore, transforming the remaining two coordinates into polar coordinates \((x_1', x_2') = (r \cos \theta, r \sin \theta)\), the three-fold integral (6.15) is then simplified to

\[
P_e(3) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{x_1'^2}{N_0}} dx_1' \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} \frac{2r}{\pi N_0} e^{-\frac{x_2'^2}{N_0}} dr d\theta
\]

\[
= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{r^2}{2\tan^2 \theta}} d\theta
\]

(6.18)

where [35, 2.562.1] has been used. Eqn. (6.18) is a new expression for the symbol error probability of 3-ary orthogonal signalling in AWGN. A different expression is given in [37, (4.93)] as

\[
P_e(M) = 1 - \int_{-\infty}^{\infty} \left[ \frac{1}{2} \text{erfc} \left( -q - \sqrt{\frac{E_s}{N_0}} \right) \right]^{M-1} \frac{e^{-q^2}}{\sqrt{\pi}} dq.
\]

Comparing the two, both expressions are in the form of a single integral. However, the previous expression requires integration of the error function over an infinite integral, whereas (6.18) requires only simpler functions, exponential and trigonometric, integrated over a finite interval; thus, (6.18) is in a simpler form and is easier to compute accurately.

By averaging the SNR variable \( \gamma \) in (6.18) over the Rayleigh distribution (6.9), we obtain

\[
P_{f.e}(3) = \frac{2}{3} - \sqrt{\frac{\Lambda}{2 + \Lambda}} + \frac{1}{\pi} \sqrt{\frac{\Lambda}{2 + \Lambda}} \tan^{-1} \left( \sqrt{\frac{3(2 + \Lambda)}{\Lambda}} \right).
\]

(6.20)

Eqn. (6.20) is a new closed-form expression for the SER of 3-ary orthogonal signalling in Rayleigh fading. To the best of our knowledge no other closed-form expressions for this symbol error probability are known.

### 6.2.3 Four-ary Orthogonal Signalling

In the 4-ary case, the integration region \( \tau_{s,2} \) in (6.8) is given as,

\[
\tau_{s,2} : \begin{cases} 
  x_2 - x_1 > \sqrt{E_s} & \text{normal vector} \quad n_{21} = \frac{1}{\sqrt{2}}(-1, 1, 0, 0) \\
  x_2 - x_3 > 0 & \text{normal vector} \quad n_{23} = \frac{1}{\sqrt{2}}(0, 1, 1, -1) \\
  x_2 - x_4 > 0 & \text{normal vector} \quad n_{24} = \frac{1}{\sqrt{2}}(0, 1, 0, -1).
\end{cases}
\]

(6.21)
Similar to the 3-ary case, a coordinate transformation has to be performed. The transformation matrix $T_4$ is chosen as

$$T_4 = (\beta_4, \beta_3, \beta_2, \beta_1) = \begin{pmatrix}
0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\
0 & \frac{2}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2}
\end{pmatrix}. \quad (6.22)$$

This choice of transformation matrix has geometric interpretations. The 3-D projection of the decision region of $s_1$ on a 3-D hyperplane perpendicular to axis $x'_4$ is illustrated in Fig. 6.2(a), and the 2-D projection a 2-D plane perpendicular to axis $x'_3$ is given in Fig. 6.2(b). In the new coordinates, the integration region $\tau_{s,2}$ is given by

$$\tau_{s,2} : \begin{cases}
x'_2 > \sqrt{2}(\frac{\sqrt{3}E_s}{2} - x'_3) \\
\sqrt{3}x'_2 - x'_1 > 0 \\
\sqrt{3}x'_2 + x'_1 > 0 \\
-\infty < x'_4 < \infty.
\end{cases} \quad (6.23)$$

The probability of symbol error of 4-ary orthogonal signaling in AWGN can be written as

$$P_e(4) = \int \int \int_{x' \in \tau_{s,2}} \frac{3}{(\pi N_0)^2} e^{-\frac{2}{4} - x'_2 - x'_3} dx'_4 dx'_2 dx'_3. \quad (6.24)$$

Eqn. (6.23) shows that the integration over $x'_4$ is decoupled from the other three dimensions. For the remaining three dimensions, transforming to cylindrical coordinates $(x'_1, x'_2, x'_3) = (r \cos \theta, r \sin \theta, z)$, we arrive at

$$P_e(4) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{z^2}{2N_0}} dz \left[ \int_{-\infty}^{\infty} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{0}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{3r^2}{4N_0}} \frac{3r}{\pi N_0} e^{-\frac{2}{N_0}} \right] dr d\theta dz. \quad (6.25)$$

The integrand in (6.25) is absolutely integrable and therefore the order of integration can be interchanged to obtain

$$P_e(4) = \int_{\frac{\sqrt{5\pi}}{2}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{2}{N_0}} dz \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{3r^2}{4N_0}} \int_{-\infty}^{\infty} \frac{3r}{\pi N_0} e^{-\frac{2}{N_0}} dr d\theta dz. \quad (6.26)$$
Further simplification of (6.26) leads to
\[ P_e(4) = Q(\sqrt{\frac{3E_s}{2N_0}}) + \int_{\frac{5\pi}{6}}^{\frac{5\pi}{6}} \int_{-\sqrt{\frac{5\pi}{2}}}^{\sqrt{\frac{5\pi}{2}}} \int_{\frac{30}{2\pi\sqrt{2+sin^2(\theta)}}}^{\frac{30}{2\pi\sqrt{2+sin^2(\theta)}}} e^{-\frac{z\sqrt{\frac{3\gamma}{2}}}{N_0\sin^2(\theta)}} dz d\theta. \] (6.27)

Finally,
\[ P_e(4) = Q(\sqrt{\frac{3\gamma}{2}}) + \int_{\frac{5\pi}{6}}^{\frac{5\pi}{6}} \frac{3\sin(\theta)}{2\pi\sqrt{2+sin^2(\theta)}} e^{-\frac{3\gamma}{4\pi\sin^2(\theta)}} \left[ 1 - Q\left(\frac{\sqrt{\frac{3\gamma}{2}}\sin(\theta)}{\sqrt{4+2\sin^2(\theta)}}\right) \right] d\theta. \] (6.28)

Eqn. (6.28) is new and involves a single integral on a finite interval of an integrand involving the \(Q\)-function, in contrast to (6.19) which involves an infinite interval integral of a product of three \(Q\)-functions as integrand.

To obtain the SER of 4-ary orthogonal signalling in Rayleigh fading, average (6.28) over the PDF of SNR \(\gamma\), i.e.,
\[ P_{f.e}(4) = \int_0^\infty P_e(4) \frac{1}{\Lambda} e^{-\frac{\gamma}{\Lambda}} d\gamma = P_{r1} + P_{r2} - P_{r3} \] (6.29)

where
\[ P_{r1} = \int_0^\infty Q\left(\sqrt{\frac{3\gamma}{2}}\right) \frac{1}{\Lambda} e^{-\frac{\gamma}{\Lambda}} d\gamma, \] (6.30)
and

\[ P_{r_2} = \int_0^\infty \int_{\frac{5\pi}{6}}^{5\pi} \frac{3 \sin \theta}{2\pi \sqrt{2 + \sin^2 \theta}} e^{-\frac{3\gamma}{4 + 2\sin^2 \theta}} \frac{1}{\Lambda} e^{-\frac{\gamma}{\Lambda}} d\theta d\gamma, \]  

(6.31)

and

\[ P_{r_3} = \int_0^\infty \int_{\frac{5\pi}{6}}^{5\pi} \frac{3 \sin \theta}{2\pi \sqrt{2 + \sin^2 \theta}} e^{-\frac{3\gamma}{4 + 2\sin^2 \theta}} \frac{1}{\Lambda} e^{-\frac{\gamma}{\Lambda}} Q\left(\frac{\sqrt{3} \gamma \sin \theta}{\sqrt{4 + 2\sin^2 \theta}}\right) d\theta d\gamma. \]  

(6.32)

By interchanging the integration order and using the alternative expression for the \( Q \)-function (6.12), we obtain

\[
P_{r_2} = \int_0^{\frac{5\pi}{6}} \int_0^{2\pi} \frac{1}{\pi} e^{-\frac{3\gamma}{4 \sin^2 \theta}} \frac{1}{\Lambda} e^{-\frac{\gamma}{\Lambda}} d\gamma d\phi
\]

\[
= \frac{1}{2} - \frac{1}{\pi} \int_0^{\frac{5\pi}{6}} \frac{4 \sin^2 \phi}{4 \sin^2 \phi + 3 \Lambda} d\phi
\]

\[
= \frac{1}{2} - \frac{1}{2} \sqrt{\frac{3 \Lambda}{4 + 3 \Lambda}}
\]  

(6.33)

where [35, 2.562.1] has been used. For the second term,

\[
P_{r_2} = \int_0^{\frac{5\pi}{6}} \frac{3 \sin \theta \sqrt{2 + \sin^2 \theta}}{\pi (4 + 2 \sin^2 \theta + 3 \Lambda)} d\theta
\]  

(6.34)

and for the third term,

\[
P_{r_3} = \int_0^{\frac{5\pi}{6}} \frac{3 \sin \theta}{2\pi \sqrt{2 + \sin^2 \theta}} \int_0^\infty \frac{1}{\Lambda} e^{-\frac{3\gamma}{4 + 2\sin^2 \theta}} \frac{1}{\pi} e^{-\frac{\gamma}{\Lambda}} \frac{3 \sin^2 \phi}{4(2 + \sin^2 \theta) \sin^2 \phi + 6 \Lambda \sin^2 \phi + 3 \Lambda \sin^2 \theta} d\phi d\gamma
\]

\[
= \int_0^{\frac{5\pi}{6}} \frac{3 \sin \theta \sqrt{2 + \sin^2 \theta}}{2\pi (4 + 2 \sin^2 \theta + 3 \Lambda)} d\theta - \int_0^{\frac{5\pi}{6}} \frac{3 \sqrt{3 \Lambda} \sin^2 \theta}{2\pi (4 + 2 \sin^2 \theta + 3 \Lambda) \sqrt{4 + 3 \Lambda}} d\theta
\]  

(6.35)

where [35, 2.562.1] has been used. Hence

\[
P_{r_2} - P_{r_3} = \int_0^{\frac{5\pi}{6}} \frac{3 \sin \theta \sqrt{2 + \sin^2 \theta}}{2\pi (4 + 2 \sin^2 \theta + 3 \Lambda)} d\theta + \int_0^{\frac{5\pi}{6}} \frac{3 \sqrt{3 \Lambda} \sin^2 \theta}{2\pi (4 + 2 \sin^2 \theta + 3 \Lambda) \sqrt{4 + 3 \Lambda}} d\theta
\]  

(6.36)

By making the change of variable \( \cos \phi = \sqrt{3} \cos \theta \) in the left integral of (6.36), we have

\[
P_{r_2} - P_{r_3} = \frac{1}{4} - \frac{3}{2} \sqrt{\frac{\Lambda}{2 + \Lambda}} + \frac{1}{2} \sqrt{\frac{3 \Lambda}{4 + 3 \Lambda}}
\]

\[
+ \frac{3}{2\pi} \sqrt{\frac{\Lambda}{2 + \Lambda}} \left[ \tan^{-1} \left( \sqrt{\frac{3(2 + \Lambda)}{\Lambda}} \right) + \tan^{-1} \left( \sqrt{\frac{2 + \Lambda}{4 + 3 \Lambda}} \right) \right]
\]  

(6.37)
where [35, 2.562.1] has been used. The symbol error probability is therefore given by

\[
P_{f,e}(4) = P_{r1} + P_{r2} - P_{r3} = \frac{3}{4} - \frac{3}{2} \sqrt{\frac{\Lambda}{2 + \Lambda}} + \frac{3}{2\pi} \sqrt{\frac{\Lambda}{2 + \Lambda}} \left[ \tan^{-1}\left( \sqrt{\frac{3(2 + \Lambda)}{\Lambda}} \right) + \tan^{-1}\left( \sqrt{\frac{2 + \Lambda}{4 + 3\Lambda}} \right) \right].
\]

(6.38)

Eqn. (6.38) is a new closed-form expression for the SER of 4-ary orthogonal signalling in Rayleigh fading. To the best of our knowledge no other closed-form expression for this SER is known.

The bit error probability is obtained in closed-form by using (6.38) with (6.11). It is

\[
P_{f,b}(4) = \frac{1}{2} - \sqrt{\frac{\Lambda}{2 + \Lambda}} + \frac{1}{\pi} \sqrt{\frac{\Lambda}{2 + \Lambda}} \left[ \tan^{-1}\left( \sqrt{\frac{3(2 + \Lambda)}{\Lambda}} \right) + \tan^{-1}\left( \sqrt{\frac{2 + \Lambda}{4 + 3\Lambda}} \right) \right].
\]

(6.39)

To the best of our knowledge, Eqn. (6.39) is new and no other closed-form expression for this BER is known.

For \( M > 4 \), the selection of \( \beta = \frac{1}{\sqrt{M}}(1, 1, \ldots, 1) \) as one of the orthonormal vectors in a transformation matrix will always decouple one dimension from the remaining \( M - 1 \) dimensions of the integration region. However, the SER expression for \( M \)-ary orthogonal signalling \((M > 4)\) in Rayleigh fading seems difficult to obtain.

### 6.3 Six-ary and Eight-ary Biorthogonal Signalling

Biorthogonal signal sets are constructed from orthogonal signal sets by adding the antipodal counterparts of the orthogonal signals; the first half of the \( M \)-ary biorthogonal signals are the \( \frac{M}{2} \)-ary orthogonal signals \( s_i, i = 1, \ldots, \frac{M}{2} \), and the second half are \( s_{\frac{M}{2} + i} = -s_i, i = 1, \ldots, \frac{M}{2} \).

Consider the AWGN channel and the same receiver structure as in Section 6.2. The probability of a symbol error for \( M \)-ary biorthogonal signalling is given by

\[
P_e(M) = \int_{z \in \mathcal{T}_r} \frac{1}{(\pi N_0)^{M/4}} e^{-\frac{|z|^2}{2N_0}} \, dz = 1 - \int_{z \in \mathcal{T}_r} \frac{1}{(\pi N_0)^{M/4}} e^{-\frac{|z|^2}{2N_0}} \, dz
\]

(6.40)
where \( z = (z_1, z_2, \ldots, z_{M/2}) \) and \( \tau_c \), the correct decision region, is given as,

\[
\tau_c : \begin{align*}
& z_2 - z_1 < 0 \\
& z_2 + z_1 > 0 \\
& \vdots \\
& z_{M/2} - z_1 < 0 \\
& z_{M/2} + z_1 > 0.
\end{align*}
\]  

(6.41)

It can be shown that \( \tau_c \) is composed of \( 2^{M/2} - 1 \) disjoint and symmetrical regions, one of which is given by

\[
\tau_s : \ z_1 > z_2 > z_3 > \cdots > z_{M/2} > 0.
\]  

(6.42)

Again, shift the origin of the signal space to \( s_1 \). Since the noise vector \( n = (x_1, x_2, \ldots, x_{M/2}) = z - s_1 \), the symbol error probability of \( M \)-ary biorthogonal signalling in AWGN is given by

\[
P_e(M) = 1 - 2^{M/2} - 1 \left( \frac{M}{2} - 1 \right)! \int_{n \in \tau_s} \frac{1}{(\pi N_0)^{M/4}} \exp \left( - \frac{|n|^2}{N_0} \right) dn
\]  

(6.43)

where \( \tau_s \) is defined as,

\[
\tau_s : \begin{align*}
& x_1 - x_2 > -\sqrt{E_s} \\
& x_2 - x_3 > 0 \\
& x_3 - x_4 > 0 \\
& \vdots \\
& x_{M/2-1} - x_{M/2} > 0.
\end{align*}
\]  

(6.44)

The BER of \( M \)-ary (\( M = 2^k \)) biorthogonal signals when complementary bit encoding is employed [37] is related to the SER by

\[
P_b(M) = \frac{1}{2} [P_e(M) + P_{s1}(M)]
\]  

(6.45)

where \( P_e(M) \) is the symbol error probability of the \( M \)-ary biorthogonal signals and \( P_{s1}(M) \) is the probability of the \( M \)-ary biorthogonal signals when the negative, \( -s_1 \), of the transmitted signal \( s_1 \) is decided.
6.3.1 Four-ary Biorthogonal Signalling

Four-ary biorthogonal signalling is more commonly known as quaternary phase shift keying (QPSK). Its error performances in both AWGN and slow Rayleigh fading have been well studied. Craig's SER expression for MPSK in AWGN (2.16) gives

\[ P_e(4) = \frac{1}{\pi} \int_0^{\frac{3\pi}{4}} e^{-\frac{2}{3\sin^2 \theta}} d\theta \]  

(6.46)

and our extension to fading leads to

\[ P_{f.e}(4) = \frac{3}{4} - \sqrt{\frac{\Lambda}{2+\Lambda}} + \frac{1}{\pi} \sqrt{\frac{\Lambda}{2+\Lambda}} \tan^{-1} \left( \frac{2+\Lambda}{\Lambda} \right). \]  

(6.47)

This technique applied to the BER of QPSK in slowly Rayleigh fading with Gray encoding results in

\[ P_{f.b}(4) = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\Lambda}{2+\Lambda}} \]  

(6.48)

as expected [39, Eqn. 14-4-38].

6.3.2 Six-ary Biorthogonal Signalling

The integration region for 6-ary biorthogonal signalling in (6.44) is given by

[\tau_r : \begin{align*}
  & x_1 - x_2 > -\sqrt{E_s} \\
  & x_2 - x_3 > 0 \\
  & x_3 > 0.
\end{align*} \]  

(6.49)

In contrast to orthogonal signalling, biorthogonal signalling has center of the mass at the origin. The decision region of signal \( s_1 \) is symmetrical about the axis of \( s_1 \). Thus, no rotation of the coordinates is required in the biorthogonal signalling case. However, the integration dimensionality \( M/2 \) cannot be reduced to \( M/2 - 1 \) by using coordinate rotation as in orthogonal signalling. Therefore, the error probability integral of \( M \)-ary biorthogonal signalling has the same dimensionality as \( (\frac{M}{2} + 1) \)-ary orthogonal signalling and it is
handled by changing the Cartesian coordinates to other coordinates. For 6-ary biorthogonal signalling, we use cylindrical coordinates, that is, \((x_1, x_2, x_3) = (z, r \cos \theta, r \sin \theta)\). The symbol error probability of 6-ary biorthogonal signals in AWGN can then be expressed as

\[
P_e(6) = 1 - \int_{-\sqrt{E_s}/\sqrt{\pi N_0}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{z^2}{2 N_0}} dz
\]

\[
= 1 - \int_{-\sqrt{E_s}/\sqrt{\pi N_0}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{z^2}{2 N_0}} \left( 1 - e^{-\frac{z^2}{N_0 \sin^2 \theta}} \right) d\theta dz.
\] (6.50)

By interchanging the integration order, we have

\[
P_e(6) = 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{z^2}{2 N_0}} dz + \int_{0}^{\frac{\pi}{4}} \int_{-\sqrt{E_s}/\sqrt{\pi N_0}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{z^2}{2 N_0}} e^{-\frac{z^2}{N_0 \sin^2 \theta}} d\theta dz.
\] (6.51)

Further simplification leads to

\[
P_e(6) = Q(\sqrt{2\gamma}) + \frac{4}{\pi} \int_{0}^{\frac{\pi}{4}} \frac{\cos \theta}{\sqrt{1 + \cos^2 \theta}} e^{-\frac{\gamma}{1 - \cos^2 \theta}} \left[ 1 - Q\left(\frac{\sqrt{2\gamma \cos \theta}}{\sqrt{1 + \cos^2 \theta}}\right) \right] d\theta.
\] (6.52)

Eqn. (6.52) is new and involves a single integral with finite integration interval and a \(Q\)-function in the integrand, as compared to the more complicated and well known expression given in [37], [39] by

\[
P_e(M) = 1 - \int_{-\sqrt{E_s}/N_0}^{\infty} \left[ 1 - \text{erfc} \left( q + \sqrt{\frac{E_s}{N_0}} \right) \right] \frac{d q}{\sqrt{\pi}}
\] (6.53)

where an infinite integral interval with a product of two \(Q\)-functions as integrand has to be evaluated. Note that (6.52) is in a similar form to (6.28).

In Rayleigh fading, the symbol error probability of 6-ary biorthogonal signalling is obtained by combining (6.52) and (6.10) as

\[
P_{f.e}(6) = \frac{5}{6} - 2 \sqrt{\frac{\Lambda}{2+\Lambda}} + \frac{2}{\pi} \sqrt{\frac{\Lambda}{2+\Lambda}} \left[ \tan^{-1} \left( \sqrt{\frac{3(2+\Lambda)}{\Lambda}} \right) + \tan^{-1} \left( \sqrt{\frac{2+\Lambda}{1+\Lambda}} \right) \right].
\] (6.54)

Eqn. (6.54) is a new closed-form expression for the SER of 6-ary biorthogonal signalling in Rayleigh fading. To the best of the authors' knowledge, no other closed-form expressions for this are known.
6.3.3 Eight-ary Biorthogonal Signalling

Transforming to new four-dimensional coordinates \((x_1, x_2, x_3, x_4) = (r_1 \cos \theta_1, r_1 \sin \theta_1, r_2 \cos \theta_2, r_2 \sin \theta_2)\) where \(r_1, r_2, \theta_1,\) and \(\theta_2\) are independent variables, the symbol error probability of 8-ary biorthogonal signalling in AWGN is given by

\[
P_e(8) = 1 - \int_0^{\pi/4} \int_0^{\pi/4} \int_0^{\pi} \int_0^{\pi} \frac{48 r_1 r_2}{(\pi N_0)^2} e^{-\frac{\pi^2}{N_0}} dr_2 d\theta_2 dr_1 d\theta_1
\]

\[
- \int_0^{\pi/4} \int_0^{\pi/4} \int_0^{\pi} \int_0^{\pi} \frac{48 r_1 r_2}{(\pi N_0)^2} e^{-\frac{\pi^2}{N_0}} dr_1 d\theta_1 dr_2 d\theta_2.
\]

(6.55)

Solving (6.55) with respect to \(r_2\) first, and then \(r_1\), we arrive at

\[
P_e(8) = 1 - \int_0^{\pi/4} \int_0^{\pi/4} \frac{48 r_1}{\pi N_0} e^{-\frac{\pi^2}{N_0}} \int_0^{\pi} d\theta_2 dr_1 d\theta_1
\]

\[
- \int_0^{\pi/4} \int_0^{\pi/4} \frac{48 r_1}{\pi N_0} e^{-\frac{\pi^2}{N_0}} \int_0^{\pi} d\theta_2 dr_1 d\theta_1
\]

\[
= 1 - \int_0^{\pi/4} \frac{6}{2\pi} d\theta_1 - \int_0^{\pi/4} \frac{6}{2\pi} (1 - e^{-\frac{\pi^2}{E_{\text{fs}}}}) d\theta_1
\]

\[
+ \int_0^{\pi/4} \int_0^{\pi/4} \frac{48 r_1}{\pi N_0} e^{-\frac{\pi^2}{N_0}} \int_0^{\pi} d\theta_2 dr_1 d\theta_2
\]

\[
+ \int_0^{\pi/4} \int_0^{\pi/4} \frac{48}{2\pi} \int_0^{\pi} e^{-\frac{\pi^2}{N_0}} e^{-\frac{\pi^2}{N_0}} d\theta_2 dr_1 d\theta_2.
\]

(6.56)

Further simplification leads to

\[
P_e(8) = 1 - 3 + \frac{3}{\pi} \int_0^{\pi} \int_0^{\pi} \frac{6r_1}{\pi N_0} e^{-\frac{\pi^2}{N_0}} d\theta_1 + \int_0^{\pi} \int_0^{\pi} \frac{12}{\pi^2} \cos^2 \theta_2 e^{-\frac{\pi^2}{N_0}} \cos^2 \theta_2 d\theta_2 d\theta_1
\]

\[
+ \int_0^{\pi} \int_0^{\pi} \frac{12}{\pi^2} \cos^2 \theta_2 e^{-\frac{\pi^2}{N_0}} \cos^2 \theta_2 + \sin^2 \theta_1 d\theta_2 d\theta_1
\]

\[
= 1 - 3 + \frac{3}{\pi} \int_0^{\pi} \int_0^{\pi} e^{-\frac{\pi^2}{N_0}} d\theta_1 + \int_0^{\pi} \int_0^{\pi} \frac{12}{\pi^2} \cos^2 \theta_2 e^{-\frac{\pi^2}{N_0}} \cos^2 \theta_2 + \sin^2 \theta_1 d\theta_2 d\theta_1
\]

\[
- \int_0^{\pi} \int_0^{\pi} \frac{12}{\pi^2} \cos^2 \theta_2 e^{-\frac{\pi^2}{N_0}} \cos^2 \theta_2 + \sin^2 \theta_1 d\theta_2 d\theta_1.
\]

(6.57)

Since the second integral in (6.57) can be solved as

\[
\int_0^{\pi} \int_0^{\pi} \frac{12}{\pi^2} \cos^2 \theta_2 e^{-\frac{\pi^2}{N_0}} \cos^2 \theta_2 + \sin^2 \theta_1 d\theta_1 d\theta_2 = \frac{12}{\pi} \int_0^{\pi} \cos \theta_2 \sqrt{1 + \cos^2 \theta_2} d\theta_2 = 2,
\]

(6.58)
the SER of 8-ary biorthogonal signaling in AWGN finally is given by

\[ P_e(8) = \frac{3}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{\gamma}{2\sin^2\theta}} d\theta - \frac{12}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta_2}{\cos^2\theta_2 + \sin^2\theta_1} e^{-\frac{(\cos^2\theta_2 + \sin^2\theta_1)\gamma}{2\cos^2\theta_2 \sin^2(\theta_1 + \frac{\pi}{4})}} d\theta_2 d\theta_1. \]  

(6.59)

Eqn. (6.59) is new and it is better suited for extension to the fading case than the conventional SER expression for 8-ary biorthogonal signalling given in [39].

To get the BER of 8-ary biorthogonal signalling when complementary bit encoding is employed, the probability \( P_{s1}(8) \) that an incorrect decision is made on the negative of \( s_1 \) has 48 symmetric integration subregions, one of which is given by

\[
\begin{cases}
  x_1 + x_2 < -\sqrt{E_x} \\
  x_2 > x_3 \\
  x_3 > x_4 \\
  x_4 > 0.
\end{cases}
\]  

(6.60)

Similar to the derivation of \( P_e(8) \), \( P_{s1}(8) \) is obtained as

\[
P_{s1}(8) = \frac{3}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{\gamma}{2\sin^2\theta}} d\theta - \frac{12}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta_2}{\cos^2\theta_2 + \sin^2\theta_1} e^{-\frac{(\cos^2\theta_2 + \sin^2\theta_1)\gamma}{2\cos^2\theta_2 \sin^2(\theta_1 + \frac{\pi}{4})}} d\theta_2 d\theta_1. \]  

(6.61)

Thus, the BER of 8-ary biorthogonal signalling in AWGN is given by

\[
P_b(8) = \frac{1}{2}[P_e(8) + P_{s1}(8)]
\]

\[
= \frac{3}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{\gamma}{2\sin^2\theta}} d\theta - \frac{6}{\pi^2} \left[ \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta_2}{\cos^2\theta_2 + \sin^2\theta_1} e^{-\frac{(\cos^2\theta_2 + \sin^2\theta_1)\gamma}{2\cos^2\theta_2 \sin^2(\theta_1 + \frac{\pi}{4})}} d\theta_2 d\theta_1 \right]
\]

\[
+ \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta_2}{\cos^2\theta_2 + \sin^2\theta_1} e^{-\frac{(\cos^2\theta_2 + \sin^2\theta_1)\gamma}{2\cos^2\theta_2 \sin^2(\theta_1 + \frac{\pi}{4})}} d\theta_2 d\theta_1 \]  

(6.62)

where the second equality results from the symmetric property of \( \exp(-\frac{\gamma}{2\sin^2\theta}) \) about \( \theta = \pi/2 \).

In slow Rayleigh fading, the SER of 8-ary biorthogonal signalling is given by

\[
P_{fe}(8) = \int_0^\infty P_e(8) f_\gamma(\gamma) d\gamma
\]

\[= P_{e1} - P_{e2} \]  

(6.63)
where

\[ P_{e1} = \int_{0}^{\infty} \frac{3}{\pi} \int_{0}^{\frac{3\pi}{2}} e^{-\frac{x}{2\sin^2 \theta}} \frac{1}{\Lambda} e^{-\frac{x}{\Lambda}} d\theta d\gamma \]

\[ = \frac{3}{\pi} \int_{0}^{\frac{3\pi}{2}} \int_{0}^{\infty} \frac{1}{\Lambda} e^{-\frac{x}{2\sin^2 \theta}} e^{-\frac{x}{\Lambda}} d\gamma d\theta \]

\[ = \frac{3}{\pi} \int_{0}^{\frac{3\pi}{2}} \frac{2\sin^2 \theta}{2\sin^2 \theta + \Lambda} d\theta \]

\[ = \frac{9}{4} - \frac{3}{\pi} \int_{0}^{\frac{3\pi}{2}} \frac{\Lambda}{2\sin^2 \theta + \Lambda} d\theta \]

and

\[ P_{e2} = \int_{0}^{\infty} \frac{12}{\pi^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{0}^{\frac{3\pi}{2}} \frac{\cos^2 \theta_2}{\cos^2 \theta_2 + \sin^2 \theta_1} e^{-\frac{(\cos^2 \theta_2 + \sin^2 \theta_1)^2}{2\cos^2 \theta_2 \sin^2 (\theta_1 - \frac{\pi}{4})}} \frac{1}{\Lambda} e^{-\frac{x}{\Lambda}} d\theta_2 d\theta_1 d\gamma \]

\[ = \frac{12}{\pi^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{0}^{\frac{3\pi}{2}} \frac{\cos^2 \theta_2}{\cos^2 \theta_2 + \sin^2 \theta_1} \frac{2\cos^2 \theta_2 \sin^2 (\theta_1 - \frac{\pi}{4})}{(\cos^2 \theta_2 + \sin^2 \theta_1) \Lambda} \cos^2 \theta_2 d\theta_2 d\theta_1 \]

\[ = \frac{12}{\pi^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{0}^{\frac{3\pi}{2}} \frac{\cos^2 \theta_2 \Lambda}{2\cos^2 \theta_2 \sin^2 (\theta_1 - \frac{\pi}{4}) + (\cos^2 \theta_2 + \sin^2 \theta_1) \Lambda} \cos^2 \theta_2 d\theta_2 d\theta_1 \]

(6.64)

Applying the integral identity given by [35, 2.562.1] to (6.65) leads to

\[ P_{e2} = \frac{9}{4} - 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{6}{\pi} \frac{\sin \theta_1}{\sqrt{1 + \sin^2 \theta_1}} d\theta_1 + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{12}{\pi^2} \frac{\sin \theta_1}{\sqrt{1 + \sin^2 \theta_1}} \tan^{-1} \left( \frac{\sqrt{1 + \sin^2 \theta_1}}{\sin \theta_1} \right) d\theta_1 \]

\[ - \frac{3}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\Lambda}{2\sin^2 (\theta_1 - \frac{\pi}{4}) + \Lambda} d\theta_1 \]

\[ + \frac{6}{\pi^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\Lambda^3 \sin \theta_1}{(2\sin^2 (\theta_1 - \frac{\pi}{4}) + \Lambda) \sqrt{2\sin^2 (\theta_1 - \frac{\pi}{4}) + (1 + \sin^2 \theta_1) \Lambda}} \]

\[ \times \left[ 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{2\sin^2 (\theta_1 - \frac{\pi}{4}) + (1 + \sin^2 \theta_1) \Lambda}{\Lambda \sin^2 \theta_1} \right) \right] d\theta_1. \]

(6.66)
Therefore, the SER of 8-ary biorthogonal signalling in Rayleigh fading is written as

\[
P_{f,e}(8) = P_{e1} - P_{e2}
\]

\[
= \frac{5}{2} - \frac{12}{\pi^2} \int_{\frac{\pi}{4}}^{\pi} \frac{\sin \theta}{\sqrt{1 + \sin^2 \theta}} \tan^{-1} \left( \frac{1 + \sin^2 \theta}{\sin^2 \theta} \right) d\theta
\]

\[
- \frac{6}{\pi} \int_{\frac{\pi}{4}}^{\pi} \frac{\Lambda^3 \sin \theta}{(2 \sin^2(\theta - \frac{\pi}{4}) + \Lambda) \sqrt{2 \sin^2(\theta - \frac{\pi}{4}) + (1 + \sin^2 \theta) \Lambda}}
\]

\[
\times \left[ 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{2 \sin^2(\theta - \frac{\pi}{4}) + (1 + \sin^2 \theta) \Lambda}{\Lambda \sin^2 \theta} \right) \right] d\theta. \quad (6.67)
\]

Further, the BER in Rayleigh fading when complementary bit encoding is used [37, p. 203] is derived from averaging (6.62) over the PDF of \( \gamma \) as

\[
P_{f,b}(8) = \frac{3}{2} - \frac{12}{\pi^2} \int_{0}^{\frac{\pi}{4}} \frac{\sin \theta}{\sqrt{1 + \sin^2(\theta)}} \tan^{-1} \left( \frac{1 + \sin^2 \theta}{\sin^2 \theta} \right) d\theta
\]

\[
- \frac{3}{\pi} \int_{\frac{\pi}{4}}^{\pi} \frac{\Lambda^3 \sin \theta}{(2 \sin^2(\theta - \frac{\pi}{4}) + \Lambda) \sqrt{2 \sin^2(\theta - \frac{\pi}{4}) + (1 + \sin^2 \theta) \Lambda}}
\]

\[
\times \left[ 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{2 \sin^2(\theta - \frac{\pi}{4}) + (1 + \sin^2 \theta) \Lambda}{\Lambda \sin^2 \theta} \right) \right] d\theta
\]

\[
- \frac{3}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\Lambda^3 \sin \theta}{(2 \sin^2(\theta + \frac{\pi}{4}) + \Lambda) \sqrt{2 \sin^2(\theta + \frac{\pi}{4}) + (1 + \sin^2 \theta) \Lambda}}
\]

\[
\times \left[ 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{2 \sin^2(\theta + \frac{\pi}{4}) + (1 + \sin^2 \theta) \Lambda}{\Lambda \sin^2 \theta} \right) \right] d\theta. \quad (6.68)
\]

Eqns. (6.67) and (6.68) are new exact expressions for the SER and BER of 8-ary biorthogonal signalling in Rayleigh fading. Both expressions are in the form of a single finite interval integral with a well behaved integrand consisting of elementary functions and are well suited to numerical evaluations.
6.4 Three-ary and Four-ary Transorthogonal Signalling

$M$-ary transorthogonal signalling, also known as simplex signalling, is constructed from $M$-ary orthogonal signalling by a translation of the signal set in signal space to achieve the most negative correlation $(-\frac{1}{M-1})$ among equally correlated signals. The performance of $M$-ary transorthogonal signalling is superior to that of $M$-ary orthogonal signalling by $10\log_{10}(\frac{M}{M-1})$ dB in SNR [37]. Hence,

$$P_f^t(M, \Lambda) = P_f(M, \frac{M}{M-1} \Lambda)$$ (6.69)

where $P_f^t(M, \Lambda)$ is the SER or BER of $M$-ary transorthogonal signalling at SNR $= \Lambda$ in Rayleigh fading and $P_f(M, \Lambda)$ is the SER or BER of $M$-ary orthogonal signalling in Rayleigh fading at SNR $= \Lambda$. The asymptotic behavior ($M \rightarrow \infty$) of transorthogonal signalling is the same as the asymptotic behaviour of orthogonal and biorthogonal signalings in both AWGN and Rayleigh fading.

6.5 Discussion

The error performances in slow Rayleigh fading of orthogonal, biorthogonal and transorthogonal signalings are shown in Figs. 6.3-6.9. The SNR per bit $\frac{E_b}{N_0}$ is related to the SNR per symbol $\Lambda = \frac{E_b}{N_0} = \frac{kE_b}{N_0}$ by a factor of $k = \log_2 M$ for $M$-ary signals, where $k$ is an integer. The asymptotic performances of $M$-ary orthogonal, biorthogonal and transorthogonal signals in Rayleigh fading in the limit as $M$ goes to infinity, are the same and are plotted in Figs. 6.3-6.5, 6.8 and 6.9 for comparison.

In Fig. 6.3, the BER's of 2-ary, 4-ary, 32-ary and asymptotic ($M \rightarrow \infty$) orthogonal signalling are shown. It has been reported in [79] that indicated by an asymptotical (SNR $\rightarrow \infty$) parameter, the BER of $M$-ary orthogonal signalling in Rayleigh fading for large SNR decreases with the increase of $M$ until $M$ reaches 32, and then increases as $M$ gets larger; thus, 32-ary orthogonal signalling achieves the lowest BER performance among arbitrary $M$ at sufficiently large SNR. Here the BER curve of 32-ary orthogonal signalling
in Rayleigh fading is numerically evaluated from a double integral with infinite integration limits and an integrand of products of Q-functions [39, (5-2-21)]. The evaluation involves intensive computation and it is difficult to achieve high accuracy, especially in the case of high SNR. On a SUN Ultra 5 workstation, it takes about 12 minutes to compute one point on the BER curve of 32-ary orthogonal signalling with sufficient precision. Our results in Fig. 6.3 show that 32-ary outperforms 2-ary, 4-ary and \( M \)-ary with \( M \to \infty \) for all practical error rates, \( P_b \leq 0.2 \). Note that for large SNR, the BER of any \( M \)-ary orthogonal signalling \((M = 2^k) > 2\) will be less than that of asymptotic \((M \to \infty)\) and greater than that of 32-ary. From Fig. 6.3, it is also clear that 4-ary orthogonal signalling achieves slightly better performance than very large \( M \) \((M \to \infty)\) and 4-ary requires about 0.84 dB more power than the optimum 32-ary orthogonal signalling at large SNR. Therefore, the closed-form BER expression of 4-ary orthogonal signalling evaluated at \((\text{SNR} - 0.43 \text{ dB})\) can be used to approximate the performance for arbitrary \( M > 4 \) with an error (in SNR) no more than 0.42 dB, for reasonably large SNR (e.g., \( \text{SNR} > 7.5 \text{ dB} \)).

The BERs of 4-ary, 8-ary and \( M \)-ary at \( M \to \infty \) biorthogonal signalling are shown in Fig. 6.4. It is observed that 4-ary biorthogonal signalling has the best BER performance among the three. In Appendix B, we apply the asymptotical parameter technique [79] to \( M \)-ary biorthogonal signalling to show that the asymptotic BER \((\text{SNR} \to \infty)\) of biorthogonal signalling increases with the increase of \( M \). For large SNR, 8-ary biorthogonal slightly underperforms 4-ary biorthogonal and outperforms asymptotic \((M \to \infty)\) biorthogonal signalling by 1.272 dB in SNR. Therefore, the exact BER expression of 8-ary biorthogonal signalling evaluated at \((\text{SNR} + 0.636 \text{ dB})\) can be used to approximate the performance for arbitrary \( M > 8 \) with an error (in SNR) of no more than 0.636 dB. By overlaying Fig. 6.4 and Fig. 6.3, it is seen that \( M \)-ary biorthogonal signalling has smaller BER than \( \frac{M}{2} \)-ary orthogonal signalling for a given signal energy per bit. In Rayleigh fading with \( M = 4 \) and 8, biorthogonal outperforms orthogonal for all \( \text{SNR} > -10 \text{ dB} \).

The BERs of \( M \)-ary transorthogonal signalling, where \( M = 2, 4, \infty \), as shown in Fig. 6.5, are lower than those of orthogonal signalling as expected. Two-ary transorthogonal sig-
Figure 6.3. Average BER of orthogonal signals in Rayleigh fading.
Figure 6.4. Average BER of biorthogonal signals in Rayleigh fading.
nalling has the same BER performance as 4-ary biorthogonal, and 4-ary transorthogonal slightly outperforms 8-ary biorthogonal in Fig. 6.4. It is also shown in Appendix B that the asymptotic BER of $M$-ary transorthogonal signalling increases with the increase of $M$ for large SNR. Hence, the closed-form BER expression of 4-ary transorthogonal signalling evaluated at $(\text{SNR} + 0.654 \text{ dB})$ can be used to approximate the performance for arbitrary $M > 4$ with an error (in SNR) of no more than 0.654 dB.

Figs. 6.6 and 6.7 depict the SER performance of $M$-ary orthogonal, biorthogonal and transorthogonal signalling in Rayleigh fading as a function of SNR per symbol, evaluated using the new expressions (6.20), (6.38), (6.54), (6.67) and (6.69). The SER of 32-ary orthogonal signaling is obtained in the same way as the 32-ary BER curve in Fig. 6.3, by directly integrating the conventional SER expression for 32-ary orthogonal signaling in AWGN over the Rayleigh PDF of SNR. It is also computationally expensive to get this curve. Fig. 6.6 shows that the SER of $M$-ary orthogonal signaling as a function of SNR per symbol increases as $M$ increases, but the increase is less pronounced for larger $M$, similar to the AWGN case. Fig. 6.7 demonstrates the same trend seen in Fig. 6.6. It also shows that 2-ary transorthogonal signaling has the best performance, expected since it is simply binary antipodal signaling. By overlaying Fig. 6.6 and Fig. 6.7, it can be seen that $M$-ary transorthogonal signaling has better SER performance than $M$-ary orthogonal and $2M$-ary biorthogonal signalings. Figs. 6.8 and 6.9 plot the average SER of orthogonal, transorthogonal and biorthogonal signals as functions of bit SNR in Rayleigh fading. These results indicate that the SER of $M$-ary orthogonal signalling in Rayleigh fading as a function of bit SNR increases as $M$ increases and approaches infinity, as does also the SER's of $M$-ary transorthogonal and biorthogonal signalling. This can also be proved for large SNR using the asymptotical parameter approach.
Figure 6.5. Average BER of transorthogonal signals in Rayleigh fading.
Figure 6.6. Average SER of orthogonal signals as a function of SNR per symbol in Rayleigh fading.
Figure 6.7. Average SER of biorthogonal and transorthogonal signals as a function of SNR per symbol in Rayleigh fading.
Figure 6.8. Average SER of orthogonal and transorthogonal signals as a function of SNR per bit in Rayleigh fading.
Figure 6.9. Average SER of biorthogonal signals as a function of SNR per bit in Rayleigh fading.
6.6 Conclusions

New SER and BER expressions for 3-ary and 4-ary orthogonal and transorthogonal signalling and 6-ary and 8-ary biorthogonal signalling in AWGN channels have been derived. These expressions are in the form of single or double integrals with finite integration limits. They are more useful than known expressions because they are conveniently evaluated with small computational effort and high precision. Closed-form SER and BER expressions for 3-ary and 4-ary orthogonal and transorthogonal signalling and 6-ary and 8-ary biorthogonal signalling in slowly Rayleigh fading channels have been derived, except that the SER and BER of 8-ary biorthogonal signalling are in the form of single integrals with finite integration interval; previously no analytic formulas for these signalings in Rayleigh fading were known. It has been shown that increasing M in orthogonal, biorthogonal and transorthogonal signalling does not necessarily result in better BER performance in Rayleigh fading, contrary to the situation in AWGN channels where Shannon-limit performance is achieved as M tends to infinity. Approximations with bounded error have been given for M-ary orthogonal, biorthogonal and transorthogonal signalling schemes.
Chapter 7

Summary and Conclusions

Higher level modulation formats \((M > 4)\) are potential candidates for future wireless communications systems because of their spectral efficiency. The performance (power) penalty due to using larger numbers of symbols in a constellation can be compensated by diversity reception techniques, well-known for improved performance. Previously no tractable analytical approaches existed for the symbol error probability of arbitrary two-dimensional coherent signalings having polygonal decision regions in a wireless environment with single or multiple channel reception. In this thesis, we have analysed the error performances of perfectly coherent arbitrary 2-D signalings, imperfectly coherent arbitrary 2-D signalings, and perfectly coherent \(M\)-dimensional orthogonal signalings. A summary of conclusions is presented in this chapter and some suggestions for further research are given.

7.1 Conclusions

1. Closed-form expressions for the symbol error probability of an arbitrary 2-D constellation with polygonal decision boundaries have been derived for slow Rayleigh fading channels without diversity reception and with selection combining diversity reception. Closed form SER expressions have also been obtained for slow Nakagami fading channels with selection combining and integer Nakagami \(m\) parameters.
2. Precise symbol error probabilities of an arbitrary 2-D constellation have been derived in a single integral form with finite integration limits and elementary functions as integrand, for slow Ricean, Rayleigh and Nakagami-m fading channels with maximal ratio combining diversity reception. Both dissimilar and iid diversity branches have been considered. The SER expressions are well suited to numerical evaluation and to achieve high accuracy.

3. Precise symbol error probabilities of an arbitrary 2-D constellation have been derived as a single integral with finite integration limits and summation of an infinite series as integrand, for Ricean, Rayleigh and Nakagami-m fading with equal gain combining diversity reception. Numerical evaluation of symbol error probability can be achieved to the accuracy of $10^{-8}$ with reasonable computation intensity.

4. Exact symbol error probabilities of an arbitrary 2-D constellation for Ricean and Nakagami-m (when $m$ is not integer) fading with selection combining have been derived resulting in a single integral with finite integration limits and a Gauss-Hermite polynomial as integrand. They can be numerically evaluated with ease.

5. A MATLAB program has been written to accept any 2-D signal set as input, then automatically draw decision regions and subregions of the constellation, and finally apply SER formulas to calculate symbol error probabilities in various environments.

6. The SER performance of six 8-ary signal sets in AWGN and slow fading with and without diversity reception have been evaluated and plotted. They are the 8PSK, rectangular, triangular, (4,4), (1,7), and max-density sets.

7. The SER performance of eleven 16-ary signal sets in AWGN and slow fading with and without diversity reception have been evaluated and plotted. They are the 16 rectangular-QAM, 16 star-QAM, triangular, hexagonal, V.29, (4,4,4,4), (4,12), (8,8), (1,5,10), (5,11) and max-density sets.
8. The amplitude and phase error tolerance of a 2-D signal set has been defined and tabulated for 8-ary and 16-ary sets as a robustness measure of the signal set to channel amplitude and phase estimation error.

9. The precise symbol error probability of a 2-D signal set in the presence of constant channel amplitude and phase estimation errors for slow Rayleigh fading has been derived. It is still in the form of a single integral with finite integration limits and elementary functions as integrand.

10. Precise symbol error probabilities (and bit error probabilities where applicable) of coherent 3-ary and 4-ary orthogonal signalling in AWGN have been derived in the form of a single integral with finite integration limits, which are better suited to numerical evaluation than conventional formulae for the SER of \( M \)-ary orthogonal signals and for achieving high accuracy.

11. Closed-form expressions for the symbol error probabilities (and bit error probabilities where applicable) of coherent 3-ary and 4-ary orthogonal signalling have been derived for slow Rayleigh fading.

12. Precise symbol error probabilities (and bit error probabilities where applicable) of coherent 6-ary and 8-ary biorthogonal signalling in AWGN have been derived in the form of a single or double integral with finite integration limits, which are well-suited to numerical evaluation.

13. Closed-form or single finite integral expressions for the symbol error probabilities (and bit error probabilities where applicable) of coherent 6-ary and 8-ary biorthogonal signalling have been derived for slow Rayleigh fading.

14. Precise symbol error probabilities (and bit error probabilities where applicable) of coherent 3-ary and 4-ary transorthogonal signalling in AWGN have been derived in the form of a single integral with finite integration limits, which are better suited to
numerical evaluation than conventional formulae for the SER of $M$-ary transorthogonal signals and for achieving high accuracy.

15. Closed-form expressions for the symbol error probabilities (and bit error probabilities where applicable) of coherent 3-ary and 4-ary transorthogonal signalling have been derived for slow Rayleigh fading.

16. The symbol error probabilities of orthogonal, biorthogonal and transorthogonal signals as functions of bit SNR and symbol SNR have also been plotted and discussed.

The relative performances of the 8-ary and 16-ary constellations studied have been shown to depend on the channel model, diversity order and signal-to-noise ratio range, but to be invariant to the diversity combining type. In general, an optimum signal set in AWGN channels for high SNR is not necessarily the best signal set in fading. In slow fading, the performances of 2-D signal sets are greatly degraded compared to those in AWGN. Diversity techniques can significantly improve the performance. There is generally not much difference in the SERs of various 8-ary signal sets or 16-ary signal sets in severe fading with single channel reception, for example, Rayleigh fading. Diversity reception, however, enlarges the performance gaps among constellations. Diversity improvement is more significant for more severe fading channels, while the improvement differences among three combining types, i.e., MRC, EGC, and SC, are more pronounced for less severe fading channels.

The optimum ring ratio parameter of a circular constellation can be readily determined by the new SER formulas for various wireless environments presented in this thesis. It has been demonstrated that the optimum ring ratio of a circular constellation in a channel is a function of SNR and approaches an asymptotic value at very large SNR. The asymptotically optimum ring ratios of a 2-D circular constellation are dependent on the channel model and the diversity order but independent of the diversity combining type. It has also been observed that the asymptotically optimum ring ratio of an arbitrary 2-D circular set in Ricean fading is invariant to the Rice $K$ factor. By optimising the ring ratios of constellation
V.29 in AWGN, a 0.63dB power savings can be achieved without sacrificing the amplitude and phase error tolerance.

In general, a 2-D constellation is rather sensitive to the channel amplitude and phase estimation error and the extent of the sensitivity depends on the constellation geometry. In the presence of a large amplitude or phase error, the SER performance of a signal set is severely degraded and increasing SNR does not help performance much. Most 2-D constellations have asymmetric performance to channel amplitude error larger or smaller than 0 dB. In general, underestimating amplitude causes less performance degradation than overestimating amplitude.

Sixteen rectangular-QAM is more robust to channel amplitude error than 16 star-QAM, while star-QAM has better performance than rectangular-QAM when phase error greater than ±0.1rad is present. In the presence of combined amplitude and phase error, 16 star-QAM achieves better performance than 16 rectangular-QAM. Circular constellations with two rings have been observed to be more robust to combined amplitude and phase error than other types of constellation structures studied in this thesis. The performance gaps among various 8-ary and 16-ary constellations are much noticeable in the presence of combined amplitude and phase error. Signal sets such as the 8-ary rectangular, the 8-ary max-density set, the 16-ary hexagonal, rectangular-QAM, V.29, (4,4,4,4), and the 16-ary max-density set are not useful modulation formats for fading channels because of their poor performance in the constant channel estimation error environment.

The effect of dynamic channel estimation errors has been investigated with three 16-ary constellations, pilot symbol aided 16 star-QAM, 16 rectangular-QAM and signalling (5, 11). It has been shown that PSAM 16 star-QAM is about half a dB more power efficient than 16 rectangular-QAM in Rayleigh fading with fading rate $f_D T = 0.01$ or $f_D T = 0.03$. In general, it is more robust to large channel estimation errors than 16 rectangular-QAM. Constellation (5, 11) has close error performances to 16 rectangular-QAM for the parameters we have studied. Its SER performance, however, outperforms that of 16 rectangular-QAM in a fast fading environment ($f_D T \geq 0.03$), while its BER performance suffers slightly
from a less ideal bit mapping. An approach for choosing a PSAM system's frame size $L$ and interpolation order $K$ has also been presented.

The bit error probability of $M$-ary orthogonal signals has been shown to descend with the increase of $M$, achieve lowest BER at $M = 32$, and start to ascend with the increase of $M > 32$ approaching asymptotic performance, for all practical error rates in slow Rayleigh fading. The BER of 4-ary orthogonal signalling is slightly lower than that of $M$-ary at infinity. The closed-form BER expression of 4-ary orthogonal signalling evaluated at (SNR - 0.42 dB) can be used to approximate the performance for arbitrary $M > 4$ with an error (in SNR) no more than 0.42 dB. The bit error probability of $M$-ary biorthogonal signals has been shown to increase with the increase of $M$ in slow Rayleigh fading. The exact BER expression of 8-ary biorthogonal signalling evaluated at (SNR + 0.636 dB) can be used to approximate the performance for arbitrary $M > 8$ with an error (in SNR) of no more than 0.636 dB. It has been observed that $M$-ary biorthogonal signalling has smaller BER than $\frac{M}{2}$-ary orthogonal signalling for a given signal energy per bit. The bit error probability of $M$-ary transorthogonal signals has been shown to increase with the increase of $M$ in slow Rayleigh fading. The closed-form BER expression of 4-ary transorthogonal signalling evaluated at (SNR + 0.654 dB) can be used to approximate the performance for arbitrary $M > 4$ with an error (in SNR) of no more than 0.654 dB. Transorthogonal signalling has been confirmed to outperform orthogonal signals, as expected.

7.2 Suggestions for Further Work

The capacity of present mobile cellular systems is primarily cochannel-interference limited. In this thesis, a cochannel interference environment is not considered. However, a performance study of various 2-D signalings in cochannel-interference environments will have practical usefulness in design of a wireless communications system. A possible application of the SER formulas derived in this thesis lies in the performance analysis of a communication system employing error correction coding. Diversity branches have been
assumed to be independent in this thesis. In practice, however, there are situations where correlated fading exists among diversity branches. Further work based on this thesis may include the performance analysis of 2-D signal sets with correlated diversity reception. The effect of weighting errors, that is, channel estimation error, on the performance of maximal ratio combining or selection combining in fading channels is also a very interesting topic. The impairments in a real wireless transceiver, such as symbol timing error, intersymbol interference, as well as the nonlinearity of power amplifiers, are to be considered in further 2-D signalling performance analysis work.
References


[56] A. Annamalai, C. Tellambura, and V. K. Bhargava, "Exact evaluation of maximal-ratio and equal-gain diversity receivers for $M$-ary QAM on Nakagami fading chan-


Appendix A

MATLAB program for SER of 2-D Signalling

In this appendix, the main MATLAB codes are provided for calculating the symbol error probability of an arbitrary 2-D constellation. The 16 star-QAM is used as an example to illustrate how to use this program.

%*****************************************************************************
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*****************************************************************************

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% Symbol error probability of generalized 2-Dimensional constellation
% version 1.1

clear all
globalv;

% Initialization part
% div: diversity order
% chn_type: channel type
% 1-AWGN, 2-Ricean, 3-Rayleigh, 4-Nakagami
% k: Rice factor K
% m: Nakagami parameter m
% SNR_max: maximum SNR value calculated
% combine: combining type
% 0-selection, 1-maximal ratio, 2-equal gain

% div chn_type k m SNR_max combine
key=[ 1 1 -100 -100 10 -100
  2 2 5 -100 45 0
  3 2 5 -100 30 1
  1 2 10 -100 30 -100
  2 4 -100 2 40 1
  3 2 10 -100 20 1
  1 3 -100 -100 50 -100

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last_key=length(key(:,1));
title=input('input name of constellation: ', 's');
first_time=1;
globalTOR1=1e-7;
globalTOR2=1e-7;

fid=fopen([title '.rep'], 'w');

for key_count=1:last_key;
    sym_err=[];
    constellation=feval(title, key_count);
    no_tot_symbol=length(constellation(:,1));
    
    if first_time
        tmp=build_constellation(1);
        Maxlength=input('maximum length of lines: ');
        SEQ=zeros(no_tot_symbol, no_tot_symbol+1);
        symbol_seq=input('input symbol to be considered in sequence: ie[1 3 4]: ');
        weight=input('input weight factor, ie. [1/8 3/8 4/8]: ');
    end;

    channel_type=key(key_count, 2);
diversity_order=key(key_count, 1);
k_factor_db=key(key_count, 3);
k_factor=10^(k_factor_db/10);
nakagami_m=key(key_count,4);
combining_type=key(key_count,6);

for symbol_index=1:length(symbol_seq);
    symbol_tag=symbol_seq(symbol_index);
    const_index=find([1:no_tot_symbol]=symbol_tag);
    SNR_db=[0:2.5:key(key_count,5)];
    SNR=10.^(SNR_db/10)*log2(no_tot_symbol);
    if first_time
        tmp=build_constellation(symbol_index);
    end;

    symbol=constellation(symbol_tag,:);
    sym_err=[sym_err calctriangle(symbol,symbol_tag,const_index)];
end; %for symbol_index
err_avg=sym_err*weight';

info=[SNR_db' err_avg];
fprintf(fid,'Constellation name: %s 
',title);
fprintf(fid,'channel type=%5d \t diversity order=%5d \n',
         channel_type,diversity_order);
fprintf(fid,'k factor (db)=%5d \t',k_factor_db);
fprintf(fid,'m factor =%6d \n',nakagami_m);
fprintf(fid, 'combining type= %5d \n',combining_type);

p=['SNR(db)        Avg_error'];
fprintf(fid,'%s \n',p);
fprintf(fid,'%4.3f 10e\n',info);
first_time=0;
fprintf(fid,'\n');
end; %for key_count

close(fid);

function y=calctriangle(symbol,symbol_tag,const_index);
%return total of one symbol

globalv;

confirm_key=’ ’;

while isempty(find(confirm_key==’y’))
    lineset=zeros(2*no_tot_symbol,2);

    for counti=const_index
        y=getlineset(symbol,constellation(counti,:),counti);
        lineset(2*counti-1:2*counti,:)= y;
    end; %for

    if first_time
        seq=input(‘input boundary line in sequence ie. [1 2 3 1]: ’);
        SEQ(symbol_tag,1:(length(seq)+1)=[length(seq) seq];
    else
        seq=SEQ(symbol_tag,2:SEQ(symbol_tag,1)+1);
    end;

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['in calctriangle'];
r_set=getintersect(lineset, seq);

if first_time
    plot(r_set(:,1),r_set(:,2),'ro');
    for counti=1:length(r_set(:,1))
        text(r_set(counti,1), r_set(counti,2),int2str(counti));
    end;
    r_set
    confirm_key=input('press y key if is correct: ','sJ);
    if isempty(find(confirm_key=='y'))
        tmp=build_constellation([]);
    end; %if empty
else
    confirm_key='y';
end; %if firsttime
end; %while

isopen=isopenbound(seq);
if open
    line_count=seq(2);
    bound1=lineset(2*line_count-1:2*line_count,:);
    line_count=seq(length(seq)-1);
    bound2=lineset(2*line_count-1:2*line_count,:);
    flip_angle=[calc_flip_angle(symbol,r_set(1,:),bound1),
                calc_flip_angle(symbol,r_set(length(r_set(:,1)),:),bound2)];
else
function y=calc_symerror(r_set,symbol,isopen,flip_angle)
%This subroutine return Prob. of error for each sub-triangles
%in vector form for specific symbol

y=[];
if isopen
    x=[symbol; r_set(2,:); r_set(1,:)];
y=[y triangle_err(x,1,flip_angle(1))];
count=length(r_set(:,1));
x=[symbol; r_set(count-1,:); r_set(count,:)];
y=[y triangle_err(x,1,flip_angle(2))];
r_set=r_set(2:count-1,:);
else
    r_set=[r_set; r_set(1,:)];
end;

for count=1:length(r_set(:,1))-1
    x=[symbol; r_set(count,:); r_set(count+1,:)];
y=[y triangle_err(x,0,[])];
end;
function y=triangle_err(x,isopen,flip_angle)
% This function calculates the error probability of a triangular
% decision region or an open triangular region defined by the
% coordinates of three points.
% Return a column vector y which contains the error probability
% of a triangular decision region at different SNR values.

global v;

y=[];
angle=calc_angle(x(1,:),x(2,:),x(3,:),isopen,flip_angle);
et=angle(1);
psi=angle(2);
x01=norm(x(1,:)-x(2,:))^2;
for snr=SNR,
    if diversity_order==1
        channel_name=['cawgn_func';
                      'cricean_func';
                      'crayleigh_func';
                      'cnakagami_func'];

        x_coeff=[x01*snr; x01*snr/(k_factor+1); x01.*snr;
                   x01*snr/nakagami_m];
        commandstr=channel_name(channel_type,:);
        commandstr=commandstr(find(commandstr==''));
        tmpy=feval('quad8',commandstr,0,eta,[1e-9],[],
                    x_coeff(channel_type),psi);
        else % diversity_order >1

        end
    end
end
if combining_type==0 % selection combining
    if channel_type==2 % Ricean
        tmpy=quad8('ricsc2_fun',0,eta,1e-3,[],snr,x01,psi);
    end

    if channel_type==3 % Rayleigh
        tmpy=0;
        for j=0:diversity_order-1,
            a=quad8('craydiv_sel',0,eta,[1e-9],[],x01*snr,psi,j);
            tmpy=tmpy+diversity_order*fac(diversity_order-1)*
                (-1)^j*a/(fac(j)*fac(diversity_order-1-j));
        end
    end
end

if channel_type==4 % Nakagami
    if diversity_order==2
        tmpy=quad8('naksc_L2',0,eta,[1e-9],[],snr,x01,psi);
    else
        tmpy=quad8('naksc_L3',0,eta,[1e-9],[],snr,x01,psi);
    end
end
end % if combining_type==0

if combining_type==1 % maximum ratio combining
    if (channel_type==2) % Ricean
        tmpy=quad8('cricdiv_mrc',0,eta,[1e-9],[],
            x01*snr/(k_factor+1).psi,diversity_order);
if channel_type==3  % Rayleigh
    tmpy=quad8('craydiv_mrc',0,eta,[1e-9],[],x01*snr,
        psi,diversity_order);
end

if (channel_type==4) %Nakagami
    tmpy=quad8('cnakdiv_mrc',0,eta,[1e-9],[],
        x01*snr/nakagami_m,psi,diversity_order);
end
end  % if combining_type==1
%
if combining_type==2  % equal gain combining
    tmpy=egc(eta,x01,psi,snr);
end  % if combining_type==2
end  % if diversity_order==1
y=[y; tmpy];
end % for

******************************************************************************

function y=build_constellation(symbol_index);

globalv;

if ~isempty(symbol_index)
    figure(symbol_index)
end;
clf;

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hold on;

plot(constellation(:,1), constellation(:,2),'+');

for counti=1:length(constellation(:,1))
    text(constellation(counti,1)+0.1,constellation(counti,2),
         int2str(counti));
end;

******************************************************************************

function y=constarl6(key_count);

% Average power is 1.
%        div  chn_typ  k  m  SNR_max  combine  ringratio
BETA=[1   1  -100 -100  10  -100  1.76537
      2   2   5  -100  45   0     1.8785 
      3   2   5  -100  30   1     1.8785 
      1   2  10  -100  30  -100   1.9137 
      2   4 -100  2.0  40   1     1.8276 
      3   2  10  -100  20   1     1.8459 
      1   3 -100 -100  50  -100   1.8459 
      2   3 -100 -100  40   1     1.8742 
      3   3 -100 -100  30   0     1.8083];

beta=BETA(key_count,7);
r=[sqrt(2/(1+b^2)) beta*sqrt(2/(1+b^2))];
theta=linspace(0,7*pi/4,8);
y=[r(1)*cos(theta') r(1)*sin(theta')];
y=[y; r(2)*cos(theta') r(2)*sin(theta')];
When MATLAB is started, take 16 star-QAM as an example. The input format is as follows:

```
>> ser
input name of constellation: constar16
maximum length of lines: 1.5
input symbol to be considered in sequence: ie[1 3 4]: [1 9]
input weight factor, ie. [1/8 3/8 4/8]:[1 1]/2
```

The 16 star-QAM constellation such as shown in Fig. A.1 will appear on the screen. The signal points are numbered. The parameter 'maximum length of lines' determines the half length of the boundary lines to be drawn on the figure. It is usually selected as something larger than the distance of the most remote symbol to the origin. Due to the symmetry of 16 star-QAM, we only need to calculate the error probability of two signal points, e.g., signal 1 and 9, as input above. And the average SER is the weighted sum of probabilities of signal 1 and 9, with the weighting factor being 0.5 and 0.5, as input above. Further, the program runs as follows:

```
input boundary line in sequence ie. [1 2 3 1]: [2 8 9 2]
```

```
r_set =

0.0000 0.0000
0.9518 -0.3942
0.9518 0.3942
```

press y key if is correct: y

The MATLAB program draws the decision region for signal point 1. The user needs to key in the line numbers that form the decision region. Since signal 1 has a closed decision
boundary, line number 2 needs to be repeated at the end. It makes no difference which line of the decision region is picked as the first entry. However, the entries describing the decision polygon must be entered in order, either in clockwise order or counter-clockwise order. Input \([2 \ 8 \ 9 \ 2], [8 \ 9 \ 2 \ 8], [9 \ 2 \ 8 \ 9], [2 \ 9 \ 8 \ 2], \) or \([9 \ 8 \ 2 \ 9]\) yield the same result. The program draws circles at the intersections of the decision region for the user to verify if the input line numbers are correct. The example is shown in Fig. A.2. If the user keys in wrong numbers, simply type 'n' at the line "press y key if correct:". Then the operator has the second chance to input the data again.

input boundary line in sequence ie. \([1 \ 2 \ 3 \ 1] \): \([10 \ 1 \ 16]\)

\[
\begin{align*}
\text{r_set} &= \\
1.0742 & 0.4450 \\
0.9518 & 0.3942 \\
0.9518 & -0.3942
\end{align*}
\]
Figure A.2. The decision region of signal point 1 of a 16 star-QAM constellation generated by MATLAB.

1.0742   -0.4450

press y key if correct: y

Similarly, signal point 9 has an open decision region. For open decision regions the first line number is not repeated as in the case of closed regions, above. Fig. A.3 demonstrates the decision region for signal point 9. Once the constellation and the decision regions are built up, the program proceeds to calculate the symbol error probability of the constellation.

The program can also be split into two parts: the construction of the constellation and decision regions, and the SER calculation. The first part outputs the necessary geometric information for SER computation in a file, and the second part uses the file as input. This method works better for complex constellations with large numbers of signal points.
Figure A.3. The decision region of signal point 9 of a 16 star-QAM constellation generated by MATLAB.
Appendix B

Asymptotical Performance of Classes of M-ary Orthogonal Signalling

Let $P_s(M, \gamma)$ denote the SER of an $M$-ary signalling in AWGN and $\gamma$ is the signal-to-noise ratio per symbol. In slow Rayleigh fading, the signal-to-noise ratio $\gamma$ has PDF given by

$$f_\gamma(\gamma) = \frac{1}{\Lambda} \exp\left(-\frac{\gamma}{\Lambda}\right)$$

where $\Lambda$ is the average SNR per symbol. The symbol error probability of the $M$-ary signalling in slow Rayleigh fading is then given by

$$P_{f,s}(M) = \int_0^\infty P_s(M, \gamma) f_\gamma(\gamma) d\gamma$$

$$= \frac{1}{\Lambda} \int_0^\infty \exp\left(-\frac{\gamma}{\Lambda}\right) P_s(M, \gamma) d\gamma$$

In [79], the asymptotical (i.e., at high average SNR's) symbol error performance of coherent $M$-ary signals over slowly Rayleigh fading ($P_{f,s}(M)$ with $\Lambda \gg 1$) is given by asymptotical parameters as

$$P_{asy,s} = \frac{\lambda_s}{\Lambda} = \frac{\lambda_b}{\Lambda_b}$$

where $\Lambda_b = \frac{E_b}{N_0} = \Lambda/\log_2 M$ is the average SNR per bit in fading, and $\lambda_s, \lambda_b$ are symbol asymptotical parameters defined by

$$\lambda_s = \lim_{\Lambda \to \infty} \left[\Lambda P_{f,s}(M)\right]$$

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and

\[ \lambda_b = \frac{\lambda_s}{\log_2 M}. \]  \hspace{1cm} (B.5)

Parameter \( \lambda_b \) is the symbol asymptotical parameter relative to the average bit SNR. Combining (B.2) and (B.4), we have

\[ \lambda_s = \int_0^\infty P_s(M, \gamma) d\gamma \]  \hspace{1cm} (B.6)

Correspondingly, the asymptotical BER performance of \( M \)-ary signals in Rayleigh fading for very large SNR is given by

\[ P_{asy,b} = \frac{\lambda^{(b)}}{\lambda_b} \]  \hspace{1cm} (B.7)

where the bit asymptotical parameter \( \lambda^{(b)} \) is defined by

\[ \lambda^{(b)} = \int_0^\infty P_b(M, \gamma) d\gamma \]  \hspace{1cm} (B.8)

where \( P_b(M, \gamma) \) is the BER of an \( M \)-ary signalling in AWGN.

For \( M \)-ary orthogonal signalling, the symbol asymptotical parameter \( \lambda_b \) is given by [79]

\[ \lambda_b = \frac{M - 1}{M \log_2 M} \frac{2}{\log_2 M} \int_{-\infty}^{\infty} y \left[ \frac{\text{erfc}(-y)}{2} - \left( \frac{\text{erfc}(-y)}{2} \right)^M \right] dy, \]  \hspace{1cm} (B.9)

and it increases from 0.5 \( \rightarrow \) 0.69 as \( M \) increases from 2 \( \rightarrow \) \( \infty \). Therefore, the SER of \( M \)-ary orthogonal signalling in Rayleigh fading at large signal-to-noise ratio per bit increases with the increase of \( M \). The bit asymptotical parameter \( \lambda^{(b)} = \lambda_b M/(2M - 2) \) shows that a minimum occurs at \( M = 32 \) [79]. Hence, the 32-ary orthogonal signalling achieves the best BER performance for arbitrary \( M \) for large SNR.

For \( M \)-ary transorthogonal signalling, we use (B.6) to derive its symbol asymptotical parameter as given by

\[ \lambda_b = \frac{(M - 1)^2}{M^2 \log_2 M} + \frac{2(M - 1)}{M \log^2 M} \int_{-\infty}^{\infty} y \left[ \frac{\text{erfc}(-y)}{2} - \left( \frac{\text{erfc}(-y)}{2} \right)^M \right] dy. \]  \hspace{1cm} (B.10)
TABLE B.1

BIT ASYMPTOTICAL PARAMETER $\lambda^{(b)}$ FOR COHERENT $M$-ARY BIOORTHOGONAL AND TRANSORTHOGONAL SIGNALLING IN RAYLEIGH FADING

<table>
<thead>
<tr>
<th>$M$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthogonal [79]</td>
<td>0.5</td>
<td>0.3419</td>
<td>0.3</td>
<td>0.2859</td>
<td>0.2819</td>
<td>0.2821</td>
<td>0.2841</td>
<td>0.345</td>
</tr>
<tr>
<td>Biorthogonal</td>
<td>0.25</td>
<td>0.2586</td>
<td>0.2661</td>
<td>0.2722</td>
<td>0.2773</td>
<td>0.2816</td>
<td>0.345</td>
<td></td>
</tr>
<tr>
<td>Transorthogonal</td>
<td>0.25</td>
<td>0.2564</td>
<td>0.2625</td>
<td>0.2680</td>
<td>0.2731</td>
<td>0.2778</td>
<td>0.2818</td>
<td>0.345</td>
</tr>
</tbody>
</table>

The bit asymptotical parameter is $\lambda^{(b)} = \lambda_b M / (2M - 2)$ and its numerical values are summarized in Table B.1. Both $\lambda_b$ and $\lambda^{(b)}$ are shown to increase with the increase of $M$ by numerical evaluation. The bit asymptotical parameter $\lambda^{(b)}$ varies from 0.25 to 0.345 as $M$ goes from 0 → $\infty$.

For $M$-ary biorthogonal signalling, we have derived the asymptotical parameters as

$$
\lambda_b = \frac{3}{4 \log_2 M} - \frac{1}{M \log_2 M} + \frac{1}{\log_2 M} \int_0^\infty \text{erfc}(-y) \left[ 1 - \left[ 1 - \text{erfc}(y) \right] \frac{y}{2} \right] dy \quad (B.11)
$$

and

$$
\lambda^{(b)} = 
\frac{3}{8 \log_2 M} + \frac{1}{2 \log_2 M} \int_0^\infty \text{erfc}(-y) \left[ 1 - \left[ 1 - \text{erfc}(y) \right] \frac{y}{2} \right] dy \\
- \frac{1}{2 \log_2 M} \int_0^\infty \text{erfc}(-y) \left[ 1 - \text{erfc}(y) \right] \frac{y}{2} dy \quad (B.12)
$$

Numerical evaluation of (B.11) and (B.12) shows that both $\lambda_b$ and $\lambda^{(b)}$ increase with the increase of $M$. Numerical results of $\lambda^{(b)}$ are tabulated in Table B.1. The bit asymptotical parameters $\lambda^{(b)}$ for $M$-ary orthogonal, biorthogonal and transorthogonal signallings have the same value of 0.345 when $M = \infty$.  

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