UNIVERSITY OF ALBERTA

REACTION-DIFFUSION EQUATIONS WITH TIME DELAY: THEORY, APPLICATION, AND NUMERICAL SIMULATION

BY

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TO MY WIFE and MY PARENTS

Abstract

This thesis emphasizes the application of theory of functional partial differential equations to the dynamics of a class of diffusive population model. In particular, we are interested in studying the dynamics of diffusive Nicholson's blowflies equation. We first establish two versions of attractivity of center manifolds for the functional partial differential equations in an abstract setting. The attractivity theorems play a crucial role in studying Hopf bifurcation as we will also investigate. Neumann boundary problems and Dirichlet boundary problems are considered separately. In the case of Neumann boundary conditions, our global attractivity results are established by using the method of a lower-upper solution pair for functional partial differential equation. We also discuss oscillating criteria of the solutions followed by an investigation of periodic solutions bifurcating from a positive equilibrium. Moreover, using the center manifold reduction method and a lengthy calculation by hand, we provide a sufficient condition of stability of the bifurcated periodic solutions. Some numerical observations are also made before the end of our study of Neumann boundary problems. We then switch our attention to Dirichlet boundary problems. Before the study of global attractivity of the steady states, we give a necessary and sufficient condition for the existence and uniqueness of a positive steady state. Under varieties of parametric ranges, global attractivity of the zero solution and the positive steady state are studied respectively. On account of non-monotonicity, we develop a new approach in order to study the global attractivity of positive steady states and a better criterion is obtained along this approach than that through the theory of monotone semiflow. In the final chapter, we propose a numerical method to compute the positive steady state of the diffusive Nicholson's blowflies equation for a one dimensional space variable. This method gets through the numerical difficulties in that there are

two solutions of the stationary equation. Finally, we present a brief description of proving the existence of the pure imaginary eigenvalues of the characteristic equation corresponding to the linearized functional partial differential equation about the positive steady state. Necessary conditions of such existence are also obtained.

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CHAPTER I

INTRODUCTION

1.1 BACKGROUND

Delayed differential equations sometimes are also called differential difference equations or differential equations with deviating arguments. However nowadays, these later two titles are seldom used; instead, the terminology of "functional differential equations" is mostly utilized. Functional differential equations are classified as of retarded, neutral, or advance type. Such a classification, first introduced by Myshkis (1951) in his monograph, lay the foundation for a general theory of linear delayed systems. The simplest general delayed differential equations can be written as the form

$$\dot{x}(t) = f(t, x(t), x(t-\tau)), \tag{1.1}$$

where f is a function satisfying certain properties.

Although the first specific example of such a general class arose in the eighteenth century, and from that time, many particular equations of such types have appeared in the mathematical literature, arising from geometric, physical, engineering, economic and biological sources. The first papers treating general classes of *linear* functional differential equation are due to Schmidt (1911) and Polossuchin (1910). Schmidt considers solutions which with their derivatives are $O(|t|^a)$ as

$|t| \to \infty$ and find a connection between these types of solutions and the characteristic equation of the linear functional differential equation. A general class of *nonlinear* delayed differential equation is first discussed by Volterra (1928, 1931) who formulates a generic nonlinear differential equation incorporating the past states of the system so as to study predator-prey models and viscoelasticity. He also clarifies some properties of the solutions by using an energy method. However, these papers are almost completely ignored and therefore do not have much immediate impact on the subject.

Beginning with the late 1940's, the theory of delayed differential equations develops rapidly and many papers have been published from that time to the present. Besides Myshkis' book, there also appear several monographs during the development of the theory of functional differential equations (for example, Bellman and Cooke (1963), El'sgol'ts and Norkin (1973), Hale (1977), Hale and Verduyn Lunel (1993), Diekmann, van Gils, Verduyn Lunel and Walther (1995), from which many valuable papers in this field can be found).

As the development of the theory of functional (ordinary) differential equations progresses, there develops an increasing interest in studying parabolic equations with time delays, because a time delay can naturally be introduced into reaction-diffusion equations. The first example is proposed by Wang (1963) who considers an automatically controlled furnace and studies the stability of the equilibrium by Lyapunov's direct method. Some systems of delayed partial differential equations have also been arising from modelling genetic repression, climate, coupled oscillators, viscoelastic materials, and structured populations. For details, we refer to Mahaffy and Pao (1984), Busenberg and Mahaffy (1985), Hale (1994), Hetzer (1995), Dyson, Villella-Bressan, and Webb (1996), Rey and Mackey (1993), and references therein.

Because of the theory of semigroups (see Pazy (1983) for details), the fundamental theory of functional partial differential equations has been set up in a semigroup setting by the pioneer work of Travis and Webb (1974, 1976, 1978). In their approach, functional partial differential equations are treated as abstract functional differential equations. Thus, some of the results in functional differential equations can be technically transplanted into the theory of functional partial differential equations. From the point of view of dynamics, however, this approach is not all encompassing. On account of the utilization of semigroups, some of the dynamics and geometry of the original problems are lost. An elegant remedy is provided by the theory of infinite dimensional monotone dynamical system. The beginnings of that theory appear in Matano (1979), which focuses on semilinear differential equations. An important idea of a strongly order preserving semiflow is introduced in Matano (1984). A paramount contribution in this field is attributed to Hirsch (1988 b), who also systematically develops the theory of monotone dynamical systems for systems of differential equations (Hirsch (1982, 1985, 1988 a, 1989, 1990, 1991)). Hirsch's and Matano's ideas are applied to functional partial differential equations by Martin and Smith (1991). Travelling wave solutions of functional partial differential equations is also an interesting topic distinguished from functional differential equations. The existence of travelling wave solutions has been investigated recently by Zou and Wu (1997) and Wu (1996, chapter 10). Wu's monograph describes many fundamental results and methods of functional partial differential equations, as well as provides a comprehensive bibliography from both mathematical and biological sources. S^1-degree is also applied to study Hopf bifurcation of functional partial differential equations (Krawcewicz, Spanily, and Wu (1994)).

1.2 MOTIVATION

As previously mentioned, the monotone method is a friendly and powerful tool in studying dynamics, especially global dynamics of functional partial differential equations. In applications, however, one tends to encounter non-monotonic situations. Nicholson's adult blowfly model proposed by Gurney, Blythe and Nisbet (1980), for instance, is the very description of a dynamical system without monotonicity. There are also other examples, as will be mentioned later. For the adult fly model, some results are obtained by introducing exponential ordering (Smith (1995, chapter 6)). Unfortunately, these results are not generalized to the diffusive blowfly equation. Therefore, one may ask the question "Is there any new approach tackling such non-monotonic dynamical systems of functional partial differential equations?" To answer this question in my thesis, I will be interested in studying dynamical systems of functional partial differential equations as follows

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - \delta u(t,x) + f(u(t-\tau,x)), \qquad \text{in } D \tag{1.2}$$

$$\frac{\partial u(t,x)}{\partial n} = 0 \text{ or } u(t,x) = 0, \qquad \text{on } \Gamma \tag{1.3}$$

$$u(\theta,x) = u_0(\theta,x) \geq 0, \qquad \text{in } D_\tau \tag{1.4}$$

where τ is the delayed time; δ is a positive constant; $x \in \Omega \subset \mathbb{R}^n$; Ω is a bounded domain with a smooth boundary $\partial\Omega$, $(t,x) \in D \equiv (0,\infty) \times \Omega$, $\Gamma \equiv (0,\infty) \times \partial\Omega$, $D_\tau \equiv [-\tau,0] \times \bar{\Omega}$; $\frac{\partial}{\partial n}$ denotes the exterior normal derivative to $\partial\Omega$; and $f(z)$ is a nonlinear function, usually with the following hypotheses:

(i) $f(0) = 0$

(ii) $\lim_{z\to\infty} f(z) = 0$

(iii) There exists $z_0 > 0$, such that $f(z)$ is monotone increasing for $z \in [0, z_0]$ and decreasing afterward.

This research is also motivated by models without diffusion appearing in several areas, including physiology, ecology and optics (See Mackey and Glass (1977), Hadeler and Tomiuk (1977), Gurney, Blythe and Nisbet (1980), May (1980), Walther (1991), Lani-Wayda and Walther (1995), and references in Mallet-Paret and Nussbaum (1986)). The general form of these models is described in the following form:

$$\dot{u}(t) = -\delta u(t) + f(u(t-\tau)), \tag{1.5}$$

where the function f is assumed to satisfy certain properties.

Since the 1970's, these models have been studied extensively by many authors. Chow (1974) and Hadeler and Tomiuk (1977) prove the existence of periodic solutions using Browder's fixed point theorem. Using the $(u(t), \dot{u}(t))$ plane, Kaplan and Yorke (1977) show the existence of a slowly oscillating periodic solution whose derivative is also slowly oscillating. They show that on the $(u(t), \dot{u}(t))$ plane there exists an asymptotically stable annulus whose boundary consists of a pair of non-trivial periodic orbits, and that all the aforementioned slowly oscillating solutions tend asymptotically to this annulus. Using circulant matrices, Nussbaum (1985) shows that equation (1.5) has no periodic solution of period $2 + \frac{1}{m}$. Uniqueness of the periodic solution is also investigated by Cao (1996).

Besides these periodic solutions, numerical studies of Wazewska-Czyzewska and Lasota (1976), Mackey and Glass (1977), Glass, Beuter and Larocque (1988), and Mackey and an der Heiden (1984) indicate the existence of apparently aperiodic (chaotic) solutions. Theoretical proof of this chaotic behavior can be found in an der Heiden and Walther (1983) for some classes of f. Walther (1991,1995, 1996) also shows the existence and smoothness of an invariant manifold of slowly oscillating solutions and the 2-dimensional attractor. Chaotic attractors are studied by Farmer (1982) through a computation of the spectrum of the Lyapunov component.

The singular perturbation version of (1.5) is

$$\epsilon\dot{u}(t) = -u(t) + f(u(t-1), \mu), \tag{1.6}$$

which is studied by Chow and Green (1985). Their numerical simulation shows how small changes in ϵ and μ give rise to chaotic behavior in solutions. For this singular perturbation model, Mallet-Paret and Nussbaum (1986) describe the asymptotic behavior of the periodic solution as $\epsilon \to 0^+$ under some assumptions on f and the global bifurcation by using a continuation method based on degree theory.

Moreover using Brouwer's degree theory, Schmitt (1979), and Martelli, Schmitt and Smith (1980) show the existence of a periodic solution even for harmonically forced delay equation of (1.5). They claim that "chaos" may be removed through external forcing. They also show via the Hopf bifurcation theorem that equation (1.5) has nontrivial periodic solutions for certain values of the parameters. Unfortunately, they cannot determine the stability of such periodic solutions.

In recent years, global attractivity of the positive equilibrium of a delay equation has been studied by Kulenovic, Ladas and Sficas (1989, 1992), Kuang (1992), So and Yu (1994), and Karakostas, Philos and Sficas (1992). Oscillation theory of equation (1.5) can also be found in Kulenovic, Ladas and Meimaridou (1987). Furthermore Mallet-Paret and Sell (1994) have developed a Poincaré-Bendxison theorem for monotone cyclic feedback system with delay, and this result can be applied to equation (1.5).

The planar delayed differential equations can be written as

$$\dot{U}(t) = -\delta U(t) + F(U(t-\tau)), \tag{1.7}$$

where $U = col(u_1, u_2) \in \mathbb{R}^2$, and $F = col(f_1, f_2)$ is a map from $\mathbb{R}^2$ to itself. The existence and global bifurcation of periodic solutions to this planar equation have also been studied by Baptistini and Táboas (1996).

Not until the late 1980's did diffusive delay equations get more and more attention, since spatial inhomogeneity exists everywhere in nature. In population dynamics, Hutchinson's equation (sometimes called Wright's equation) with diffusion is often considered. We should mention here that the Hutchinson's equation

$$\dot{y}(t) = -\alpha y(t-1)(1+y(t)), \qquad \alpha > 0$$

can be transformed into

$$\dot{x}(t) = \alpha(1 - e^{x(t-1)}), \qquad \alpha > 0 \tag{1.8}$$

by using the transformation $x(t) = \ln(1 + y(t))$. The general form of equation (1.8) is

$$\dot{x}(t) = f(x(t-1)). \tag{1.9}$$

This equation is simpler than (1.5). It turns out that only a few results can be found for the delay equation (1.5) with diffusion. Murakami (1995) and Murakami and Hamaya (1995) study the global attractivity of the steady state for a diffusive generalized Wazewska and Lasota's model with Neumann boundary conditions, in which the nonlinear term is monotone decreasing. Dirichlet boundary conditions are even more scarcely considered. Some results of global dynamics are obtained by Cooke and Huang (1992), in which the generalized diffusive Hutchinson equation is considered under the assumption of a non-delay term dominating the system. Busenberg and Huang (1996) have studied Hopf bifurcation of the diffusive Hutchinson equation. More general types of diffusive delay equations with Dirichlet boundary conditions have also been investigated by Freitas (1997). But the bifurcation analysis in his paper is restricted to a particular case, i.e. the characteristic equation has nothing to do with the spatial variable. In equation (1.2) with assumptions (i)–(iii) on f, however, the nonlinear term is not monotone. This non-monotonicity may result in some difficulties in the research and can also be expected to give rise to some additional phenomena in its dynamics. Furthermore, in this equation the characteristic equation of the linearized equation about the positive steady state explicitly contains the spatial variable. This makes the bifurcation analysis considerably more complicated. Therefore in this thesis, I will make an effort to study the diffusive functional differential equation for the varieties of dynamics without monotonicity.

1.3 MAIN CONTRIBUTION AND ORGANIZATION

The major contribution of this thesis lies in the following three aspects.

Firstly, we provide detailed proofs of two versions of attractivities of center manifold. Although the idea comes from the corresponding results in differential equations or functional differential equations, our proofs are original and nontrivial.

Secondly, for Neumann boundary value problems of the diffusive Nicholson's blowflies equation, we obtain the existence of stable periodic solutions via Hopf bifurcation analysis. This is also new, since up to now, even the stability analysis of Hopf bifurcation for the non-diffusive Nicholson's blowflies equation has not been carried out yet.

Lastly, for Dirichlet boundary value problems of the diffusive Nicholson's blowflies equation, we develop a new approach to deal with the global attractivities of the positive steady state in the case of non-monotonicity. Our idea is creative. This approach should be applicable to other Dirichlet boundary value problems of functional partial differential equations. Therefore, we expect that our contribution will have impact on the studies of dynamics of functional partial differential equations.

The thesis is organized as follows. In chapter 2 , we will first introduce some basic results in functional partial differential equations whose proofs can be found elsewhere. Later, we will present two versions of attractivity of center manifolds with detailed proofs. The attractivity of the center manifold is crucial in studying the stability of periodic solutions bifurcating from positive steady states (Hopf bifurcation). From chapter 3 on, we will choose $f(z) = Pze^{-az}$ in order to carry out every proof and calculation, where P and a are constants. Notice that without diffusion, equation (1.2) together with such choice of f is exactly modelling the population of adult blowflies. Although we use this specific f throughout our proofs, we should point out that, without any difficulties, some of the proofs are also applicable to a general f satisfying (i)–(iii), as we will remark at the end of each chapter. Chapter 3 will focus on Neumann boundary problems. We first present global attractivities of equilibria followed by a discussion of oscillation criteria and Hopf bifurcation. Criteria for stability of the bifurcated periodic solutions are also given through a lengthy calculation. In chapter 4, we will consider Dirichlet boundary problems. A new approach is introduced in dealing with the non-monotonic dynamical system of functional partial differential equations. Global attractivities of positive steady state will be proved via this approach. The results are better than those derived from the theory of monotone semiflow. This new approach should be applicable to other Dirichlet boundary problems. Chapter 5 contains some numerical simulations of positive steady states. We also briefly describe the ideas of proving the existence of pure imaginary eigenvalues of the eigenvalue problems. Necessary conditions are also provided. Finally at the end of the thesis, we attach an appendix where Nicholson's blowflies experiments and models are briefly described, together with a collection of all the mathematical results in the studies of dynamics of the Nicholson's blowflies equation. A problem is also addressed for further research in this field.

1.4 DISCUSSION

Throughout this thesis, we have studied the dynamics of the diffusive Nicholson's blowflies equations. In this thesis, one can see, that dynamics of Neumann boundary value problems with large diffusion rate is very similar to those of the corresponding non-diffusive model. This can be understood in the following way.

By making a change of variables, we get a unit space region. Then the diffusion rate (denoted by d) is proportional to $\frac{1}{A^2}$, where A is the real size of the space area. Therefore, when A is small, the diffusion d is large. Since Nicholson's data come from his laboratory experiments, the real size of the space area cannot be large. The diffusion rate therefore cannot be small. In this case, we are not strange that our results agree with Nicholson's data. When the real size of the space area is very large, however, the spatial patterns of our results are no longer simple, as indicates in our numerical simulation. In this case, Nicholson's data disagree with our equation. Nonetheless, our studies of the diffusive Nicholson's blowflies equation should be still important in the ecological problems, since the diffusive term and the time delay term in our equation are quite representative.

Besides Neumann boundary value problems, we also study Dirichlet boundary value problems. Our results on the global attractivities of the positive steady state for Dirichlet boundary value problems are stronger than those for Neumann boundary value problems. However, unlike Neumann boundary value problems, Hopf bifurcation analysis for Dirichlet boundary value problems is far from complete. This is due to the lack information of the positive steady state. Another difference between the results of Dirichlet boundary value problems and those of Neumann boundary value problems is that the results of the former problems are related to the first eigenvalue of Laplace operator $-\Delta$.

Ecologically, the diffusive Nicholson's blowflies equation may be referred to as "educational" rather than "practical". The significance of our studies lies in the fact that this simple equation provides a process for gaining insight, expressing ideas, and eventually extending to more complex diffusion models.

In ecology, spatial dispersion is important, because only when populations of organisms are considered in both time and space can the ecological situation be understood. Experimental investigation of the phenomenon of animal dispersion develops first from insects. Since the famous experiments of Dobzhansky and Wright (1943, 1947) on the release of Drosophila flies, a variety of excellent research has been conducted. Nowadays, people realize that a model of dispersion must consider the forces operating between population individuals, and it cannot be limited to the simple random walk (simple diffusion). One method of accounting for these forces is to include an advection in the diffusion equation (Shigesada and Teramoto (1979)). An advection-diffusion equation models are expected to be able to explain some particular behavior of animals, such as insect swarming and fish schooling. Therefore, it is natural to require mathematicians to study advection-diffusion equations with time delay.

CHAPTER II

ATTRACTIVITIES OF CENTER MANIFOLDS

2.1 INTRODUCTION

Center manifold theory, which plays an important role in understanding the dynamics of nonlinear systems near an equilibrium, has been studied by many authors. We refer to Carr (1981), Diekmann and van Gils (1991), Hale (1985), Kelley (1967) and Lin, So and Wu (1992) and references therein for details of the subject.

The existence of center manifolds of functional partial differential equations has been set up by Lin, So and Wu(1992). Smoothness also has been obtained by So, Wu and Yang (1998). In this chapter, we focus on the discussion of attractivity of center manifolds for functional partial differential equations.

Attractivity of center manifolds, plays a vital role in studying the stability of Hopf bifurcation, and so is of importance. In ordinary differential equations, there are two versions of attractivities of center-unstable manifolds, see Chow and Hale (1982, p.320–p.321). In Hale and Verduyn Lunel (1993, p.316), one version of attractivity of center–unstable manifolds for functional differential equations of retarded or neutral type, is stated with an outlined proof. Unfortunately, in this book, the set-up of equation (10.2.14) is far from easy to follow (and might have some errors), and also the fixed point mapping in equation (10.2.19) need to be modified by putting a negative sign in front of the integration and the matrix B_{cu}.

The objective of this chapter is to establish two versions of attractivities of center manifolds for reaction-diffusion equations with time delay.

The rest of this chapter is organized as follows. In Section 2.2, we recall some basic results with some notations. The attractivity theorems and their proofs are in Section 2.3.

2.2 PRELIMINARIES

In this section, we will recall some results on a functional partial differential equation in the form

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)[Lu_\tau + g(u_\tau)]\, d\tau, \qquad -\infty < s \le t. \qquad (2.1)$$

where, $u : \mathbb{R} \to X$ is a continuous function and X is a Banach space over the reals $\mathbb{R}$ with a norm$|\cdot|$; u_τ is the usual notation for the element of $\mathcal{C} := C([-r, 0]; X)$ defined by $u_\tau(\theta) = u(\tau+\theta)$ for $-r \le \theta \le 0$; $\mathcal{C}$ is the Banach space of all continuous X–valued functions defined on $[-r, 0]$, equipped with the supremum norm $|\cdot|$.

Throughout this chapter, we need to pick up the following assumptions upon occasion.

(A1) $r > 0$ is a fixed constant.

(A2) $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup on X satisfying $|T(t)| \leq e^{\omega t}$ $(t \geq 0)$ for some constant $\omega \in \mathbb{R}$.

(A3) $T(t) : X \to X$ is compact for $t > 0$.

(A4) $g \in C(\mathcal{C}; X)$, $\quad g(0) = 0$ and

$$|g|_{0,1} := \sup_{\phi, \psi \in \mathcal{C},\ \phi \neq \psi} \frac{|g(\phi) - g(\psi)|}{|\phi - \psi|} < \infty.$$

(A5) $L : \mathcal{C} \to X$ is a bounded linear operator.

We denote the space of all bounded linear operators on X equipped with the operator norm $|\cdot|$ by $\mathcal{L}(X; X)$. For any $\eta > 0$, we will also use $BC^\eta(\mathbb{R}; X)$, defined as

$$BC^\eta(\mathbb{R}; X) := \left\{ h \in C(\mathbb{R}; X), \quad \sup_{t \in \mathbb{R}} e^{-\eta|t|} |h(t)| < \infty \right\},$$

endowed with the weighted supremum norm:

$$|h|_\eta := \sup_{t \in \mathbb{R}} e^{-\eta|t|} |h(t)|.$$

Clearly, $BC^\eta(\mathbb{R}; X)$ together with the norm $|\cdot|_\eta$ is a Banach space. The existence and uniqueness of functional partial differential equations have been investigated by many authors. Theorems 2.1 and 2.2 following are due to Travis and Webb (1974, 1978).

THEOREM 2.1. *Suppose that* **(A1)** *and* **(A2)** *are satisfied, and*

(A6) *$F : \mathcal{C} \to X$ is Lipschitz continuous, i.e. there exists a constant $K > 0$ such that*

$$|F(\phi_1) - F(\phi_2)| \le K|\phi_1 - \phi_2|, \qquad \textit{for all} \qquad \phi_1, \phi_2 \in \mathcal{C}. \tag{2.2}$$

Then for every $\phi \in \mathcal{C}$, there exists a unique continuous mapping $u : [-r, \infty) \to X$, sometimes also denoted by $u(\phi)$, satisfying

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)F(u_s)\,ds, \qquad t \ge 0 \tag{2.3}$$

$$u_0 = \phi. \tag{2.4}$$

If we further assume **(A3)**,

then for each fixed $t > r$, the mapping $\phi \in \mathcal{C} \mapsto u_t(\phi) \in \mathcal{C}$ is compact.

Finally, if in addition to **(A1)**, **(A2)** *and* **(A6)**, *we also assume that*

(A7) *$F : \mathcal{C} \to X$ is continuously differentiable and there exists a constant $M > 0$ such that the Fréchet derivative DF satisfies*

$$|DF(\psi_1) - DF(\psi_2)| \le M|\psi_1 - \psi_2|, \qquad \textit{for all} \quad \psi_1, \psi_2 \in \mathcal{C},$$

then for each $\phi \in \mathcal{C}$ satisfying

$$\phi(0) \in D(A_T), \quad \dot{\phi} \in \mathcal{C} \quad \textit{and} \quad \dot{\phi}^-(0) = A_T\phi(0) + F(\phi), \tag{2.5}$$

where $A_T : D(A_T) \subset X \to X$ denotes the infinitesimal generator of $\{T(t)\}_{t \ge 0}$, the solution $u(\phi) : [0, \infty) \to X$ of (2.3)-(2.4) is continuously differentiable and *satisfies the abstract functional differential equation*

$$\frac{du(t)}{dt} = A_T u(t) + F(u_t), \qquad t \geq 0. \tag{2.6}$$

In the literature, solutions of the integral equation (2.3) are usually referred to as *mild* solutions of the differential equation (2.6). According to Fitzgibbon (1978) and Martin and Smith (1990), every mild solution satisfies (2.6) for $t > r$, if $\{T(t)\}_{t \geq 0}$ is analytic.

Let us now consider the linearized equation of (2.3)–(2.4) as follows:

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)L(u_s)\,ds, \qquad t \geq 0 \tag{2.7}$$

$$u_0 = \phi. \tag{2.8}$$

Correspondingly, one defines the solution semiflow $W(t) : \mathcal{C} \to \mathcal{C}$ by $W(t)\phi = u_t(\phi)$, for $t \geq 0$ and $\phi \in \mathcal{C}$, where $u(t)$ is the solution of (2.7)–(2.8). Moreover for each $\lambda \in \mathbb{C}$, one can define a linear operator $\Delta(\lambda) : D(A_T) \to X$ by

$$\Delta(\lambda)x = A_T x - \lambda x + L(e^{\lambda \cdot} x), \qquad x \in D(A_T).$$

The equation

$$\Delta(\lambda)x = 0 \tag{2.9}$$

is called the *characteristic equation* of (2.6). The nontrivial solution pair (λ, x) of (2.9), which means $x \neq 0$ in $D(A_T)$, is called an *eigen-pair*, where $\lambda \in \mathbb{C}$

is called a *characteristic value* of (2.6). Information on the characteristic value makes it possible to decompose the space $\mathcal{C}$ by applying some operator algebra. More specifically; one has :

THEOREM 2.2. *$\{W(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathcal{C}$ with infinitesimal generator $A : D(A) \subset \mathcal{C} \to \mathcal{C}$ given by*

$$(A\phi)(\theta) = \dot{\phi}(\theta), \qquad -r \leq \theta \leq 0,$$

$$D(A) = \{\phi \in \mathcal{C} \ : \ \dot{\phi} \in \mathcal{C}, \ \phi(0) \in D(A_T), \ \dot{\phi}^{-}(0) = A_T\phi(0) + L\phi\}.$$

Moreover, there exist three linear subspaces $\mathcal{U}, \mathcal{N}$ and $\mathcal{S}$ of $\mathcal{C}$ such that $\mathcal{C} = \mathcal{U} \oplus \mathcal{N} \oplus \mathcal{S}$ and

(i) $dim(\mathcal{U}) + dim(\mathcal{N}) < \infty$;

(ii) for $\phi \in \mathcal{U} \oplus \mathcal{N}$, $W(t)\phi$ *can be extended to all of* $t \in \mathbb{R}$;

(iii) $W(t)\mathcal{U} \subset \mathcal{U}$, $\quad W(t)\mathcal{N} \subset \mathcal{N}$ *for all* $t \in \mathbb{R}$ *and* $W(t)S \subset S$ *for all* $t \geq 0$; *and*

(iv) there exist constants $\gamma_+, \gamma_- > 0$ *such that for any* $0 < \epsilon < \min\{\gamma_+, \gamma_-\}$, *there exists a constant* $K(\epsilon) > 0$ *such that*

$$\begin{aligned} |W(t)\phi| &\leq K(\epsilon)e^{(\gamma_- - \epsilon)t}|\phi| && \textit{for} \quad t \leq 0, \ \phi \in \mathcal{U}; \\ |W(t)\phi| &\leq K(\epsilon)e^{\epsilon|t|}|\phi| && \textit{for} \quad t \in \mathbb{R}, \ \phi \in \mathcal{N}; \\ |W(t)\phi| &\leq K(\epsilon)e^{-(\gamma_+ - \epsilon)t}|\phi| && \textit{for} \quad t \geq 0, \ \phi \in \mathcal{S}. \end{aligned} \tag{2.10}$$

Basically, a variation-of-constants formula is a fundamental tool for studying a dynamical system with external forces or with a nonlinear perturbation. In order to present a variation-of-constants formula for (2.3) due to Memory (1991), one needs to extend $\mathcal{C}$ to the metric space

$$\widehat{\mathcal{C}} = \{\widehat{\phi} : [-r, 0] \to X \ : \text{ there exists} a \in [-r, 0] \text{ such that } \widehat{\phi} \text{ is continuous on } [-r, a),$$

$$\lim_{s \to a^-} \widehat{\phi}(s) \in X \text{ exists if } a \neq -r \quad \text{and} \quad \widehat{\phi} \text{ is continuous on } [a, 0] \text{ if } a \neq 0\}$$

equipped with the supremum metric. Assume that $L : \mathcal{C} \to X$ can be extended to a continuous linear operator $\widehat{L} : \widehat{\mathcal{C}} \to X$ on $\widehat{\mathcal{C}}$. Using the same argument as that of Travis and Webb (1978), one can show that the existence and uniqueness of solutions to (2.3)-(2.4) on the extended space $\widehat{\mathcal{C}}$. Moreover, for each $\widehat{\phi} \in \widehat{\mathcal{C}}$, the unique solution $u(\widehat{\phi})$, when restricted on $[0, \infty)$, is continuous. Furthermore, one can also define by $W(t)$ the solution semiflow of the linearized system (2.7)-(2.8) for the extended space. We then have $W(t) : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ for $t \geq 0$ and $W(t)(\widehat{\mathcal{C}}) \subset \mathcal{C}$ for $t > r$. The projections $P_{\mathcal{N}}$, $P_{\mathcal{U}}$ and $P_{\mathcal{S}}$ of $\mathcal{C}$ onto $\mathcal{N}$, $\mathcal{U}$ and $\mathcal{S}$ respectively, can be applied to functions $\widehat{\phi} \in \widehat{\mathcal{C}}$ (c.f. Memory(1991)).

Now $\widehat{L}$ is denoted again by L. Let $X_0 : [-r, 0] \to \mathcal{L}(X; X)$ be defined by

$$X_0(\theta) = \begin{cases} 0 & -r \leq \theta < 0 \\ I & \theta = 0, \end{cases}$$

where 0 (resp. I) denotes the zero (resp. identity) operator on X. For $x \in X$, one denotes

$$X_0^{\mathcal{U}} x = P_{\mathcal{U}} X_0 x, \quad X_0^{\mathcal{N}} x = P_{\mathcal{N}} X_0 x, \quad X_0^{\mathcal{S}} x = P_{\mathcal{S}} X_0 x,$$

where $X_0 x \in \widehat{\mathcal{C}}$ is defined by $(X_0 x)(\theta) = X_0(\theta) x$ for $\theta \in [-r, 0]$. After the preparation above, one has a variation-of-constants formula as follows.

THEOREM 2.3.

(i) For any continuous function $h : [0, \infty) \to X$, *the solution of*

$$\begin{aligned} u(t) &= T(t)\phi(0) + \int_0^t T(t-s)[Lu_s + h(s)]\, ds, \qquad t \geq 0 \\ u_0 &= \phi \end{aligned} \tag{2.11}$$

can be expressed as

$$u_t = W(t)\phi + \int_0^t W(t-s)X_0 h(s)\, ds, \quad t \geq 0.$$

(ii) Assume **(A4)**. *Then the solution of the initial-value problem*

$$\begin{aligned} u(t) &= T(t)\phi(0) + \int_0^t T(t-s)[Lu_s + g(u_s)]\, ds, \qquad t \geq 0 \\ u_0 &= \phi \in \mathcal{C}, \end{aligned} \tag{2.12}$$

satisfies

$$\begin{aligned} P_{\mathcal{U}} u_t &= W(t)P_{\mathcal{U}}\phi + \int_0^t W(t-s)X_0^{\mathcal{U}} g(u_s)\, ds, \\ P_{\mathcal{N}} u_t &= W(t)P_{\mathcal{N}}\phi + \int_0^t W(t-s)X_0^{\mathcal{N}} g(u_s)\, ds, \qquad \textit{and} \\ P_{\mathcal{S}} u_t &= W(t)P_{\mathcal{S}}\phi + \int_0^t W(t-s)X_0^{\mathcal{S}} g(u_s)\, ds \end{aligned} \tag{2.13}$$

for all $t \geq 0$.

(iii) For $x \in X$,

$$\begin{aligned} &W(t)X_0^{\mathcal{U}}x, \quad W(t)X_0^{\mathcal{N}}x \in \mathcal{C} \qquad \textit{for} \quad t \in \mathbb{R}, \\ &W(t)X_0^{\mathcal{S}}x \in \mathcal{C} \qquad\qquad\qquad\quad\ \textit{for} \quad t \geq r. \end{aligned} \tag{2.14}$$

(iv) For any $0 < \epsilon < \min\{\gamma_+, \gamma_-\}$, *there exists* $K(\epsilon) > 0$ *such that for all* $x \in X$,

one has

$$
\begin{aligned}
&|W(t)X_0^{\mathcal{U}}x| \le K(\epsilon)e^{(\gamma_- - \epsilon)t}|x| && \textit{for} \quad t \le 0;\\
&|W(t)X_0^{\mathcal{N}}x| \le K(\epsilon)e^{\epsilon|t|}|x| && \textit{for} \quad t \in \mathbb{R};\\
&|W(t)X_0^{\mathcal{S}}x| \le K(\epsilon)e^{-(\gamma_+ - \epsilon)t}|x| && \textit{for} \quad t \ge 0.
\end{aligned}
\tag{2.15}
$$

The existence and invariance of the center manifold were established in Lin, So and Wu (1992). A simplified proof of such existence and invariance was given in So, Wu and Yang (1998) by applying the technique of the contraction on a Banach space. Followed is a brief description.

Let η and ϵ be such that $0 < \epsilon < \eta < \min\{\gamma_+, \gamma_-\} - \epsilon$ and let $BC^{\eta}(\mathbb{R}; X)$ be the contracted Banach space over X as defined in the beginning. Let us define a linear operator $\mathcal{K}$ on $BC^{\eta}(\mathbb{R}; X)$ by

$$
\begin{aligned}
(\mathcal{K}h)(t) := &\int_0^t u(X_0^{\mathcal{N}}h(s))(t-s)\,ds\\
&+ \int_{-\infty}^t u(X_0^{\mathcal{S}}h(s))(t-s)\,ds + \int_{\infty}^t u(X_0^{\mathcal{U}}h(s))(t-s)\,ds,
\end{aligned}
$$

where $u(\phi)(\cdot)$ is the solution of (2.3)–(2.4) with $\phi \in \widehat{C}$.

According to (2.15), one obtains that $\mathcal{K} : BC^{\eta}(\mathbb{R}; X) \to BC^{\eta}(\mathbb{R}; X)$ is a bounded linear operator with

$$
|\mathcal{K}| \le K(\epsilon)\left(\frac{1}{\eta - \epsilon} + \frac{1}{\gamma_+ - \epsilon - \eta} + \frac{1}{\gamma_- - \epsilon - \eta}\right). \tag{2.16}
$$

From now on, let us consider system (2.1) with assumptions **(A1)**, **(A2)**,

(A3), **(A4)** holding. Define N_g on $BC^\eta(\mathbb{R};X)$ by

$$N_g(u)(t) = g(u_t), \qquad \text{for } t \in \mathbb{R} \text{ and } u \in BC^\eta(\mathbb{R};X).$$

One can show that N_g maps $BC^\eta(\mathbb{R};X)$ into itself and thus the mapping $\mathcal{R}$: $BC^\eta(\mathbb{R};X) \times P_{\mathcal{N}}\mathcal{C} \to BC^\eta(\mathbb{R};X)$ given by

$$\mathcal{R}(u,\phi)(t) = (W(t)\phi)(0) + (\mathcal{K}(N_g(u)))(t) \tag{2.17}$$

is well-defined.

THEOREM 2.4. *If $0 < \epsilon < \eta < \min\{\gamma_+, \gamma_-\} - \epsilon$ and*

$$K(\epsilon)\left(\frac{1}{\eta-\epsilon} + \frac{1}{\gamma_+ - \eta - \epsilon} + \frac{1}{\gamma_- - \eta - \epsilon}\right) e^{\eta r}|g|_{0,1} < 1, \tag{2.18}$$

then for every $\phi \in P_{\mathcal{N}}\mathcal{C}$, the fixed point equation

$$u = \mathcal{R}(u,\phi) \tag{2.19}$$

has a unique solution $u^*(\phi)$ in $BC^\eta(\mathbb{R};X)$ and the center manifold of (2.1) is defined by $M_g := \{(u^*(\phi))_0 \ : \ \phi \in P_{\mathcal{N}}\mathcal{C}\} \subset \mathcal{C}$, which satisfies the following properties:

(i) The mapping $\phi \in P_{\mathcal{N}}\mathcal{C} \mapsto u^*(\phi) \in BC^\eta(\mathbb{R};X)$ is Lipschitz continuous.

(ii) $u^*(\phi)$ is the unique solution in $BC^\eta(\mathbb{R};X)$ of (2.1) with $P_{\mathcal{N}}u_0 = \phi$

(iii) The centre manifold M_g is invariant under the flow defined by (2.1), that is, if u is a solution of (2.1) with $u_0 \in M_g$ then $u_\tau \in M_g$ for all $\tau \in \mathbb{R}$.

2.3 THEOREMS OF ATTRACTIVITY AND THEIR PROOFS

In this section, we suppose that the unstable subspace of $\mathcal{C}$ is trivial, i.e. $\mathcal{U} = \{0\}$. This implies $\mathcal{C} = \mathcal{N} \oplus \mathcal{S}$ according to Theorem 2.2. Recall that for any solution $u(t)$ of (2.1) we have by Theorem 2.3

$$\begin{aligned} u_t^{\mathcal{S}}(\phi) &= W(t-\sigma)u_\sigma^{\mathcal{S}}(\phi) + \int_\sigma^t W(t-s)X_0^{\mathcal{S}} g(u_s(\phi))\, ds, \quad t \geq \sigma > -\infty \\ u_t^{\mathcal{N}}(\phi) &= W(t-\sigma)u_\sigma^{\mathcal{N}}(\phi) + \int_\sigma^t W(t-s)X_0^{\mathcal{N}} g(u_s(\phi))\, ds, \quad t \geq \sigma > -\infty, \end{aligned} \tag{3.1}$$

where $u_t^{\mathcal{S}}(\phi) = P_{\mathcal{S}} u_t(\phi)$ and $u_t^{\mathcal{N}}(\phi) = P_{\mathcal{N}} u_t(\phi)$.

We first make the assumption that $|g|_{0,1}$ can be as small as we like so that there exists a constant Δ satisfying:

(A8)

$$\gamma_+ - 2\epsilon - 2K(\epsilon)|g|_{0,1}(\Delta + 1) > 0;$$

(A9)

$$\frac{2K^2(\epsilon)|g|_{0,1}}{\gamma_+ - 2\epsilon - 2K(\epsilon)|g|_{0,1}(\Delta+1)} < 1; \quad \text{and}$$

(A10)

$$\frac{K^2(\epsilon)|g|_{0,1}(\Delta+1)}{\gamma_+ - 2\epsilon - K(\epsilon)|g|_{0,1}(\Delta+1)} \leq \Delta.$$

Our first version of the attractivity of center manifolds is motivated by Hale and Verduyn Lunel (1993), (see also Chow and Hale (1982)).

THEOREM 3.1. *Let assumptions* **(A8)** $-$ **(A10)** *be satisfied. Let*

$$M_g := \{(u^*(\phi))_0 \ : \ \phi \in P_{\mathcal{N}}\mathcal{C}\} \subset \mathcal{C}$$

be the center manifold of (2.1), where u^* *is defined in Theorem 2.4. Then there exists a mapping* $M^* : \mathcal{C} = \mathcal{N} \oplus \mathcal{S} \to \mathcal{N}$ *such that*

(i) *if for fixed* $\phi \in \mathcal{N}$, *we define* $H(\phi,\psi) = \phi + M^*(\phi,\psi), \quad \psi \in \mathcal{S}$, *then the set* $\mathcal{W}^{\mathcal{S}}(\phi)$, *defined by*

$$\mathcal{W}^{\mathcal{S}}(\phi) = \{(H(\phi,\psi),\psi) : \quad \psi \in \mathcal{S}\} \subset \mathcal{C},$$

is an invariant manifold in the following sense: if

$$\bar{\phi} = H(\phi,\psi), \tag{3.2}$$

then

$$u_t^{\mathcal{N}}(\cdot,\bar{\phi}+\psi) = H(z(t,\phi), u_t^{\mathcal{S}}(\cdot,\bar{\phi}+\psi)); \tag{3.3}$$

in other words, if $(\bar{\phi},\psi) \in \mathcal{W}^{\mathcal{S}}(\phi)$, *then* $u_t(\bar{\phi}+\psi) \in \mathcal{W}^{\mathcal{S}}(z(t,\phi))$, *where,* $u_t(\cdot,\bar{\phi}+\psi)$ *is the unique solution of (2.1) with initial condition* $u_0(\cdot) = \bar{\phi}+\psi \in \mathcal{C}$ *and* $z(t) = z(t,\phi)$ *is the solution of the following equation*

$$z(t) = W(t)\phi + \int_0^t W(t-s)X_0^{\mathcal{N}} g(u_0^*(z(s)))\,ds, \quad t \in \mathbb{R}, \tag{3.4}$$

where $u_0^*(z(s)) = (u^*(z(s))_0 \in M_g$ *is the solution of (2.1) on the center manifold,* *(ii) the following estimates*

$$|u_t^{\mathcal{N}}(\bar{\phi}+\psi) - z(t,\phi)| \leq \Delta K(\epsilon) e^{-[\gamma_+ - \epsilon - K(\epsilon)|g|_{0,1}(\Delta+1)]t} |\psi - u_0^{*\mathcal{S}}(\phi)|$$

$$|u_t^{\mathcal{S}}(\bar{\phi}+\psi) - u_0^{*\mathcal{S}}(z(t,\phi))| \leq K(\epsilon) e^{-[\gamma_+ - \epsilon - K(\epsilon)|g|_{0,1}(\Delta+1)]t} |\psi - u_0^{*\mathcal{S}}(\phi)|$$

hold for $t \geq 0$.

PROOF. First of all, we will show that there exists a mapping M^* satisfying:

(P1) $M^*(\phi, u_0^{*\mathcal{S}}(\phi)) = 0 \quad$ for $\phi \in \mathcal{N}$.

(P2) $M^*(\phi,\psi)$ is uniformly Lipschitz continuous with respect to the second component and the Lipschitz constant is bounded by Δ, that is,

$$|M^*(\phi,\psi_1) - M^*(\phi,\psi_2)| \leq \Delta|\psi_1 - \psi_2|$$

where $\phi \in \mathcal{N}$ and $\psi_1, \psi_2 \in \mathcal{S}$.

(P3) For any $\phi \in \mathcal{N}$ and $\psi \in \mathcal{S}$, let $\bar{\phi} = \phi + M^*(\phi,\psi) \in \mathcal{N}$. Then

$$u_t^{\mathcal{N}}(\cdot, \bar{\phi}+\psi) = z(t,\phi) + M^*(z(t,\phi), u_t^{\mathcal{S}}(\cdot,\bar{\phi}+\psi)). \tag{3.5}$$

Let $\mathcal{F}(\Delta)$ be the set of all mappings from $\mathcal{C}$ to $\mathcal{N}$ with the properties of (P1) and (P2). Define a metric on $\mathcal{F}$ by

$$d_{\mathcal{F}}(M_1, M_2) = \sup_{\phi\in\mathcal{N}, u_0^{*\mathcal{S}}(\phi)\neq\psi\in\mathcal{S}} \frac{|M_1(\phi,\psi) - M_2(\phi,\psi)|}{|\psi - u_0^{*\mathcal{S}}(\phi)|}.$$

Then with this metric, $\mathcal{F}(\Delta)$ is a complete space. Now, for any $M \in \mathcal{F}(\Delta)$ and $\phi \in \mathcal{N}, \psi \in \mathcal{S}$, let $\bar{\phi}_M = \phi + M(\phi,\psi)$. We denote by $u_t^{\mathcal{S}}(\cdot, \bar{\phi}_M + \psi, M)$ the solution of

$$
\begin{aligned}
&u_t^S(\cdot, \bar{\phi}_M + \psi, M) \\
&= W(t)\psi + \int_0^t W(t-s) X_0^S g\left(\bar{\phi}_0(s, z, u^S, M) + u_s^S(\cdot, \bar{\phi}_M + \psi, M)\right)\, ds,
\end{aligned} \tag{3.6}
$$

where,

$$
\bar{\phi}_0(s, z, u^S, M) = z(s, \phi) + M\left(z(s, \phi), u_s^S(\cdot, \bar{\phi}_M + \psi, M)\right).
$$

Now for $s \geq t \geq 0$, we denote

$$
\begin{aligned}
\bar{\phi}_t(s, z, u^S, M) &:= z(s, z(t, \phi)) \\
&\quad + M\left(z(s, z(t, \phi)), u_s^S(\cdot, \bar{\phi}_0(t, z, u^S, M) + u_t^S(\cdot, \bar{\phi}_M + \psi, M), M)\right).
\end{aligned}
$$

Then one can define a mapping T as follows:

$$
\begin{aligned}
&(TM)(\phi, \psi) \\
&:= -\int_0^\infty W(-s) X_0^{\mathcal{N}} g\left(\bar{\phi}_0(s, z, u^S, M) + u_s^S(\cdot, \bar{\phi}_M + \psi, M)\right)\, ds \\
&\quad + \int_0^\infty W(-s) X_0^{\mathcal{N}} g(u_0^*(z(s, \phi)))\, ds.
\end{aligned} \tag{3.7}
$$

It is sufficient to show that mapping T has a fixed point. For this purpose, we first note that since $u_t^S(\cdot, \bar{\phi}_M + \psi, M)$ satisfies (3.6) and $u_0^*(z(t, \phi))$ is the solution of (2.1) on the center manifold, we have

$$
\begin{aligned}
&|u_t^S(\cdot, \bar{\phi}_M + \psi, M) - u_0^{*S}(z(t, \phi))| \\
&\leq K(\epsilon) e^{-(\gamma_+ - \epsilon)t} |\psi - u_0^{*S}(\phi)| \\
&+ K(\epsilon) |g|_{0,1} (\Delta + 1) \int_0^t e^{-(\gamma_+ - \epsilon)(t-s)} |u_s^S(\cdot, \bar{\phi}_M + \psi, M) - u_0^{*S}(z(s, \phi))|\, ds,
\end{aligned}
$$

which implies by Gronwall's inequality

$$
\begin{aligned}
&|u_t^S(\cdot,\bar{\phi}_M+\psi,M)-u_0^{*S}(z(t,\phi))| \\
&\le K(\epsilon)e^{-[\gamma_+-\epsilon-K(\epsilon)|g|_{0,1}(\Delta+1)]t}|\psi-u_0^{*S}(\phi)|
\end{aligned}
\tag{3.8}
$$

for $t\ge 0$. Particularly when $\psi=u_0^{*S}(\phi)$, one obtains

$$
u_t^S(\cdot,\bar{\phi}_M+\psi,M)=u_0^{*S}(z(t,\phi))
$$

for any $t\ge 0$. Moreover $TM(\phi,u_0^{*S}(\phi))=0$ by (3.7). Hence, mapping TM satisfies (P1). Using similar calculations as above, one has

$$
\begin{aligned}
&|u_t^S(\cdot,\bar{\phi}_{1M}+\psi_1,M)-u_t^S(\cdot,\bar{\phi}_{2M}+\psi_2,M)| \\
&\le K(\epsilon)e^{-[\gamma_+-\epsilon-K(\epsilon)|g|_{0,1}(\Delta+1)]t}|\psi_1-\psi_2|,
\end{aligned}
\tag{3.9}
$$

where $\bar{\phi}_{1M}=\phi+M(\phi,\psi_1)$ and $\bar{\phi}_{2M}=\phi+M(\phi,\psi_2)$.

Next we will show that TM satisfies (P2). After some calculations, using (3.7) and the estimate (3.9) one arrives at

$$
\begin{aligned}
&|(TM)(\phi,\psi_1)-(TM)(\phi,\psi_2)| \\
&\le K(\epsilon)|g|_{0,1}(\Delta+1)\int_0^\infty e^{\epsilon s}|u_s^S(\cdot,\bar{\phi}_{1M}+\psi_1,M)-u_s^S(\cdot,\bar{\phi}_{2M}+\psi_2,M)|\,ds \\
&\le K^2(\epsilon)|g|_{0,1}(\Delta+1)|\psi_1-\psi_2|\int_0^\infty e^{-[\gamma_+-2\epsilon-K(\epsilon)|g|_{0,1}(\Delta+1)]s}\,ds \\
&=\frac{K^2(\epsilon)|g|_{0,1}(\Delta+1)}{\gamma_+-2\epsilon-K(\epsilon)|g|_{0,1}(\Delta+1)}|\psi_1-\psi_2|.
\end{aligned}
$$

So by **(A10)**, TM satisfies (P2).

Now one can show that $T:\mathcal{F}(\Delta)\to\mathcal{F}(\Delta)$ is a contraction. In fact by (3.7) one has

$$\begin{aligned}
&|(TM_1)(\phi,\psi)-(TM_2)(\phi,\psi)| \\
&\le K(\epsilon)|g|_{0,1}\int_0^\infty e^{\epsilon s}[|u_s^S(\cdot,\bar\phi_{M_1}+\psi,M_1)-u_0^{*S}(z(s,\phi))|d_{\mathcal{F}}(M_1,M_2) \\
&\qquad +(\Delta+1)|u_s^S(\cdot,\bar\phi_{M_1}+\psi,M_1)-u_s^S(\cdot,\bar\phi_{M_2}+\psi,M_2)|]\,ds.
\end{aligned} \tag{3.10}$$

Using (3.6), (3.8), and the Gronwall inequality, one has

$$\begin{aligned}
&|u_t^S(\cdot,\bar\phi_{M_1}+\psi,M_1)-u_t^S(\cdot,\bar\phi_{M_2}+\psi,M_2)| \\
&\le \frac{K(\epsilon)|\psi-u_0^{*S}(\phi)|d_{\mathcal{F}}(M_1,M_2)}{\Delta+1}e^{-[\gamma-\epsilon-2K(\epsilon)|g|_{0,1}(\Delta+1)]t}.
\end{aligned} \tag{3.11}$$

Substituting (3.8) and (3.11) into (3.10) one obtains

$$|(TM_1)(\phi,\psi)-(TM_2)(\phi,\psi)|\le\frac{2K^2(\epsilon)|g|_{0,1}d_{\mathcal{F}}(M_1,M_2)}{\gamma_+-2\epsilon-2K(\epsilon)|g|_{0,1}(\Delta+1)}|\psi-u_0^{*S}(\phi)|$$

so that

$$d_{\mathcal{F}}(TM_1,TM_2)\le\frac{2K^2(\epsilon)|g|_{0,1}}{\gamma_+-2\epsilon-2K(\epsilon)|g|_{0,1}(\Delta+1)}d_{\mathcal{F}}(M_1,M_2)$$

and hence T is a contraction by **(A9)**. Now by the fixed point theorem of a contraction mapping, there exists $M^*\in\mathcal{F}(\Delta)$ such that

$$TM^*(\phi,\psi)=M^*(\phi,\psi),\qquad \text{for any } (\phi,\psi)\in\mathcal{C}. \tag{3.12}$$

For such a fixed point M^* and any $(\phi,\psi)\in\mathcal{C}$, one need to show (3.5) holds. Using

(3.7) and (3.12) one has

$$\begin{aligned}
&M^*(\phi,\psi)\\
&:= -\int_0^\infty W(-s)X_0^{\mathcal{N}} g\left(\bar{\phi}_0(s,z,u^{\mathcal{S}},M^*) + u_s^{\mathcal{S}}(\cdot,\bar{\phi}_{M^*}+\psi,M^*)\right)\,ds \\
&\quad + \int_0^\infty W(-s)X_0^{\mathcal{N}} g(u_0^*(z(s,\phi)))\,ds.
\end{aligned} \tag{3.13}$$

This implies

$$\begin{aligned}
&M^*(z(t,\phi),u_t^{\mathcal{S}}(\cdot,\bar{\phi}_{M^*}+\psi,M^*))\\
&= -\int_0^\infty W(-s)X_0^{\mathcal{N}} g(\bar{\phi}_t(s,z,u^{\mathcal{S}},M^*)\\
&\qquad\qquad + u_s^{\mathcal{S}}(\cdot,\bar{\phi}_0(t,z,u^{\mathcal{S}},M^*) + u_t^{\mathcal{S}}(\cdot,\bar{\phi}_{M^*}+\psi,M^*),M^*))\,ds\\
&\quad + \int_0^\infty W(-s)X_0^{\mathcal{N}} g(u_0^*(z(s,z(t,\phi))))\,ds\\
&= -\int_t^\infty W(t-s)X_0^{\mathcal{N}} g(\bar{\phi}_t(s-t,z,u^{\mathcal{S}},M^*)\\
&\qquad\qquad + u_{s-t}^{\mathcal{S}}(\cdot,\bar{\phi}_0(t,z,u^{\mathcal{S}},M^*) + u_t^{\mathcal{S}}(\cdot,\bar{\phi}_{M^*}+\psi,M^*),M^*))\,ds\\
&\quad + \int_t^\infty W(t-s)X_0^{\mathcal{N}} g(u_0^*(z(s-t,z(t,\phi))))\,ds.
\end{aligned} \tag{3.14}$$

Claim: For $s \geq t \geq 0$,

$$u_{s-t}^{\mathcal{S}}(\cdot,\bar{\phi}_0(t,z,u^{\mathcal{S}},M^*) + u_t^{\mathcal{S}}(\cdot,\bar{\phi}_{M^*}+\psi,M^*),M^*) = u_s^{\mathcal{S}}(\cdot,\bar{\phi}_{M^*}+\psi,M^*) \tag{3.15}$$

Proof: According to (3.6), $u_{s-t}^{\mathcal{S}}(\cdot,\bar{\phi}_0(t,z,u^{\mathcal{S}},M^*) + u_t^{\mathcal{S}}(\cdot,\bar{\phi}_{M^*}+\psi,M^*),M^*)$ satisfies

$$
\begin{aligned}
&u^S_{s-t}(\cdot,\bar\phi_0(t,z,u^S,M^*)+u^S_t(\cdot,\bar\phi_{M^*}+\psi,M^*),M^*)\\
&=W(s-t)u^S_t(\cdot,\bar\phi_{M^*}+\psi,M^*)\\
&+\int_0^{s-t}W(s-t-\tau)X_0^S g(\bar\phi_t(\tau,z,u^S,M^*)\\
&\qquad+u^S_\tau(\cdot,\bar\phi_0(t,z,u^S,M^*)+u^S_t(\cdot,\bar\phi_{M^*}+\psi,M^*),M^*))\,d\tau.
\end{aligned}
\tag{3.16}
$$

Again by (3.6) and properties of semigroup $W(t)$ one has

$$
\begin{aligned}
&W(s-t)u^S_t(\cdot,\bar\phi_{M^*}+\psi,M^*)=W(s)\psi\\
&+\int_0^t W(s-\tau)X_0^S g\left(\bar\phi_0(\tau,z,u^S,M^*)+u^S_\tau(\cdot,\bar\phi_{M^*}+\psi,M^*)\right)\,d\tau.
\end{aligned}
\tag{3.17}
$$

Substituting (3.17) into (3.16) and carrying out a few calculations, we obtain

$$
\begin{aligned}
&u^S_{s-t}(\cdot,\bar\phi_0(t,z,u^S,M^*)+u^S_t(\cdot,\bar\phi_{M^*}+\psi,M^*),M^*)=W(s)\psi\\
&+\int_0^t W(s-\tau)X_0^S g\left(\bar\phi_0(\tau,z,u^S,M^*)+u^S_\tau(\cdot,\bar\phi_{M^*}+\psi,M^*)\right)\,d\tau\\
&+\int_t^s W(s-\tau)X_0^S g(\bar\phi_t(\tau-t,z,u^S,M^*)\\
&\qquad+u^S_{\tau-t}(\cdot,\bar\phi_0(t,z,u^S,M^*)+u^S_t(\cdot,\bar\phi_{M^*}+\psi,M^*),M^*))\,d\tau.
\end{aligned}
\tag{3.18}
$$

On the other hand, since $u^S_s(\cdot,\bar\phi_{M^*}+\psi,M^*)$ is the solution of (3.6), it satisfies

$$
\begin{aligned}
&u^S_s(\cdot,\bar\phi_{M^*}+\psi,M^*)=W(s)\psi\\
&+\int_0^s W(s-\tau)X_0^S g\left(\bar\phi_0(\tau,z,u^S,M^*)+u^S_\tau(\cdot,\bar\phi_{M^*}+\psi,M^*)\right)\,d\tau.
\end{aligned}
\tag{3.19}
$$

Combining (3.18) and (3.19) and noticing that $z(s-t,z(t,\phi))=z(s,\phi)$, one obtains

$$
\begin{aligned}
&|u^S_{s-t}(\cdot,\bar\phi_0(t,z,u^S,M^*)+u^S_t(\cdot,\bar\phi_{M^*}+\psi,M^*),M^*)-u^S_s(\cdot,\bar\phi_{M^*}+\psi,M^*)|\\
&\le K_g^\Delta(\epsilon)\int_t^s e^{-\gamma_+(\epsilon)(s-\tau)}|u^S_{\tau-t}(\cdot,\bar\phi_0(t,z,u^S,M^*)+u^S_t(\cdot,\bar\phi_{M^*}+\psi,M^*),M^*)\\
&\qquad-u^S_\tau(\cdot,\bar\phi_{M^*}+\psi,M^*)|\,d\tau,
\end{aligned}
$$

where $K_g^{\Delta}(\epsilon) := K(\epsilon)|g|_{0,1}(\Delta+1)$ and $\gamma_+(\epsilon) := \gamma_+ - \epsilon$. Hence by applying Gronwall's inequality, we obtain (3.15). This completes the proof of the claim.

Using (3.4), (3.14), (3.15), and (3.13) one has

$$
\begin{aligned}
& z(t,\phi) + M^*(z(t,\phi), u_t^{\mathcal{S}}(\cdot, \bar{\phi}_{M^*} + \psi, M^*)) \\
&= W(t)\phi + \int_0^{\infty} W(t-s) X_0^{\mathcal{N}} g(u_0^*(z(s,\phi)))\, ds \\
&\qquad - \int_t^{\infty} W(t-s) X_0^{\mathcal{N}} g\left(\bar{\phi}_0(s,z,u^{\mathcal{S}},M^*) + u_s^{\mathcal{S}}(\cdot, \bar{\phi}_{M^*} + \psi, M^*)\right)\, ds \\
&= W(t)(\phi + M^*(\phi,\psi)) \\
&\qquad + \int_0^{t} W(t-s) X_0^{\mathcal{N}} g\left(\bar{\phi}_0(s,z,u^{\mathcal{S}},M^*) + u_s^{\mathcal{S}}(\cdot, \bar{\phi}_{M^*} + \psi, M^*)\right)\, ds.
\end{aligned}
$$

This shows that

$$
z(t,\phi) + M^*(z(t,\phi), u_t^{\mathcal{S}}(\cdot, \bar{\phi}_{M^*} + \psi, M^*)) + u_t^{\mathcal{S}}(\cdot, \bar{\phi}_{M^*} + \psi, M^*)
$$

is the solution of (2.1) for $t \geq 0$. By uniqueness of the solution of (2.1), one has

$$
u_t^{\mathcal{S}}(\cdot, \bar{\phi}_{M^*} + \psi, M^*) = u_t^{\mathcal{S}}(\cdot, \bar{\phi} + \psi)
$$

and hence

$$
\begin{aligned}
u_t^{\mathcal{N}}(\cdot, \bar{\phi} + \psi) &= z(t,\phi) + M^*(z(t,\phi), u_t^{\mathcal{S}}(\cdot, \bar{\phi}_{M^*} + \psi, M^*)) \\
&= z(t,\phi) + M^*(z(t,\phi), u_t^{\mathcal{S}}(\cdot, \bar{\phi} + \psi)).
\end{aligned}
$$

Therefore, equation (3.5) holds.

Now for any $(\phi,\psi) \in \mathcal{C} = \mathcal{N} \oplus \mathcal{S}$, $\phi \in \mathcal{N}$ and $\psi \in \mathcal{S}$, define $H : \mathcal{C} \to \mathcal{N}$ by

$$
H(\phi,\psi) := \phi + M^*(\phi,\psi).
$$

We still define $\bar{\phi}$ as in (P3). Then $\bar{\phi} = H(\phi, \psi)$. Moreover, by (3.5),

$$H\left(z(t,\phi), u_t^S(\cdot, \bar{\phi} + \psi)\right) = z(t,\phi) + M^*\left(z(t,\phi), u_t^S(\cdot, \bar{\phi} + \psi)\right)$$
$$= u_t^{\mathcal{N}}(\cdot, \bar{\phi} + \psi).$$

Hence, (3.3) is also proved to be true.

So far, we have shown (i) of the theorem. Further, the estimates of (ii) in the theorem immediately follow from (3.8), (3.5) and the properties (P1) and (P2) about mapping M^*. This completes the proof.

Next, we will present another version of attractivity of the center manifold. Basically, this part is excerpted from So, Wu, and Yang (1998). Before invoking the proof of this version, we also need some assumptions as follows. Choose η such that $\epsilon < \eta < \gamma_+ - 2\epsilon$. This η satisfies the requirements of Theorem 2.4. By taking $|g|_{0,1}$ sufficiently small, we can assume that there exist constants $\Delta, \beta > 0$ satisfying

(A11)

$$\eta + 3K(\epsilon)|g|_{0,1} + \epsilon < \beta < \gamma_+ - \epsilon - K(\epsilon)|g|_{0,1}(\Delta + 1);$$

(A12)

$$K^2(\epsilon)|g|_{0,1}^2\Delta(\Delta+1)\Pi_1^{-1} + K(\epsilon) + \frac{K(\epsilon)|g|_{0,1}\Delta}{\gamma_+ - \epsilon - \beta} \leq \Delta;$$

(A13)

$$\frac{K^2(\epsilon)|g|_{0,1}(\Delta+1)}{\gamma_+ - 2\epsilon - K(\epsilon)|g|_{0,1}(\Delta+1)} \leq \Delta; \qquad \text{and}$$

(A14)

$$\frac{K(\epsilon)|g|_{0,1}}{\gamma_+ - \epsilon} + K^2(\epsilon)|g|_{0,1}(\Delta+1)\Pi_2^{-1} < 1;$$

where

$$\Pi_1 := [\gamma_+ - \epsilon - K(\epsilon)|g|_{0,1}(\Delta+1) - \beta][\gamma_+ - 2\epsilon - K(\epsilon)|g|_{0,1}(\Delta+1)]$$

$$\Pi_2 := [\gamma_+ - 2\epsilon - K(\epsilon)|g|_{0,1}(\Delta+1)][\gamma_+ - \epsilon - K(\epsilon)|g|_{0,1}(\Delta+1)]$$

LEMMA 3.2. *There exists a continuous mapping $\mathcal{J} : \mathbb{R}_+ \times \mathcal{C} \to \mathcal{S}$ such that if $u_t(\phi+\psi)$ ($\phi \in \mathcal{N}$ and $\psi \in \mathcal{S}$) is a solution of (2.1), then $u_t(\phi+\psi)$ exists for all $t \geq 0$ and*

$$u_t^{\mathcal{S}}(\phi+\psi) = \mathcal{J}(t, u_t^{\mathcal{N}}(\phi+\psi), \psi). \tag{3.20}$$

Moreover, $\mathcal{J}$ satisfies the inequality

$$|\mathcal{J}(t,\bar{\phi}_1,\psi_1) - \mathcal{J}(t,\bar{\phi}_2,\psi_2)| \leq \Delta(|\bar{\phi}_1 - \bar{\phi}_2| + e^{-\beta t}|\psi_1 - \psi_2|) \tag{3.21}$$

for all $\bar{\phi}_1, \bar{\phi}_2 \in \mathcal{N}$, $\psi_1, \psi_2 \in \mathcal{S}$, and $t \geq 0$.

PROOF. It suffices to show that there exist continuous $\mathcal{J} : \mathbb{R}_+ \times \mathcal{C} \to \mathcal{S}$ and $v : D \times \mathcal{C} \to \mathcal{N}$, where $D := \{(\tau, t) \in \mathbb{R}_+^2 : 0 \leq \tau \leq t\}$, J satisfies (3.21) and such that for given $\bar{\phi} \in \mathcal{N}$ and $\psi \in \mathcal{S}$, one has

$$\begin{aligned}\mathcal{J}(t,\bar{\phi},\psi) &= W(t)\psi \\ &\quad + \int_0^t W(t-s)X_0^{\mathcal{S}} g(v(s,t,\bar{\phi},\psi) + \mathcal{J}(s, v(s,t,\bar{\phi},\psi),\psi))\, ds,\end{aligned} \tag{3.22}$$

and

$$
\begin{aligned}
v(\tau, t, \bar{\phi}, \psi) &= W(\tau - t)\bar{\phi} \\
&\quad - \int_{\tau}^{t} W(\tau - s)X_0^{\mathcal{N}} g(v(s, t, \bar{\phi}, \psi) + \mathcal{J}(s, v(s, t, \bar{\phi}, \psi), \psi))\, ds,
\end{aligned}
\tag{3.23}
$$

for $0 \le \tau \le t$. Indeed, suppose such J and v exist. We will show that

$$
\mathcal{J}(t, u_t^{\mathcal{N}}(\phi + \psi), \psi) = u_t^{\mathcal{S}}(\phi + \psi) \qquad \text{for } t \ge 0.
$$

Let $\bar{\phi} = u_t^{\mathcal{N}}(\phi + \psi)$, and denote

$$
\begin{aligned}
\mathcal{J}_u^v(s, t) &:= u_s^{\mathcal{S}}(\phi + \psi) - \mathcal{J}(s, v(s, t, \bar{\phi}, \psi), \psi) \\
v^{\mathcal{J}}(s, t) &:= v(s, t, \bar{\phi}, \psi) + \mathcal{J}(s, v(s, t, \bar{\phi}, \psi), \psi).
\end{aligned}
\tag{3.24}
$$

Since

$$
\begin{aligned}
u_t^{\mathcal{S}}(\phi + \psi) &= W(t)\psi + \int_0^t W(t - s)X_0^{\mathcal{S}} g(u_s^{\mathcal{N}}(\phi + \psi) + u_s^{\mathcal{S}}(\phi + \psi))\, ds \qquad \text{and} \\
u_\tau^{\mathcal{N}}(\phi + \psi) &= W(\tau - t)\bar{\phi} - \int_\tau^t W(\tau - s)X_0^{\mathcal{N}} g(u_s^{\mathcal{N}}(\phi + \psi) + u_s^{\mathcal{S}}(\phi + \psi))\, ds
\end{aligned}
$$

for $0 \le \tau \le t$, we have

$$
\begin{aligned}
&|u_t^{\mathcal{S}}(\phi + \psi) - \mathcal{J}(t, \bar{\phi}, \psi)| \\
&= \left| \int_0^t W(t - s)X_0^{\mathcal{S}} \left[g(u_s^{\mathcal{N}}(\phi + \psi) + u_s^{\mathcal{S}}(\phi + \psi)) - g(v^{\mathcal{J}}(s, t)) \right] ds \right| \\
&\le K(\epsilon)|g|_{0,1} \int_0^t e^{-(\gamma_+ - \epsilon)(t - s)} \left[\left| u_s^{\mathcal{N}}(\phi + \psi) - v(s, t, \bar{\phi}, \psi) \right| + |\mathcal{J}_u^v(s, t)| \right] ds
\end{aligned}
\tag{3.25}
$$

and for $0 \le \tau \le t$

$$
\begin{aligned}
&|u_\tau^{\mathcal{N}}(\phi + \psi) - v(\tau, t, \bar{\phi}, \psi)| \\
&= \left| \int_\tau^t W(\tau - s)X_0^{\mathcal{N}} \left[g(u_s^{\mathcal{N}}(\phi + \psi) + u_s^{\mathcal{S}}(\phi + \psi)) - g(v^{\mathcal{J}}(s, t)) \right] ds \right| \\
&\le K(\epsilon)|g|_{0,1} \int_\tau^t e^{\epsilon(s - \tau)} \left[\left| u_s^{\mathcal{N}}(\phi + \psi) - v(s, t, \bar{\phi}, \psi) \right| + |\mathcal{J}_u^v(s, t)| \right] ds.
\end{aligned}
$$

Thus

$$e^{\epsilon\tau}|u_\tau^{\mathcal{N}}(\phi+\psi)-v(\tau,t,\bar{\phi},\psi)| \le K(\epsilon)|g|_{0,1}\int_\tau^t e^{\epsilon s}\left|u_s^{\mathcal{N}}(\phi+\psi)-v(s,t,\bar{\phi},\psi)\right| ds$$
$$+ K(\epsilon)|g|_{0,1}\int_\tau^t e^{\epsilon s}\left|\mathcal{J}_u^v(s,t)\right| ds.$$

Using Gronwall's inequality, we have

$$\begin{aligned}&e^{\epsilon\tau}|u_\tau^{\mathcal{N}}(\phi+\psi)-v(\tau,t,\bar{\phi},\psi)|\\ &\le K(\epsilon)|g|_{0,1}e^{K(\epsilon)|g|_{0,1}(t-\tau)}\int_\tau^t e^{\epsilon s}\left|\mathcal{J}_u^v(s,t)\right| ds.\end{aligned}$$

This implies, for $0\le s\le t$,

$$\begin{aligned}&|u_s^{\mathcal{N}}(\phi+\psi)-v(s,t,\bar{\phi},\psi)|\\ &\le K(\epsilon)|g|_{0,1}e^{K(\epsilon)|g|_{0,1}(t-s)}\int_s^t e^{\epsilon(\xi-s)}\left|\mathcal{J}_u^v(\xi,t)\right| d\xi.\end{aligned} \tag{3.26}$$

Substituting (3.26) into (3.25), we obtain

$$\begin{aligned}&|u_t^{\mathcal{S}}(\phi+\psi)-\mathcal{J}(t,\bar{\phi},\psi)|\\ &= K(\epsilon)^2|g|_{0,1}^2\int_0^t e^{-(\gamma_+-\epsilon)(t-s)}e^{K(\epsilon)|g|_{0,1}(t-s)}\int_s^t e^{\epsilon(\xi-s)}\left|\mathcal{J}_u^v(\xi,t)\right| d\xi\, ds\\ &\quad + K(\epsilon)|g|_{0,1}\int_0^t e^{-(\gamma_+-\epsilon)(t-s)}\left|\mathcal{J}_u^v(s,t)\right| ds.\end{aligned}$$

By interchanging the order of integration in the first term, we have

$$
\begin{aligned}
&|u_t^{\mathcal{S}}(\phi+\psi) - \mathcal{J}(t,\bar{\phi},\psi)| \\
&= \frac{K(\epsilon)^2|g|_{0,1}^2}{\gamma_+^g(\epsilon)} \int_0^t e^{-(\gamma_+-\epsilon-K(\epsilon)|g|_{0,1})(t-\xi)} \left[1 - e^{-\gamma_+^g(\epsilon)\xi}\right] |\mathcal{J}_u^v(\xi,t)|\, d\xi \\
&\quad + K(\epsilon)|g|_{0,1} \int_0^t e^{-(\gamma_+-\epsilon)(t-s)} |\mathcal{J}_u^v(s,t)|\, ds \\
&\leq \frac{K(\epsilon)^2|g|_{0,1}^2}{\gamma_+^g(\epsilon)} \int_0^t e^{-(\gamma_+-\epsilon-K(\epsilon)|g|_{0,1})(t-\xi)} |\mathcal{J}_u^v(\xi,t)|\, d\xi \\
&\quad + K(\epsilon)|g|_{0,1} \int_0^t e^{-(\gamma_+-\epsilon)(t-s)} |\mathcal{J}_u^v(s,t)|\, ds \\
&\leq K_1 \int_0^t e^{-K_2(t-s)} |\mathcal{J}_u^v(s,t)|\, ds
\end{aligned}
$$

for some constants $K_1, K_2 > 0$, where $\gamma_+^g(\epsilon) := \gamma_+ - 2\epsilon - K(\epsilon)|g|_{0,1}$. Thus

$$
\begin{aligned}
&|u_t^{\mathcal{S}}(\phi+\psi) - \mathcal{J}(t, v(t,t,\bar{\phi},\psi),\psi)| \\
&\leq K_1 \int_0^t e^{-K_2(t-s)} \left|u_s^{\mathcal{S}}(\phi+\psi) - \mathcal{J}(s, v(s,t,\bar{\phi},\psi),\psi)\right|\, ds,
\end{aligned}
$$

since, according to (3.23), $v(t,t,\bar{\phi},\psi) = \bar{\phi} = u_t^{\mathcal{N}}(\phi+\psi)$. Using Gronwall's inequality again, we have, $u_t^{\mathcal{S}}(\phi+\psi) - \mathcal{J}(t, v(t,t,\bar{\phi},\psi),\psi) = 0$. This implies $u_t^{\mathcal{S}}(\phi+\psi) = \mathcal{J}(t, u_t^{\mathcal{N}}(\phi+\psi),\psi)$.

We will now establish the existence of $\mathcal{J}$ and v. Let

$$
\begin{aligned}
\mathcal{F} \equiv \mathcal{F}(\Delta,\beta) :=\{\mathcal{J} \;:\; \mathbb{R}_+ \times \mathcal{C} \to \mathcal{S};\, &\mathcal{J} \text{ is a continuous mapping and} \\
&|\mathcal{J}(t,\bar{\phi}_1,\psi_1) - \mathcal{J}(t,\bar{\phi}_2,\psi_2)| \leq \Delta(|\bar{\phi}_1 - \bar{\phi}_2| + e^{-\beta t}|\psi_1 - \psi_2|)\}.
\end{aligned}
$$

Clearly, $\mathcal{F}$ is a complete metric space under the metric

$$
\rho(\mathcal{J}_1,\mathcal{J}_2) \equiv |\mathcal{J}_1 - \mathcal{J}_2|_{\mathcal{F}} := \sup_{t,\bar{\phi},\psi} |\mathcal{J}_1(t,\bar{\phi},\psi) - \mathcal{J}_2(t,\bar{\phi},\psi)|.
$$

Now for any $\mathcal{J} \in \mathcal{F}$ and $t \geq 0$, let $v(\tau, t, \bar{\phi}, \psi; \mathcal{J})$ denote the solution of (3.23) for $0 \leq \tau \leq t$, i.e.

$$v(\tau, t, \bar{\phi}, \psi; \mathcal{J}) = W(\tau - t)\bar{\phi} - \int_{\tau}^{t} W(\tau - s) X_0^{\mathcal{N}} g(v^{\mathcal{J}}(s, t; \mathcal{J}))\, ds, \qquad (3.27)$$

where

$$v^{\mathcal{J}}(s, t; \mathcal{J}) := v(s, t, \bar{\phi}, \psi; \mathcal{J}) + \mathcal{J}(s, v(s, t, \bar{\phi}, \psi; \mathcal{J}), \psi).$$

Such a solution does exist and is continuous in $(\tau, t, \bar{\phi}, \psi)$ since (3.23) is a (finite dimensional) ODE in v. Let $\mathcal{J} \in \mathcal{F}$ be given. We define $T\mathcal{J}$ by

$$(T\mathcal{J})(t, \bar{\phi}, \psi) = W(t)\psi + \int_0^t W(t - s) X_0^{\mathcal{S}} g(v^{\mathcal{J}}(s, t; \mathcal{J}))\, ds, \quad t \geq 0.$$

We will show that $T : \mathcal{F} \to \mathcal{F}$ has a fixed point by using the contraction mapping principle. First we will show $T(\mathcal{F}) \subset \mathcal{F}$. In fact, let $\mathcal{J} \in \mathcal{F}$. Then

$$\begin{aligned} &|(T\mathcal{J})(t, \bar{\phi}_1, \psi_1) - (T\mathcal{J})(t, \bar{\phi}_2, \psi_2)| \\ &\leq \left(K(\epsilon) e^{-(\gamma_+ - \epsilon)t} + \frac{K(\epsilon)|g|_{0,1} \Delta e^{-\beta t}}{\gamma_+ - \epsilon - \beta} \right) |\psi_1 - \psi_2| \\ &\quad + K_g^{\Delta}(\epsilon) e^{-(\gamma_+ - \epsilon)t} \int_0^t e^{(\gamma_+ - \epsilon)s} |v(s, t, \bar{\phi}_1, \psi_1; \mathcal{J}) - v(s, t, \bar{\phi}_2, \psi_2; \mathcal{J})|\, ds, \end{aligned}$$

where $K_g^{\Delta}(\epsilon) := K(\epsilon)|g|_{0,1}(\Delta + 1)$. On the other hand, from (3.27) and by using

Gronwall's inequality

$$
\begin{aligned}
&|v(s,t,\bar{\phi}_1,\psi_1;\mathcal{J}) - v(s,t,\bar{\phi}_2,\psi_2;\mathcal{J})| \\
&\leq \left[K(\epsilon)e^{\epsilon t}|\bar{\phi}_1 - \bar{\phi}_2| + \frac{K(\epsilon)|g|_{0,1}\Delta|\psi_1-\psi_2|}{\beta-\epsilon}\left(e^{-(\beta-\epsilon)s} - e^{-(\beta-\epsilon)t}\right)\right] \\
&e^{K(\epsilon)|g|_{0,1}(\Delta+1)t}e^{-[K(\epsilon)|g|_{0,1}(\Delta+1)+\epsilon]s}, \qquad \text{for} \quad 0 \leq s \leq t.
\end{aligned}
$$

After a few simple calculations, one has

$$
\begin{aligned}
&|(T\mathcal{J})(t,\bar{\phi}_1,\psi_1) - (T\mathcal{J})(t,\bar{\phi}_2,\psi_2)| \\
&\leq \left(K(\epsilon) + \frac{K(\epsilon)|g|_{0,1}\Delta}{\gamma_+ - \epsilon - \beta} + C(\Delta,\beta)\right)e^{-\beta t}|\psi_1-\psi_2| + \frac{K^2(\epsilon)|g|_{0,1}(\Delta+1)}{\gamma_+^g(\epsilon)}|\bar{\phi}_1 - \bar{\phi}_2|
\end{aligned}
$$

where,

$$C(\Delta,\beta) := K^2(\epsilon)|g|_{0,1}^2\Delta(\Delta+1)\Pi_1^{-1}.$$

Using **(A12)**–**(A13)**, one obtains

$$|(T\mathcal{J})(t,\bar{\phi}_1,\psi_1) - (T\mathcal{J})(t,\bar{\phi}_2,\psi_2)| \leq \Delta(|\bar{\phi}_1 - \bar{\phi}_2| + e^{-\beta t}|\psi_1-\psi_2|)$$

and hence $T\mathcal{J} \in \mathcal{F}$. Next one will show that T is a contraction on $\mathcal{F}$. In fact, using arguments similar to the above, one could show that

$$|(T\mathcal{J}_1)(t,\bar{\phi},\psi) - (T\mathcal{J}_2)(t,\bar{\phi},\psi)| \leq \left(\frac{K(\epsilon)|g|_{0,1}}{\gamma_+ - \epsilon} + C(\Delta)\right)|\mathcal{J}_1 - \mathcal{J}_2|_{\mathcal{F}}$$

where,

$$C(\Delta) := K^2(\epsilon)|g|_{0,1}(\Delta+1)\Pi_2^{-1}.$$

Thus,

$$|T\mathcal{J}_1 - T\mathcal{J}_2|_{\mathcal{F}} \le \left(\frac{K(\epsilon)|g|_{0,1}}{\gamma_+ - \epsilon} + C(\Delta)\right)|\mathcal{J}_1 - \mathcal{J}_2|_{\mathcal{F}}.$$

According to assumptions **(A11)–(A14)**, T is a contraction mapping. This completes the proof of the lemma.

The following theorem shows that the centre manifold is attracting.

THEOREM 3.3. *Suppose assumptions* **(A11)–(A14)** *hold. Let* $u_t(\phi + \psi)$ *be the solution of (2.1) with initial condition* $\phi + \psi \in \mathcal{N} \oplus \mathcal{S}$. *Then there exist constants* $K_1 > 0$ *and* $\beta_1 > 0$ *with* K_1 *depending on* ϕ *and* ψ, *such that*

$$|u_t^{\mathcal{S}}(\phi + \psi) - u_0^{*\mathcal{S}}(u_t^{\mathcal{N}}(\phi + \psi))| \le K_1 e^{-\beta_1 t}, \qquad t \ge 0 \tag{3.28}$$

where $u_0^{*\mathcal{S}}(\cdot) = P_{\mathcal{S}}(u_0^*(\cdot))$ *and* $u_0^*(\cdot)$ *was introduced in Theorem 2.4.*

PROOF. Fix any $t \ge 0$. The solution of (2.1) through $u_t^{\mathcal{N}}(\phi+\psi)+u_0^{*\mathcal{S}}(u_t^{\mathcal{N}}(\phi+\psi))$ is defined for all time and lies on the centre manifold M_g, since $\mathcal{M}_g$ is invariant. Therefore, we can find a point $u_0^*(\phi^*) \in \mathcal{M}_g$, where $\phi^* \in \mathcal{N}$, such that

$$u_t(u_0^*(\phi^*)) = u_t^{\mathcal{N}}(\phi + \psi) + u_0^{*\mathcal{S}}(u_t^{\mathcal{N}}(\phi + \psi)).$$

By Lemma 3.2,

$$
\begin{aligned}
&|u_t^{\mathcal{S}}(\phi+\psi) - u_0^{*\mathcal{S}}(u_t^{\mathcal{N}}(\phi+\psi))| \\
&= |\mathcal{J}(t, u_t^{\mathcal{N}}(\phi+\psi), \psi) - \mathcal{J}(t, u_t^{\mathcal{N}}(\phi+\psi), u_0^{*\mathcal{S}}(\phi^*))| \\
&\leq \Delta e^{-\beta t}|\psi - u_0^{*\mathcal{S}}(\phi^*)| \leq \Delta e^{-\beta t}(|\psi| + |u_0^{*\mathcal{S}}(\phi^*)|).
\end{aligned}
$$

On the other hand, since $\mathcal{U} = \{0\}$, by (2.17) and (2.19), $u_\tau^*(\bar{\phi})$ satisfies, for $\bar{\phi} \in \mathcal{N}$,

$$
u_\tau^*(\bar{\phi}) = W(\tau)\bar{\phi} + \int_{-\infty}^{\tau} W(\tau-s)X_0^{\mathcal{S}} g(u_s^*(\bar{\phi})) + \int_0^{\tau} W(\tau-s)X_0^{\mathcal{N}} g(u_s^*(\bar{\phi}))\, ds, \quad \tau \in \mathbb{R}.
$$

If we take $\bar{\phi} = u_t^{\mathcal{N}}(\phi+\psi)$, then $u_{-t}^*(\bar{\phi}) = u_0^*(\phi^*)$. Using the fact that any solution $u_t(\phi+\psi)$ of (2.1) satisfies the estimate

$$
|u_t^{\mathcal{N}}(\phi+\psi)| + |u_t^{\mathcal{S}}(\phi+\psi)| \leq K(\epsilon)(|\phi| + |\psi|)e^{[2K(\epsilon)|g|_{0,1}+\epsilon]t},
$$

and using Theorem 2.4, a few simple calculations show that $|u_0^{*\mathcal{S}}(\phi^*)|$ satisfies the estimate

$$
|u_0^{*\mathcal{S}}(\phi^*)| \leq (K(\epsilon) + C(\eta, r))K(\epsilon)(|\phi| + |\psi|)e^{[\eta+3K(\epsilon)|g|_{0,1}+\epsilon]t},
$$

where

$$
C(\eta, r) := \frac{K^2(\epsilon)|g|_{0,1}e^{\eta r}}{(\gamma_+ - \epsilon - \eta)\left[1 - K(\epsilon)|g|_{0,1}\left(\frac{1}{\eta-\epsilon} + \frac{1}{\gamma_+-\epsilon-\eta}\right)e^{\eta r}\right]}.
$$

Hence by assumptions **(A11)–(A14)**, there exist constants $K_1 > 0$ and $\beta_1 \in (0, \beta)$ such that (3.28) holds. This completes the proof of the theorem.

REMARK. Both estimates (ii) in Theorem 3.1 and (3.27) in Theorem 3.3 imply that $\mathrm{dist}(u_t(\phi+\psi), M_g) \to 0$ as $t \to \infty$, since $u_0^*(u_t^{\mathcal{N}}(\phi+\psi)) \in M_g$. Throughout this chapter, one can see that the center manifold actually is a local feature of dynamical systems since assumptions **(A8)–(A14)** are based on the cut-off trick, see for example Hale and Verduyn Lunel (1993, p.314–p.315). Locally, the center manifold theorem allows us to reduce an infinite dimensional dynamical system into a finite dimensional system on the center manifold. The attractivity theorem therefore implies that stability of the equilibrium (or the periodic solution) of this finite dynamical system guarantees the stability of the infinite dynamical system. In the next chapter, we will make use of the center manifold reduction method to study the stability of periodic solutions bifurcating from a positive equilibrium.

CHAPTER III

NEUMANN BOUNDARY VALUE PROBLEMS

3.1 INTRODUCTION

In this chapter, our aim is to study the global dynamics of functional partial differential equations with Neumann boundary conditions. More specifically, we will restrict our attention to

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) - \delta u(t,x) + f(u(t-\tau,x)), && \text{in } D \\ \frac{\partial u(t,x)}{\partial n} &= 0, && \text{on } \Gamma \\ u(\theta,x) &= u_0(\theta,x) \geq 0, && \text{in } D_\tau \end{aligned}$$

where, τ is the delayed time; δ is a positive constant; $x \in \Omega \subset \mathbb{R}^n$, Ω is a bounded domain with a smooth boundary $\partial\Omega$, $(t,x) \in D \equiv (0,\infty) \times \Omega$, $\Gamma \equiv (0,\infty) \times \partial\Omega$, $D_\tau \equiv [-\tau, 0] \times \bar{\Omega}$; $\frac{\partial}{\partial n}$ denotes the exterior normal derivative to $\partial\Omega$; $u_0(\theta,x)$ is Hölder continuous in D_τ with $u_0(0,x) \in C^1(\bar{\Omega})$; and $f(z)$ is a nonlinear function satisfying the following hypotheses:

(i) $f(0) = 0$

(ii) $\lim_{z\to\infty} f(z) = 0$

(iii) there exists $z_0 > 0$, such that $f(z)$ is monotone increasing for $z \in [0, z_0]$ and decreasing afterwards.

As we mentioned in chapter 1, we will choose $f(z) = Pze^{-az}$ as the carrier in all the proofs and calculations. For this particular choice of a nonlinear term, the above equation without diffusion is known for modelling a population of adult flies. As a model to describe the dynamics of Nicholson's blowflies experiments (1954), the non diffusive delay equation was first proposed by Gurney, Blythe and Nisbet (1980) in the form of:

$$\dot{N}(t) = -\delta N(t) + PN(t-\tau)e^{-aN(t-\tau)}, \qquad t > 0, \tag{1.1}$$

together with the initial condition

$$N(\theta) = \phi(\theta) \geq 0, \qquad \theta \in [-\tau, 0], \tag{1.2}$$

where $N(t)$ is the size of the (adult) blowflies population at time t; P is the maximum per capita daily egg production rate; $\frac{1}{a}$ is the size at which the blowflies population reproduces at its maximum rate; δ is the per capita daily adult death rate; and τ is the generation time. For this equation, global attractivity and oscillation of solutions have been investigated by several authors, with the following results.

(i) So and Yu (1994): If $0 < \frac{P}{\delta} \leq 1$, then every solution $N(t)$ tends to zero as $t \to \infty$.

(ii) So and Yu (1994) and Karakostas, Philos and Sficas (1992): If $1 < \frac{P}{\delta} < e$, or if $\frac{P}{\delta} > 1$ and $(e^{\delta\tau} - 1)\ln\frac{P}{\delta} \le 1$, the positive equilibrium $N^* \equiv \frac{1}{a}\ln\frac{P}{\delta}$ is a global attractor.

(iii) Kulenovic and Ladas (1987): If $\frac{P}{\delta} > e$ and $\delta\tau e^{\delta\tau}(\ln\frac{P}{\delta} - 1) > \frac{1}{e}$, then every non-trivial solution $N(t)$ oscillates about the positive equilibrium N^*.

In addition, Kuang (1992) proves the global attractivity of the positive equilibrium under the condition $1 < \frac{P}{\delta} < e^2$, with no restriction on the delay. He also illustrates the global existence of periodic solution in a generalized setting. We also should mention Li (1996) and Luo and Liu (1996) for recent progress in the studies of Nicholson's blowflies equation.

Recently there has been an increasing interest in studying parabolic equations with time delays. It is the object of this chapter to extend the above results to the case where spatial diffusion is taken into consideration, i.e. we consider the equation

$$\frac{\partial N(t,x)}{\partial t} = \Delta N(t,x) - \delta N(t,x) + PN(t-\tau,x)e^{-aN(t-\tau,x)}, \qquad \text{in } D \quad (1.3)$$

$$\frac{\partial N(t,x)}{\partial n} = 0, \qquad \text{on } \Gamma \quad (1.4)$$

$$N(\theta,x) = \phi(\theta,x) \ge 0, \qquad \text{in } D_\tau. \quad (1.5)$$

It appears that only a few papers have been published concerning the oscillatory behavior of solutions for diffusive functional differential equations. We mention here the work of Erbe, Kong, and Zhang (1993), Fu and Zhuang (1995), Kreith and Ladas (1985), Yoshida (1986, 1992) and the references therein. The earliest work seems to be Bykov and Kultaev (1983). In most of these papers, the nonlinear term $f(u)$ is assumed to be odd and convex. As a result their methods cannot be directly applied to (1.3)–(1.5). In this chapter, we will develop an oscillation theory for (1.3)–(1.5) parallel to the one in Kulenovic and Ladas (1987) and Kulenovic, Ladas and Meimaridou (1987) for delay differential equations.

Our global attractivity results are established by using the method of lower–upper solutions pair for functional partial differential equations. We refer the interested reader to Bebernes and Ely (1983), Redlinger (1984) and the references therein for details. This method has been used by Gourley and Britton (1993) and Redlinger (1985) for a similar purpose.

As to periodic solutions, we obtain the existence of periodic solutions bifurcating from a positive equilibrium. Through a lengthy calculation by hand, we show that these periodic solutions are stable. The center manifold theorem and the attractivity of the center manifold play an essential role.

The rest of the chapter is organized as follows. First, the attractivity of equilibria will be considered in Section 2. In Section 3, we discuss the oscillatory behavior of solutions (about the positive equilibrium N^*). A Hopf bifurcation analysis is carried out in Section 4. Finally in Section 5, some numerical observations are made.

3.2 GLOBAL ATTRACTIVITY OF EQUILIBRIA

In this section, we will first show the global attractivity of the zero solution for equations (1.3)–(1.5) when $0 < \frac{P}{\delta} \leq 1$. Next, we will show that the positive equilibrium is a global attractor when $1 < \frac{P}{\delta} \leq e$. We begin with the following lemma:

LEMMA 2.1. *The solution $N(t,x)$ of (1.3)–(1.5) satisfies $N(t,x) \geq 0$ for $(t,x) \in (0,\infty) \times \bar{\Omega}$.*

PROOF. We first show that $N(t,x) \geq 0$ on $(0,\tau] \times \bar{\Omega}$. Suppose not. Then there exists $(t_0, x_0) \in (0,\tau] \times \bar{\Omega}$ such that $N(t_0,x_0) < 0$. We can find $(t^*, x^*) \in (0,\tau] \times \bar{\Omega}$ such that

$$N(t^*,x^*) = \min_{(t,x)\in[0,\tau]\times\bar{\Omega}} N(t,x) < 0.$$

Since $N(\theta,x) \geq 0$ for $(\theta,x) \in [-\tau,0] \times \bar{\Omega}$, by (1.3) we have

$$\Delta N - \delta N - \frac{\partial N}{\partial t} \leq 0, \qquad \text{on } (0,\tau] \times \Omega.$$

Therefore if $(t^*,x^*) \in (0,\tau] \times \Omega$, then the minimum principle shows that for $0 \leq t \leq t^*$, $N(t,x) \equiv N(t^*,x^*) < 0$, which is impossible since $N(0,x) = \phi(0,x) \geq 0$. Hence we must have $(t^*,x^*) \in (0,\tau] \times \partial\Omega$. However in this case, the strong minimum principle of Hopf implies that $\frac{\partial N}{\partial n}|_{(t^*,x^*)} < 0$. This contradicts the boundary condition $\frac{\partial N}{\partial n} = 0$ on $\partial\Omega$. Therefore, $N(t,x) \geq 0$ for $(0,\tau] \times \bar{\Omega}$.

Applying this argument repeatedly (the method of steps), one can easily show that $N(t,x) \geq 0$ on $(0,\infty) \times \bar{\Omega}$. This completes the proof.

Next we will introduce the concept of a lower–upper solutions pair due to Redlinger as adapted to (1.3)–(1.5).

DEFINITION 2.2. *A lower-upper solutions pair for (1.3)–(1.5) is a pair of suitably smooth functions v and w such that:*

(i) $v \leq w$ *in* $\bar{D}$;

(ii) v *and* w *satisfy*

$$\begin{aligned} &\frac{\partial v}{\partial t} \leq \Delta v - \delta v + P\psi(t-\tau,x)e^{-a\psi(t-\tau,x)}, && \text{in } D \\ &\frac{\partial v}{\partial n} \leq 0, && \text{on } \Gamma \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial w}{\partial t} \geq \Delta w - \delta w + P\psi(t-\tau,x)e^{-a\psi(t-\tau,x)}, && \text{in } D \\ &\frac{\partial w}{\partial n} \geq 0, && \text{on } \Gamma \end{aligned}$$

for all $\psi \in C(D_\tau \cap \bar{D})$ *with* $v \leq \psi \leq w$ *in* $D_\tau \cup \bar{D} = [-\tau,\infty) \times \bar{\Omega}$; *and*

(iii) $v(\theta,x) \leq \phi(\theta,x) \leq w(\theta,x)$ *on* D_τ.

The following lemma is a special case of Redlinger (1984, Theorem 3.4).

LEMMA 2.3. *Let (v, w) be a lower–upper solution pair for the initial boundary value problem (1.3)–(1.5). Then there exists a unique regular solution $N(t, x)$ of (1.3)–(1.5) such that $v \leq N \leq w$ on $D_\tau \cup \bar{D}$.*

The next lemma gives us boundedness of the solution $N(t, x)$.

LEMMA 2.4. *There exists a constant $K = K(\phi) \geq 0$ such that $N(t, x) \leq K$ on $D_\tau \cup \bar{D}$.*

PROOF. Let $w(t)$ be the solution of the ordinary differential equation (ode)

$$\frac{dw}{dt} = -\delta w + \frac{P}{ae}, \qquad t > 0, \tag{2.1}$$

satisfying the initial condition

$$w(0) = \max_{(\theta,x)\in D_\tau} \phi(\theta, x). \tag{2.2}$$

Define $\bar{w}(t)$ by:

$$\bar{w}(t) = \begin{cases} w(0), & \text{for } t \in [-\tau, 0] \\ w(t), & \text{for } t > 0. \end{cases}$$

Then $(0, \bar{w})$ is a lower–upper solution pair of (1.3)–(1.5) since $0 \leq \phi \leq \bar{w}$ and for all $\psi \in C(D_\tau \cap \bar{D})$ with $0 \le \psi \le \bar{w}$ in $D_\tau \cup \bar{D}$ we have

$$\begin{aligned}
&\frac{\partial \bar{w}}{\partial t} - \Delta \bar{w} + \delta \bar{w} - P\psi(t-\tau, x)e^{-a\psi(t-\tau,x)} \\
&\ge \frac{\partial \bar{w}}{\partial t} - \Delta \bar{w} + \delta \bar{w} - \frac{P}{ae} \\
&= \frac{d\bar{w}}{dt} + \delta \bar{w} - \frac{P}{ae} \\
&= 0.
\end{aligned}$$

Hence by Lemma 2.3, $0 \le N(t,x) \le \bar{w}(t)$ for all $(t,x) \in D_\tau \cup \bar{D}$. Now since $\lim_{t\to\infty} \bar{w}(t) = \frac{P}{\delta ae}$, there exists a constant $K > 0$ such that $\bar{w}(t) \le K$ for $t \in [-\tau, \infty)$. This in turn implies $0 \le N(t,x) \le K$ for $(t,x) \in [-\tau,\infty) \times \bar{\Omega}$. This completes the proof.

LEMMA 2.5. *If $\phi(\theta, x) \not\equiv 0$ on D_τ, then $N(t,x) > 0$ for $(t,x) \in (\tau, \infty) \times \bar{\Omega}$.*

PROOF. It was shown in Lemma 2.1 that $N(t,x) \ge 0$ for $(t,x) \in (0,\infty) \times \bar{\Omega}$. There are two cases to consider.

Case 1: $\phi(0,x) \not\equiv 0$. Then (1.3)–(1.5) implies

$$\begin{aligned}
\Delta N - \delta N - \frac{\partial N}{\partial t} &\le 0, && \text{in } D \\
\frac{\partial N}{\partial n} &= 0, && \text{on } \Gamma \quad \text{and} \\
N(0,x) = \phi(0,x) &\ge 0, && \text{for all } x \in \Omega.
\end{aligned}$$

We now show that $N(t,x) > 0$ for all $(t,x) \in (0,\infty) \times \bar{\Omega}$. Suppose not, then there exists $(\bar{t}, \bar{x}) \in (0, \infty) \times \bar{\Omega}$ such that $N(\bar{t}, \bar{x}) = 0$. But this is impossible according to the minimum principle and the strong minimum principle of Hopf.

Case 2: $\phi(0, x) \equiv 0$ for $x \in \bar{\Omega}$. We first show that $N(t, x) \not\equiv 0$ for $(t, x) \in (0, \tau] \times \bar{\Omega}$. Suppose not. From (1.3)-(1.5), we have $\phi(\theta, x) \equiv 0$ for $(\theta, x) \in D_\tau$, which contradicts the assumption $\phi(\theta, x) \not\equiv 0$ on D_τ. Therefore there exists $t_0 \in (0, \tau]$ such that $N(t_0, x) \not\equiv 0$ for $x \in \bar{\Omega}$. Now following the same argument as in Case 1 one shows that $N(t, x) > 0$ for $(t, x) \in (t_0, \infty) \times \bar{\Omega}$. This completes the proof.

We are now ready to state the main result of this section.

THEOREM 2.6.

(i) If $0 < \frac{P}{\delta} \leq 1$, *then the solution* $N(t, x)$ *of (1.3)-(1.5) tends to zero (uniformly in* x*) as* $t \to \infty$.

(ii) If $1 < \frac{P}{\delta} \leq e$, *then any non-trivial solution* $N(t, x)$ *of (1.3)-(1.5) satisfies*

$$\lim_{t \to \infty} N(t, x) = N^*, \qquad \text{uniformly in } x,$$

where $N^* = \frac{1}{a} \ln \frac{P}{\delta}$ *is the positive equilibrium.*

PROOF. Let $\bar{w}(t)$ be the solution of the delay equation

$$\frac{dw}{dt} = -\delta w + P w(t - \tau), \qquad t > 0$$

with initial condition

$$w(\theta) = \max_{x\in\bar{\Omega}} \phi(\theta, x), \qquad \theta \in [-\tau, 0].$$

Then $(0, \bar{w}(t))$ is a lower–upper solution pair since for all $\psi \in C(D_\tau \cap \bar{D})$ such that $0 \leq \psi \leq \bar{w}$ on $D_\tau \cup \bar{D}$, we have

$$\begin{aligned}\frac{d\bar{w}}{dt} &= -\delta\bar{w} + P\bar{w}(t-\tau),\\ &\geq -\delta\bar{w} + P\psi(t-\tau, x),\\ &\geq -\delta\bar{w} + P\psi(t-\tau, x)e^{-a\psi(t-\tau,x)}.\end{aligned}$$

Hence by Lemma 2.3, $0 \leq N(t,x) \leq \bar{w}(t)$. Consider the case when $0 < \frac{P}{\delta} < 1$. Then $\lim_{t\to\infty} \bar{w}(t) = 0$ (c.f. El'sgol'ts and Norkin (1973, p.131)). Consequently $\lim_{t\to\infty} N(t,x) = 0$ uniformly in x. Next we consider the case when $1 < \frac{P}{\delta} < e$. In this case we have a positive equilibrium $N^* = \frac{1}{a}\ln\frac{P}{\delta}$. Let $\underline{N}(t) = \min_{x\in\bar{\Omega}} N(t,x)$, $\overline{N}(t) = \max_{x\in\bar{\Omega}} N(t,x)$, $\underline{N} = \underline{\lim}_{t\to\infty} \underline{N}(t)$, and $\overline{N} = \overline{\lim}_{t\to\infty} \overline{N}(t)$. Then $[\underline{N}, \overline{N}] \subset [0, \frac{P}{\delta ae}] \subset [0, \frac{1}{a}]$, according to the proof of Lemma 2.4. We would like to improve the upper solution so that we have a better upper bound for $N(t,x)$. Let $\gamma_1 = \frac{P}{\delta ae}$. Since the function $f(z) = \frac{z}{e} - \ln z$ is decreasing for $1 \leq z \leq e$ and $f(e) = 0$, we have $\frac{P}{\delta ae} > \frac{1}{a}\ln\frac{P}{\delta}$ for $1 < \frac{P}{\delta} < e$, that is $\gamma_1 > N^*$. For any sufficiently small $\epsilon > 0$, there exists $t_0 \geq 0$ such that

$$\overline{N}(t) \leq \frac{P}{\delta ae} + \epsilon < \frac{1}{a} \qquad \text{for all } t \geq t_0.$$

This implies $N(t,x) \le \frac{P}{\delta a e} + \epsilon < \frac{1}{a}$ on $[t_0, \infty) \times \bar{\Omega}$. Define w_1^ϵ to be the solution of the ode

$$\frac{dw_1^\epsilon}{dt} = -\delta w_1^\epsilon + P(\gamma_1 + \epsilon)e^{-a(\gamma_1+\epsilon)}, \qquad t > t_0 + \tau$$

satisfying

$$w_1^\epsilon(t_0 + \tau) = \gamma_1 + \epsilon$$

and set

$$w_1^\epsilon(t) = \begin{cases} \max\{\max_{t\in[-\tau,t_0]} \bar{w}(t), \gamma_1 + \epsilon\} =: K_0, & \text{for } -\tau \le t < t_0, \\ K_0 + \frac{\gamma_1+\epsilon-K_0}{\tau}(t - t_0), & \text{for } t_0 \le t \le t_0 + \tau. \end{cases}$$

Clearly $(0, w_1^\epsilon)$ is a lower–upper solution pair. Consequently,

$$0 \le N(t,x) \le w_1^\epsilon(t)$$

and

$$0 \le \underline{N} \le \overline{N} \le \lim_{t\to\infty} w_1^\epsilon(t) = \frac{P}{\delta}(\gamma_1 + \epsilon)e^{-a(\gamma_1+\epsilon)}$$

Since ϵ can be made arbitrarily small, we have

$$0 \le \underline{N} \le \overline{N} \le \frac{P}{\delta}\gamma_1 e^{-a\gamma_1}.$$

Let $\gamma_2 = \frac{P}{\delta}\gamma_1 e^{-a\gamma_1}$. Then $\gamma_2 < \gamma_1$, since $\gamma_1 > N^*$ implies $\frac{P}{\delta}e^{-a\gamma_1} < 1$. By considering the function $f(z) = \frac{P}{\delta}ze^{-az} - N^*$, we find that $\gamma_2 > N^*$. Repeating the above procedure, we obtain a sequence $\{\gamma_n\}$ satisfying

$$\gamma_{n+1} = \frac{P}{\delta}\gamma_n e^{-a\gamma_n}$$

and

$$0 \leq \underline{N} \leq \overline{N} \leq \gamma_{n+1} < \gamma_n < \cdots < \gamma_1.$$

In the limit we have

$$0 \leq \underline{N} \leq \overline{N} \leq \lim_{n\to\infty} \gamma_n = N^*.$$

Next we would like to improve on the lower solution. For $\epsilon > 0$ sufficiently small, there exists $t_1 > \tau$ such that $\overline{N}(t) \leq N^* + \epsilon < \frac{1}{a}$ for $t \geq t_1$. Let

$$\delta_0 := \frac{1}{2}\min\left\{\min_{(t,x)\in[t_1,t_1+\tau]\times\bar{\Omega}} N(t,x), N^*\right\}.$$

Let $v_1^\epsilon(t)$ be the solution of the ode

$$\frac{dv_1}{dt} = -\delta v_1 + P\delta_0 e^{-a\delta_0}, \qquad t > t_1 + \tau$$

satisfying

$$v_1(t_1+\tau) = \delta_0,$$

and set

$$v_1^\epsilon(t) = \begin{cases} 0, & \text{for } -\tau \leq t < t_1, \\ \frac{\delta_0}{\tau}(t - t_1), & \text{for } t_1 \leq t \leq t_1 + \tau. \end{cases}$$

Define $w^\epsilon(t)$ by

$$w^\epsilon(t) = \begin{cases} \max\{\max_{t\in[-\tau,t_1]} \bar{w}(t), N^* + \epsilon\} =: K_1, & \text{for } -\tau \le t < t_1, \\ K_1 + \frac{N^*+\epsilon-K_1}{\tau}(t - t_1), & \text{for } t_1 \le t \le t_1 + \tau, \\ N^* + \epsilon, & \text{for } t \ge t_1 + \tau. \end{cases}$$

It is easy to verify that $(v_1^\epsilon, w^\epsilon)$ is a lower–upper solution pair. Thus $\frac{P}{\delta}\delta_0 e^{-a\delta_0} \le \underline{N} \le \overline{N} \le N^* + \epsilon$. Let $\delta_1 = \frac{P}{\delta}\delta_0 e^{-a\delta_0}$. Then $N^* > \delta_1 > \delta_0$. As before we obtain an increasing sequence $\{\delta_n\}$ satisfying $\lim_{n\to\infty} \delta_n = N^*$. Hence $\underline{N} = \overline{N} = N^*$. The above approach can also be applied to the case when $\frac{P}{\delta} = 1$ or when $\frac{P}{\delta} = e$. This completes the proof.

3.3 OSCILLATION ABOUT THE POSITIVE EQUILIBRIUM

In this section we will consider the case $\frac{P}{\delta} > e$. We will show that under some additional restrictions on the time delay τ, all non–trivial solutions of (1.3)–(1.5) oscillate about the positive equilibrium N^*. First we introduce the change of variables $N(t, x) = N^* + \frac{1}{a}u(t, x)$. Then equation (1.3)–(1.5) can be rewritten as

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - \delta u(t,x) - \delta F(u(t-\tau, x)), \qquad \text{in } D \tag{3.1}$$

$$\frac{\partial u}{\partial n} = 0, \qquad \text{on } \Gamma \tag{3.2}$$

$$u(\theta, x) = (\phi(\theta, x) - N^*)a, \qquad \text{in } D_\tau \tag{3.3}$$

where $F(z) = aN^*(1 - e^{-z}) - ze^{-z}$. Note that we are only interested in those solutions $u(t, x)$ of (3.1)–(3.3) such that $u(t, x) \ge -aN^*$.

Definition 3.1. *The solution u of (3.1)-(3.3) is said to oscillate in the domain $\mathbb{R}_+ \times \Omega$ if for each $r > 0$, there exists a point $(t_0, x_0) \in [r, \infty) \times \Omega$ such that $u(t_0, x_0) = 0$.*

Definition 3.2. *System (3.1)-(3.2) is said to be an oscillatory system if every solution of this system oscillates.*

Clearly the solution $N(t,x)$ of (1.3)–(1.5) oscillates about N^* if and only if the solution $u(t,x)$ of (3.1)–(3.3) oscillates about zero. In order to show the oscillation of solutions of (3.1)–(3.3), we need to consider a linear partial differential equation with time delay of the form

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - \delta u(t,x) - \gamma u(t-\tau, x), \qquad \text{in } D \tag{3.4}$$

$$\frac{\partial u}{\partial n} = 0, \qquad \text{on } \Gamma \tag{3.5}$$

$$u(\theta, x) = (\phi(\theta, x) - N^*)a. \qquad \text{in } D_\tau \tag{3.6}$$

For system (3.4)–(3.5), we have the following lemma.

Lemma 3.3. *The following statements are equivalent.*

(i) The first characteristic equation of (3.4)-(3.5)

$$\lambda + \delta + \gamma e^{-\lambda\tau} = 0 \tag{3.7}$$

has no real roots.

(ii) System (3.4)–(3.5) is oscillatory.

PROOF. (i) $\Rightarrow$ (ii). Integrating (3.4) over Ω, we have

$$\frac{d}{dt}\int_\Omega u(t,x)dx = \int_\Omega \Delta u(t,x)dx - \delta\int_\Omega u(t,x)dx - \gamma\int_\Omega u(t-\tau,x)dx.$$

Let $U(t) = \int_\Omega u(t,x)dx$. Since $\int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial n} = 0$, we have

$$\frac{dU(t)}{dt} + \delta U(t) + \gamma U(t-\tau) = 0. \tag{3.8}$$

The characteristic equation of (3.8) is exactly (3.7), which has no real roots. Thus every solution of (3.8) oscillates (cf. Ladas, Sficas, and Stavroulakis (1983)). This implies that every solution $u(t,x)$ of (3.4)–(3.6) oscillates.

(ii) $\Rightarrow$ (i). If λ_0 is a real root of equation (3.7), then $u(t,x) = e^{\lambda_0 t}$ is a solution of (3.4)–(3.5) which does not oscillate. This completes the proof.

REMARK. For equation (3.4) with Dirichlet boundary condition, the first characteristic equation is

$$\lambda + \lambda_1 + \delta + \gamma e^{-\lambda\tau} = 0, \tag{3.9}$$

where λ_1 is the smallest eigenvalue corresponding to the operator $-\Delta$ with Dirichlet boundary condition. The conclusion in Lemma 3.3 still holds if we replace (i) by "equation (3.9) has no real roots". The proof is similar to the Neumann case by multiplying (3.4) by $\eta(x)$, the eigenfunction corresponding to the smallest eigenvalue λ_1, and integrating over Ω and noting that

$$\int_\Omega \eta \Delta u = \int_\Omega u \Delta \eta = -\lambda_1 \int_\Omega u\eta.$$

To study oscillation of system (3.1)–(3.2) we need to consider the relationship between the following two equations

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - Q(t,x)u(t-\tau,x), \tag{3.10}$$

and

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - qu(t-\tau,x), \tag{3.11}$$

under Neumann boundary conditions, where

$$\lim_{t\to\infty} Q(t,x) = q, \qquad \text{uniformly in } x. \tag{3.12}$$

The following lemma is analogous to Theorem 2 in Kulenovic, Ladas, and Meimaridou (1987). Note that the first characteristic equation of (3.11) is

$$\lambda + qe^{-\lambda\tau} = 0. \tag{3.13}$$

LEMMA 3.4. *Assume that (3.13) has no real roots. Then equation (3.10) is oscillatory.*

PROOF. By (3.12) there exists $t_0 > 0$ such that $Q(t,x) - \frac{1}{2}q \geq 0$ for $(t,x) \in [t_0, \infty) \times \bar{\Omega}$. Suppose that (3.10) has an eventually positive solution. Then for sufficiently large t, we have

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) - Q(t,x)u(t-\tau,x) \\ &= \Delta u(t,x) - \frac{1}{2}qu(t-\tau,x) - \left(Q(t,x) - \frac{1}{2}q\right)u(t-\tau,x) \\ &\leq \Delta u(t,x) - \frac{1}{2}qu(t-\tau,x).\end{aligned}$$

Integrating the above inequality over Ω and letting $U(t) = \int_\Omega u(t,x)dx$, we get

$$\dot{U}(t) + \frac{1}{2}qU(t-\tau) \leq 0. \tag{3.14}$$

Let Λ be the set of all $\lambda \geq 0$ for which there exists $t_0 \geq 0$ such that $\dot{U}(t) + \lambda U(t) \leq 0$ for all $t \geq t_0$. By following the idea in Kulenovic, Ladas, and Meimaridou (1987), we get two contradictory properties for Λ: one of which is that Λ is bounded above and the other is that $\lambda \in \Lambda \Rightarrow (\lambda + \frac{m}{2}) \in \Lambda$, where $m = \min_{\lambda \in \mathbb{R}}\{\lambda + qe^{-\lambda\tau}\} > 0$. This completes the proof.

REMARK. Consider the equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - \delta u(t,x) - Q(t,x)u(t-\tau,x), \qquad \text{in } D \tag{3.15}$$

$$\frac{\partial u}{\partial n} = 0, \qquad \text{on } \Gamma \tag{3.16}$$

where $Q(t,x)$ satisfies (3.12). Let $u(t,x) = e^{-\lambda t}v(t,x)$. Then

$$\frac{\partial v(t,x)}{\partial t} = \Delta v(t,x) - e^{\delta\tau}Q(t,x)v(t-\tau,x).$$

Since the change of variables preserves the oscillatory property of solutions, we have :

COROLLARY 3.5. *If the equation*

$$\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) - \delta u(t,x) - qu(t-\tau,x), && \textit{in } D \\
\frac{\partial u}{\partial n} &= 0, && \textit{on } \Gamma
\end{aligned}$$

is oscillatory, so is (3.15)–(3.16).

We are now ready to prove our main result of this section.

THEOREM 3.6. *If $\frac{p}{\delta} > e$ and $\delta\tau e^{\delta\tau}(\ln\frac{p}{\delta} - 1) > \frac{1}{e}$ hold, then system (3.1)–(3.2) is oscillatory or equivalently, every solution $N(t,x)$ of (1.3)–(1.5) oscillates about the positive equilibrium N^*.*

PROOF. Suppose for the purpose of contradiction that (3.1)–(3.2) has an eventually positive solution $u(t,x)$. Then $F(u(t-\tau,x)) \geq 0$ eventually because the function $F(z) = aN^*(1-e^{-z}) - ze^{-z}$ is increasing for $z \geq 0$ and $F(0) = 0$.

Therefore we have eventually

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - \delta u(t,x) - \delta F(u(t-\tau,x))$$

$$\leq \Delta u(t,x) - \delta u(t,x),$$

which implies, by the maximum principle, that

$$\lim_{t\to\infty} u(t,x) = 0 \qquad \text{uniformly in } x. \tag{3.17}$$

Next we assume (3.1) has an eventually negative solution $u(t,x)$. Without loss of generality we assume $-aN^* < u(t,x) < 0$ for all $(t,x) \in (\tau,\infty)\times\bar{\Omega}$. Since $F(-aN^*) = aN^* > 0$ and since $F(1-aN^*) = \ln\frac{P}{\delta} - \frac{1}{e}\frac{P}{\delta} < 0$, which is the minimum, there exists $A < 0$ such that $F(A) = 0$. Note that A is the unique zero of $F(z)$ for $z < 0$ because the function $F(z)$ is decreasing on $z \leq 1 - aN^* < 0$.

Consider the delay equation

$$\frac{dv(t)}{dt} = -\delta v(t) - \delta F^*(v(t-\tau))$$

satisfying the initial condition

$$v(\theta) = \min_{x\in\bar{\Omega}} u(\theta,x)$$

where

$$F^*(z) = \begin{cases} F(z), & \text{for } z \in [-aN^*, A] \\ 0, & \text{for } z \in (A, 0]. \end{cases}$$

We will show that $v(t) \le u(t,x)$ for all $(t,x) \in [-\tau, \infty) \times \bar{\Omega}$. This can be done in steps. First we consider $(t,x) \in [0,\tau] \times \bar{\Omega}$. Let $w(t,x) = u(t,x) - v(t)$. Then

$$\frac{\partial w(t,x)}{\partial t} = \Delta w(t,x) - \delta w(t,x) - \delta(F(u(t-\tau,x)) - F^*(v(t-\tau))).$$

We claim that

$$F(u(t-\tau,x)) - F^*(v(t-\tau)) \le 0.$$

Indeed, since $u(t-\tau,x) \ge v(t-\tau)$ for $(t,x) \in [0,\tau] \times \bar{\Omega}$ we have altogether three cases: (i) $A \ge u(t-\tau,x) \ge v(t-\tau)$, (ii) $u(t-\tau,x) \ge A \ge v(t-\tau)$ and (iii) $u(t-\tau,x) \ge v(t-\tau) \ge A$.

In Case (i) the claim is true because $F^*(v(t-\tau)) = F(v(t-\tau))$ and $F(z)$ is decreasing. In Case (ii) the claim also holds since $F^*(v(t-\tau)) = F(v(t-\tau)) \ge 0$, while $F(u(t-\tau,x)) \le 0$. In Case (iii) the claim is again valid since $F^*(v(t-\tau)) = 0$ and $F(u(t-\tau,x)) \le 0$.

From this we have $\frac{\partial w(t,x)}{\partial t} \ge \Delta w(t,x) - \delta w(t,x)$. Since $w(0,x) \ge 0$, one can show by the minimum principle that $w(t,x) \ge 0$, that is, $u(t,x) \ge v(t)$. We will now show $v(t) \to 0$ as $t \to \infty$. Note that $-aN^* < v(t) < 0$ and $\lim_{t\to\infty} v(t) \ne -aN^*$. There are two cases to consider.

Case 1: $\lim_{t\to\infty} v(t) = \alpha$ exists. We claim that $\alpha = 0$. In fact if $\alpha < A < 0$, then eventually

$$-\delta\alpha - \delta a N^*(1 - e^{-\alpha}) - \alpha e^{-\alpha} = 0,$$

which implies $\alpha = 0$. Therefore $\alpha \geq A$. If $\alpha = A$, then we have $-\delta A = 0$ which implies $A = 0$, which is a contradiction. So $\alpha > A$, in which case $-\delta\alpha = 0$.

Case 2: $0 \geq \overline{\lim}_{t\to\infty} v(t) \equiv \bar{l} > \underline{l} \equiv \underline{\lim}_{t\to\infty} v(t)$. If $\underline{l} > A$ then eventually $v(t) \geq \underline{l} - \epsilon \geq A$ for sufficiently small ϵ and hence eventually $\dfrac{dv(t)}{dt} = -\delta v(t)$. This implies $v(t) \to 0$. Consequently $\underline{l} \leq A < 0$ and eventually

$$\dot{v}(t) = -\delta v(t) - \delta F(v(t-\tau)).$$

Applying Lemma 3.2 in So and Yu (1994), we have $v(t) \to 0$.

Hence for an eventually negative solution $u(t,x)$, we also obtain

$$\lim_{t\to\infty} u(t,x) = 0 \qquad \text{uniformly in } x.$$

Rewriting (3.1)–(3.2) as

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - \delta u(t,x) - Q(t,x)u(t-\tau,x), \qquad \text{in } D \qquad (3.18)$$

$$\frac{\partial u}{\partial n} = 0, \qquad \text{on } \Gamma \qquad (3.19)$$

where

$$Q(t,x) = \frac{a\delta N^*(1-e^{-u(t-\tau,x)})}{u(t-\tau,x)} - \delta e^{-u(t-\tau,x)},$$

and

$$\lim_{t\to\infty} Q(t,x) = a\delta N^* - \delta = \delta(aN^*-1) > 0 \qquad \text{uniformly in } x.$$

The first characteristic equation of the equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - \delta u(t,x) - \delta(aN^* - 1)u(t-\tau,x), \qquad \text{in } D \tag{3.20}$$

with Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0, \qquad \text{on } \Gamma \tag{3.21}$$

is

$$\lambda + \delta + \delta(aN^* - 1)e^{-\lambda\tau} = 0,$$

which has no real solution since $\delta\tau e^{\delta\tau}(\ln\frac{P}{\delta} - 1) > \frac{1}{e}$. Hence system (3.20)–(3.21) oscillates. This implies by Corollary 3.5 that (3.18)–(3.19) is oscillatory and thus it contradicts the assumption that (3.18)–(3.19) has an eventually positive (or negative) solution. This completes the proof.

3.4 HOPF BIFURCATION AND STABLE PERIODIC SOLUTIONS

In this section, we will restrict our attention to the case when $\Omega \subset \mathbb{R}$ and $\frac{P}{\delta} > e^2$. After an appropriate change of variable in space and time, the equation becomes

$$\frac{\partial u}{\partial t} = du_{xx} - \tilde{\tau}u - \tilde{\tau}[\beta(1 - e^{-u(t-1,x)}) - u(t-1,x)e^{-u(t-1,x)}] \tag{4.1}$$

with $\Omega = (0,1)$, where $\beta = \ln \frac{P}{\delta}$ and $\tilde{\tau} = \tau\delta$ and d is treated as a parameter independent of $\tilde{\tau}$ and β. P and δ are parameters from (3.1). Note that the equilibrium $u \equiv 0$ of (4.1) corresponds to the positive equilibrium N^* of (1.3). The linearized equation (of (4.1)) about $u = 0$ is

$$\frac{\partial u}{\partial t} = du_{xx} - \tilde{\tau}u - \tilde{\tau}(\beta - 1)u(t-1, x). \tag{4.2}$$

Moreover, the characteristic equation of (4.2) is

$$\lambda + dk^2 + \tilde{\tau} + \tilde{\tau}(\beta - 1)e^{-\lambda} = 0. \tag{4.3.k}$$

We know that for fixed d, (4.3.0) has a pair of simple characteristic values $\lambda = \pm i(\pi - \arccos\frac{1}{\beta-1})$ at $\tilde{\tau} = \tau_0 := \frac{1}{\sqrt{\beta(\beta-2)}}(\pi - \arccos\frac{1}{\beta-1})$ for $\beta > 2$. Let β_0 be the unique real solution of $\pi - \arccos\frac{1}{\beta-1} = \sqrt{\beta(\beta-2)}$. Then $\tau_0 > 1$, for $2 < \beta < \beta_0$. Note that $\beta_0 \approx 3.26 > 3$. To apply the Hopf bifurcation theorem as developed for abstract functional differential equations in Wu (1996), we need to show that all the other characteristic values of (4.3.k) have negative real parts. First of all, we will consider the case when $k \geq 1$. This can be done by employing the following well-known lemma whose proof can be found in Hale and Verduyn Lunel (1993).

LEMMA 4.1. *All roots of the equation $(z+\alpha)e^z + \xi = 0$, where α and ξ are* *real numbers, have negative real parts if and only if*

$$\alpha > -1,$$

$$\alpha + \xi > 0,$$

$$\rho \sin \rho - \alpha \cos \rho > \xi,$$

where $\rho = \frac{\pi}{2}$ *if* $\alpha = 0$ *and* ρ *is the root of* $\rho = -\alpha \tan \rho$ *in* $(0, \pi)$ *if* $\alpha \neq 0$.

Therefore we just need to verify that for $k \geq 1$, $\rho \sin \rho - (dk^2 + \tau_0) \cos \rho > \tau_0(\beta - 1)$, where ρ is the root of $\rho = -(dk^2 + \tau_0) \tan \rho$ in $(0, \pi)$. Note that since $\alpha = dk^2 + \tau_0 > 0$, $\rho \in (\frac{\pi}{2}, \pi)$. Let us consider the function $f(z) = z + (\tau_0 + dk^2) \tan z$ which is continuous and increasing for $z \in (\frac{\pi}{2}, \pi)$. Since $\rho \sin \rho - (dk^2 + \tau_0) \cos \rho = -\frac{dk^2+\tau_0}{\cos \rho}$, it suffices to show $\cos \rho > -\frac{dk^2+\tau_0}{\tau_0(\beta-1)}$. Clearly this is true for $\frac{dk^2+\tau_0}{\tau_0(\beta-1)} > 1$. For the case when $\frac{dk^2+\tau_0}{\tau_0(\beta-1)} \leq 1$, we have $\frac{\pi}{2} < \pi - \arccos \frac{dk^2+\tau_0}{\tau_0(\beta-1)} < \pi$ and

$$\begin{aligned} f\left(\pi - \arccos \frac{dk^2 + \tau_0}{\tau_0(\beta - 1)}\right) &= \pi - \arccos \frac{dk^2 + \tau_0}{\tau_0(\beta - 1)} - \sqrt{\tau_0^2(\beta - 1)^2 - (\tau_0 + dk^2)^2} \\ &> \pi - \arccos \frac{\tau_0}{\tau_0(\beta - 1)} - \sqrt{\tau_0^2 \beta(\beta - 2)} \\ &= 0 \end{aligned}$$

which implies $\pi - \arccos \frac{dk^2+\tau_0}{\tau_0(\beta-1)} > \rho$ since $f(\rho) = 0$ and $f(z)$ is increasing. Consequently,

$$\cos \rho > \cos \left(\pi - \arccos \frac{dk^2 + \tau_0}{\tau_0(\beta - 1)}\right) = -\frac{dk^2 + \tau_0}{\tau_0(\beta - 1)}.$$

Secondly, we will show that at $\tilde{\tau} = \tau_0$ all the roots of (4.3.0), except $\pm i(\pi - \arccos \frac{1}{\beta-1})$, have negative real parts. Let $\lambda = p + qi$, where p, q are real and $q > 0$. Then (4.3.0) can be rewritten as

$$p + \tau_0 + \tau_0(\beta - 1)e^{-p}\cos q = 0, \tag{4.4}$$

$$q - \tau_0(\beta - 1)e^{-p}\sin q = 0. \tag{4.5}$$

Suppose (p, q) is a solution of (4.4)–(4.5) with $p > 0$. Then $\sin q > 0$ and $\cos q < 0$ which imply that the angle q is in the second quadrant. Let us consider the functions $f(z) = z - \tau_0(\beta - 1)e^{-p}\sin z$ and $g(z) = p + \tau_0 + \tau_0(\beta - 1)e^{-p}\cos z$. Clearly, $f(z)$ is increasing and $g(z)$ decreasing for z in the second quadrant. Note that for $\omega_0 = \pi - \arccos \frac{1}{\beta-1}$,

$$f(\omega_0) = \omega_0 - \tau_0(\beta - 1)e^{-p}\sin\omega_0 > \omega_0 - \tau_0(\beta - 1)\sin\omega_0 = 0 = f(q)$$

and that

$$g(\omega_0) = p + \tau_0 + \tau_0(\beta - 1)e^{-p}\cos\omega_0 > p + \tau_0 + \tau_0(\beta - 1)\cos\omega_0 = p > 0 = g(q)$$

which is impossible. Lastly we need to verify the transversality condition $p'(\tau_0) \neq 0$. In fact, according to the implicit function theorem

$$p'(\tau_0) = \frac{\tau_0(\beta - 2)\beta}{1 + 2\tau_0 + \tau_0^2(\beta - 1)^2} > 0.$$

Therefore the Hopf bifurcation theorem in Wu (1996, p.189) (see also Hassard, Kazarinoff, and Wan (1981)) is applicable and the system has a family of periodic solutions bifurcating from $u = 0$ when $\tilde{\tau}$ is near τ_0.

One can also consider the stability of these (bifurcated) periodic solutions by calculating the normal form on the center manifold. Let $A_0(\tilde{\tau})$ be the infinitesimal generator of the semiflow of the following delay equation

$$\frac{du}{dt} = -\tilde{\tau}u - \tilde{\tau}(\beta - 1)u(t-1). \tag{4.6}$$

Let $A_0^*(\tau_0)$ denote the formal adjoint of $A_0(\tau_0)$ under the bilinear pairing

$$(\psi, \phi)_0 = \psi(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\zeta - \theta)d\eta(\theta)\phi(\zeta)d\zeta,$$

where

$$d\eta(\theta) = [-\tau_0\delta(\theta) - \tau_0(\beta - 1)\delta(\theta + 1)]d\theta.$$

Let $\Phi = (\phi_1, \phi_2) \in C([-1, 0]; \mathbb{R}^2)$ be such that

$$A_0(\tau_0)\Phi = \Phi \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}.$$

Then we have $\Phi = (\cos\omega_0\theta, \sin\omega_0\theta)$ where $\omega_0 = \pi - \arccos\frac{1}{\beta-1}$. Similarly, let $\Psi^* = (\psi_1^*, \psi_2^*)^T \in C^*([0, 1]; \mathbb{R}^2)$ be such that

$$A_0^*(\tau_0)\Psi^{*T} = \Psi^{*T} \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix},$$

which gives $\psi_1^* = \sin\omega_0 s$ and $\psi_2^* = \cos\omega_0 s$. Then

$$\Psi^*\Phi = \begin{pmatrix} (\psi_1^*, \phi_1)_0 & (\psi_1^*, \phi_2)_0 \\ (\psi_2^*, \phi_1)_0 & (\psi_2^*, \phi_2)_0 \end{pmatrix} = \begin{pmatrix} -\frac{\tau_0}{2}\sqrt{\beta(\beta-2)} & \frac{1}{2}(\tau_0+1) \\ \frac{1}{2}(\tau_0+1) & \frac{\tau_0}{2}\sqrt{\beta(\beta-2)} \end{pmatrix}.$$

Now define $\Psi = (\psi_1, \psi_2)^T = (\Psi^*\Phi)^{-1}(\psi_1^*, \psi_2^*)^T$. Then

$$\Psi(0) = (\psi_1(0), \psi_2(0))^T = \frac{2}{1+2\tau_0+\tau_0^2(\beta-1)^2}\left(\tau_0+1, \tau_0\sqrt{\beta(\beta-2)}\right)^T.$$

We also define an inner product on $X = C([0,1];\mathbb{R})$

$$\langle f, g\rangle = \int_0^1 fg, \qquad \text{for } f, g \in X.$$

Now consider the "suspended" system

$$\begin{aligned} \begin{pmatrix} \dot{u}(t) \\ \dot{\tilde{\tau}}(t) \end{pmatrix} &= \begin{pmatrix} d\Delta & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ \tilde{\tau} \end{pmatrix} + \begin{pmatrix} -\tau_0 u - \tau_0(\beta-1)u(t-1,x) \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} (-\tilde{\tau}+\tau_0)u + (-\tilde{\tau}+\tau_0)(\beta-1)u(t-1,x) \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} -\tilde{\tau}(1-\frac{\beta}{2})u^2(t-1,x) - \tilde{\tau}(\frac{\beta}{6}-\frac{1}{2})u^3(t-1,x) + \cdots \\ 0 \end{pmatrix}. \end{aligned}$$

By applying the center manifold theorem, we conclude that the flow on the center manifold is given as follows:

$$(x_1(t), x_2(t))^T = (\Psi, \langle u_t, 1\rangle)_0,$$

$$u_t = \Phi(x_1(t), x_2(t))^T \cdot 1 + h(\tilde{\tau}, x_1(t), x_2(t)) \in C = C([-1,0];X),$$

and

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \Psi(0)F(\tilde{\tau}, x_1(t), x_2(t)), \tag{4.7}$$

where

$$\begin{aligned} & F(\tilde{\tau}, x_1, x_2) \\ &= (-\tilde{\tau} + \tau_0)\langle(\Phi(0) + \Phi(-1))(x_1, x_2)^T \cdot 1, 1\rangle \\ &+ 2(-\tilde{\tau} + \tau_0)\langle h(\tilde{\tau}, x_1, x_2)(-1), 1\rangle \\ &- \tilde{\tau}\langle(1 - \frac{\beta}{2})(\Phi(-1)(x_1, x_2)^T \cdot 1 + h(\tilde{\tau}, x_1, x_2)(-1))^2, 1\rangle \\ &- \tilde{\tau}\langle(\frac{\beta}{6} - \frac{1}{2})(\Phi(-1)(x_1, x_2)^T \cdot 1 + h(\tilde{\tau}, x_1, x_2)(-1))^3 + \cdots, 1\rangle. \end{aligned} \tag{4.8}$$

The existence, smoothness, and attractivity of the center manifolds for functional partial differential equations were discussed in Lin, So and Wu (1992) and So, Wu and Yang (1998). See also chapter 2 for the attractivity of the center manifold. Now let $z = x_1 - ix_2$ and $\lambda(\tilde{\tau})$ be the eigenvalue of matrix $\tilde{M}(\tilde{\tau})$ defined by

$$\tilde{M}(\tilde{\tau}) = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} + (-\tilde{\tau} + \tau_0)\Psi(0)(\Phi(0) + \Phi(-1)).$$

Then system (4.7)–(4.8) can be rewritten as

$$\begin{aligned} \dot{z} = \lambda(\tilde{\tau})z + 2(-\tilde{\tau} + \tau_0)(\psi_1(0) - i\psi_2(0))\langle w(\tilde{\tau}, z, \bar{z})(-1), 1\rangle \\ - \tilde{\tau}(\psi_1(0) - i\psi_2(0))SD(\tilde{\tau}) \\ - \tilde{\tau}(\psi_1(0) - i\psi_2(0))TD(\tilde{\tau}), \end{aligned}$$

where

$$SD(\tilde{\tau}) := \langle (1-\frac{\beta}{2})(\frac{1}{2}\Phi(-1)(z+\bar{z}, i(z-\bar{z}))^T \cdot 1 + w(\tilde{\tau}, z, \bar{z})(-1))^2, 1\rangle,$$

$$TD(\tilde{\tau}) := \langle (\frac{\beta}{6}-\frac{1}{2})(\frac{1}{2}\Phi(-1)(z+\bar{z}, i(z-\bar{z}))^T \cdot 1 + w(\tilde{\tau}, z, \bar{z})(-1))^3 + \cdots, 1\rangle,$$

and

$$w(\tilde{\tau}, z, \bar{z}) = h\left(\tilde{\tau}, \frac{z+\bar{z}}{2}, \frac{i(z-\bar{z})}{2}\right).$$

At $\tilde{\tau} = \tau_0$, the above equation becomes

$$\begin{aligned} \dot{z} &= i\omega_0 z - \tau_0(\psi_1(0) - i\psi_2(0))SD(\tau_0) \\ &\quad - \tau_0(\psi_1(0) - i\psi_2(0))TD(\tau_0), \end{aligned} \tag{4.9}$$

where

$$SD(\tau_0) := \langle (1-\frac{\beta}{2})(\frac{1}{2}\Phi(-1)(z+\bar{z}, i(z-\bar{z}))^T \cdot 1 + w(z, \bar{z})(-1))^2, 1\rangle,$$

$$TD(\tau_0) := \langle (\frac{\beta}{6}-\frac{1}{2})(\frac{1}{2}\Phi(-1)(z+\bar{z}, i(z-\bar{z}))^T \cdot 1 + w(z, \bar{z})(-1))^3 + \cdots, 1\rangle,$$

and

$$w(z, \bar{z}) = w(\tau_0, z, \bar{z}) = w_{20}\frac{z^2}{2} + w_{11}z\bar{z} + w_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{4.10}$$

Note that $w_{ij} \in P_S C$.

Since the center manifold is invariant under the semiflow, $w(z(t), \bar{z}(t))$ satisfies

$$\begin{aligned} \dot{w} = A_U w - \tau_0 X_0^S [&(1-\frac{\beta}{2})(\frac{1}{2}\Phi(-1)(z+\bar{z}, i(z-\bar{z}))^T \cdot 1 + w(z, \bar{z})(-1))^2 \\ &+ (\frac{\beta}{6}-\frac{1}{2})(\frac{1}{2}\Phi(-1)(z+\bar{z}, i(z-\bar{z}))^T \cdot 1 + w(z, \bar{z})(-1))^3 + \cdots], \end{aligned} \tag{4.11}$$

for all $t \in \mathbb{R}$, where A_U denotes the infinitesimal generator of the solution semi-group of (4.2), $X_0^{\mathcal{S}}$ is defined as

$$X_0^{\mathcal{S}} = X_0 g - \Phi(\Psi, \langle X_0 g, 1 \rangle)_0 \cdot 1$$

for any $g \in X$ and $X_0 : [-1, 0] \to \mathcal{L}(\mathcal{C}([0,1]), \mathcal{C}([0,1]))$ is given by

$$X_0(\theta) = \begin{cases} 0, & \text{for } \theta \in [-1, 0) \\ I, & \text{for } \theta = 0. \end{cases}$$

Let

$$\begin{aligned} & - \tau_0 X_0^{\mathcal{S}} [(1 - \frac{\beta}{2})(\frac{1}{2}\Phi(-1)(z + \bar{z}, i(z - \bar{z}))^T \cdot 1 + w(z, \bar{z})(-1))^2 \\ & + (\frac{\beta}{6} - \frac{1}{2})(\frac{1}{2}\Phi(-1)(z + \bar{z}, i(z - \bar{z}))^T \cdot 1 + w(z, \bar{z})(-1))^3 + \cdots] \qquad (4.12) \\ & = H_{20} \frac{z^2}{2} + H_{11} z\bar{z} + H_{02} \frac{\bar{z}^2}{2} + \cdots , \end{aligned}$$

where $H_{ij} \in P_{\mathcal{S}}\mathcal{C}$. Note that on the center manifold, we have

$$\dot{w} = \frac{\partial w}{\partial z}\dot{z} + \frac{\partial w}{\partial \bar{z}}\dot{\bar{z}}.$$

This equation together with (4.11)–(4.12) gives

$$(2i\omega_0 - A_U) w_{20} = H_{20}$$

$$-A_U w_{11} = H_{11}$$

$$\vdots$$

from which w_{ij} can be solved through H_{ij}. Using a near identity transformation of the form

$$z = \xi + a_{20}\frac{\xi^2}{2} + a_{11}\xi\bar{\xi} + a_{02}\frac{\bar{\xi}^2}{2} + \cdots,$$

where a_{20}, a_{11}, and a_{02} are appropriately chosen, we arrive at the normal form

$$\dot{\xi} = i\omega_0\xi + c_1(\tau_0)\xi|\xi|^2 + o(|\xi|^2),$$

where

$$c_1(\tau_0) = \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}.$$

Let $q(\theta) = e^{i\omega_0\theta}$. Then $\Phi(\theta) = (\text{Re } q(\theta), \text{Im } q(\theta))$ and g_{ij} can be expressed as

$$\begin{aligned}
g_{20} &= -\frac{\tau_0}{2}(1 - \frac{\beta}{2})(\psi_1(0) - i\psi_2(0))q^2(-1),\\
g_{11} &= -\frac{\tau_0}{2}(1 - \frac{\beta}{2})(\psi_1(0) - i\psi_2(0)),\\
g_{02} &= -\frac{\tau_0}{2}(1 - \frac{\beta}{2})(\psi_1(0) - i\psi_2(0))\bar{q}^2(-1),\\
\frac{g_{21}}{2} &= -\tau_0(\psi_1(0) - i\psi_2(0))\tilde{g}_{21},
\end{aligned}$$

where

$$\tilde{g}_{21} := (1 - \frac{\beta}{2})\langle\bar{q}(-1)\frac{w_{20}(-1)}{2} + q(-1)w_{11}(-1), 1\rangle + \frac{3}{8}(\frac{\beta}{6} - \frac{1}{2})q(-1).$$

After a lengthy calculation, we finally obtain

$$
\begin{aligned}
&\mathrm{Re}\ c_1(\tau_0)\\
&= -\frac{\tau_0(1-\frac{\beta}{2})^2 D^2(1+\tau_0(\beta-1)^2)}{(\beta-1)^2}\\
&\quad - \frac{2\tau_0 D D_1(1+\tau_0(\beta-1)^2)}{(\beta-1)^2}\\
&\quad - \frac{\tau_0(1-\frac{\beta}{2})^2 D^2 H}{2}(\tau_0^2(\beta-1)^2-1)\left[\tau_0(\beta-1)(\beta^2-3)+(2\beta+1)(3-\beta)\right]\\
&\quad - \frac{\tau_0(1-\frac{\beta}{2})^2 D^2 H}{3(\beta-1)^2}\left[(\beta^2-3)G-2(\beta-2)K\right],
\end{aligned}
$$

where

$$
\begin{aligned}
G &= (2\beta-3)(1+\tau_0(1+2\beta(\beta-2)))(1-\tau_0(\beta-1))\\
&\quad +\beta^2(\tau_0+2)(1+\tau_0(\beta-1)),\\
K &= (2\beta-3)\beta(\beta-2)(\tau_0+2)(1-\tau_0(\beta-1))\\
&\quad -\beta^2(1+\tau_0(\beta-1))(1+\tau_0(1+2\beta(\beta-2))),\\
D &= \frac{1}{1+2\tau_0+\tau_0^2(\beta-1)^2},\\
D_1 &= (1-\frac{\beta}{2})^2\frac{D}{2\beta}(\tau_0(\beta-1)-1)(\tau_0(\beta-1)^2+1+\beta)+\frac{(3-\beta)(\beta-1)}{16},\\
H &= \frac{(\beta-1)^2}{\beta(\beta^2-3)^2+4\beta^2(\beta-2)^3},
\end{aligned}
$$

and where

$$
\begin{aligned}
&(\beta^2-3)G-2(\beta-2)K \\
&=(\beta^2-3)(2\beta^2+2\beta-3)+2(\beta-2)\beta^2 \\
&+\tau_0\left\{(2\beta-1)(\beta^2-3)[(2\beta-3)(\beta-2)+\beta^2]+2\beta^3(\beta-2)(2\beta-3)\right\} \\
&+\tau_0^2(\beta-1)\left\{(\beta^2-3)[(\beta-1)^2+2]+2\beta(\beta-2)[(\beta-2)(5-\beta)+1]\right\} \\
&+2(\beta-2)^2(2\beta-3)\beta(\tau_0+2)(\tau_0(\beta-1)-1) \\
&>0.
\end{aligned}
$$

Here we have used the fact that $\tau_0(\beta-1)-1>0$, since $\tau_0(\beta-1)=\frac{\omega_0}{\sin\omega_0}>1$. Hence for $2<\beta<3$, we have $\mathrm{Re}\, c_1(\tau_0)<0$.

Summarizing the above discussion, we have the result :

THEOREM 4.2. *Assume $\frac{P}{\delta}>e^2$. Then there exists τ_0 at which spatial homogeneous periodic solutions of (4.1) bifurcate from the positive equilibrium. Moreover, there exists $c_0>e^3$ such that for $e^2<\frac{P}{\delta}<c_0$, the bifurcated periodic solutions are stable.*

3.5 SECONDARY BIFURCATION AND EFFECT OF DIFFUSION

It is well-known that for the Hutchinson's equation with diffusion, reducing the diffusion rate can drive the instability of the spatial homogeneous periodic solutions which bifurcated from the positive equilibrium through a Hopf bifurcation. We refer the interested reader to Yoshida (1982) and Morita (1984) for details. Such a destabilization effect also occurs in our model. In fact, for small enough d, a second bifurcation occurs by fixing $\tilde{\tau} > \tau_0$ and using d as the bifurcation parameter. Let us consider equation (4.3.1). Set $\lambda = bi$, where $b > 0$. Then

$$b^2 = \tilde{\tau}^2(\beta-1)^2 - (d+\tilde{\tau})^2$$

so that b can be solved for provided $\tilde{\tau}(\beta-1) > d+\tilde{\tau}$, that is, $\tilde{\tau}(\beta-2) > d$. Moreover, we have

$$d + \tilde{\tau} + \tilde{\tau}(\beta-1)\cos\sqrt{\tilde{\tau}^2(\beta-1)^2 - (d+\tilde{\tau})^2} = 0. \tag{5.1}$$

Solve (5.1) for $d = d(\tilde{\tau})$ with $d(\tau_0) = 0$ and we get

$$d'(\tau_0) = \frac{\tau_0\beta(\beta-2)}{1+\tau_0} > 0. \tag{5.2}$$

Equations (5.1)-(5.2) imply that for fixed $\tilde{\tau} > \tau_0$ there exists $\tilde{d} = d(\tilde{\tau}) > 0$ such that (4.3.1) has a pair of simple roots $\lambda = \pm i\sqrt{\tilde{\tau}^2(\beta-1)^2 - (\tilde{d}+\tilde{\tau})^2}$. In this case, one could also use the center manifold reduction method to show that Hopf bifurcation occurs resulting in the existence of spatial inhomogeneous periodic solutions. However, these periodic solutions would be unstable (at least for $\tilde{\tau} > \tau_0$ and sufficiently close to τ_0) for in this case equation (4.3.0) has a root with positive real parts.

Using the semi-discrete scheme

$$\dot{u}_i = \frac{d}{h^2}(u_{i-1} - 2u_i + u_{i+1}) - \tilde{\tau}u_i - \tilde{\tau}\left[\beta(1 - e^{-u_i(t-1)}) - u_i(t-1)e^{-u_i(t-1)}\right] \quad (5.3)$$

where $i = 1, 2, \cdots, n-1$, coupled with

$$\begin{aligned} \dot{u}_0 &= \frac{2d}{h^2}(u_1 - u_0) - \tilde{\tau}u_0 - \tilde{\tau}\left[\beta(1 - e^{-u_0(t-1)}) - u_0(t-1)e^{-u_0(t-1)}\right] \\ \dot{u}_n &= \frac{2d}{h^2}(u_{n-1} - u_n) - \tilde{\tau}u_n - \tilde{\tau}\left[\beta(1 - e^{-u_n(t-1)}) - u_n(t-1)e^{-u_n(t-1)}\right] \end{aligned} \quad (5.4)$$

together with a standard routine for solving systems of delayed differential equations, we performed some numerical simulations. Here, n is the number of (equal) partitions of the interval $[0, 1]$, $h = \frac{1}{n}$, and $u_i = u(t, ih)$. The results indicate the possibility of complicated dynamics (besides spatially inhomogeneous periodic solutions). By choosing $\beta = 3$ and $\tilde{\tau} = 10.25$ (away from τ_0), Fig 3.1 shows that for $d = 0.00015$ (small), chaotic behavior takes place both in time and space; while for larger d, say $d = 0.015$ and $d = 0.15$ as in Fig 3.2 and Fig 3.3 resp., chaotic behavior seems deduced and finally for $d = 8.5$ (c.f. Fig 3.4), chaotic behavior is replaced by periodic behavior (spatial homogeneity).

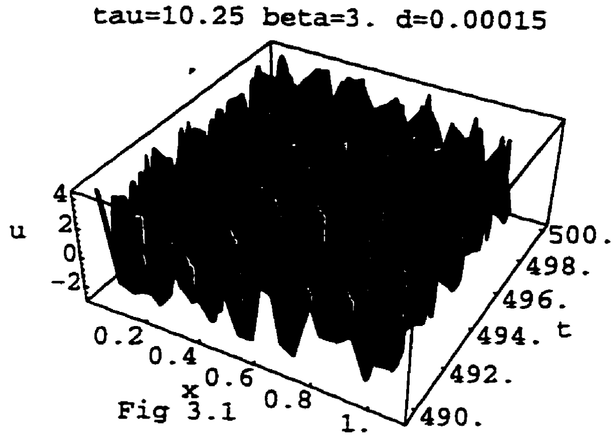

Fig 3.1

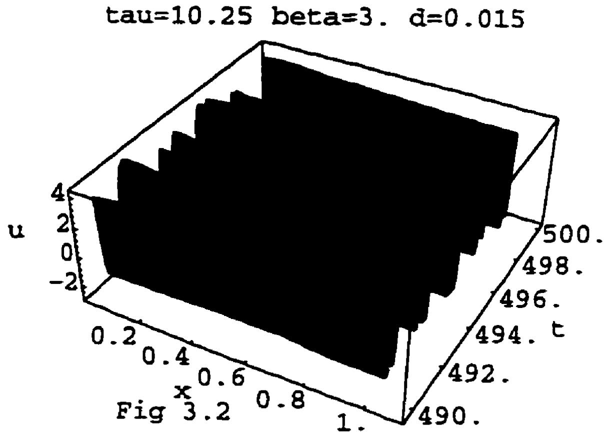

Fig 3.2

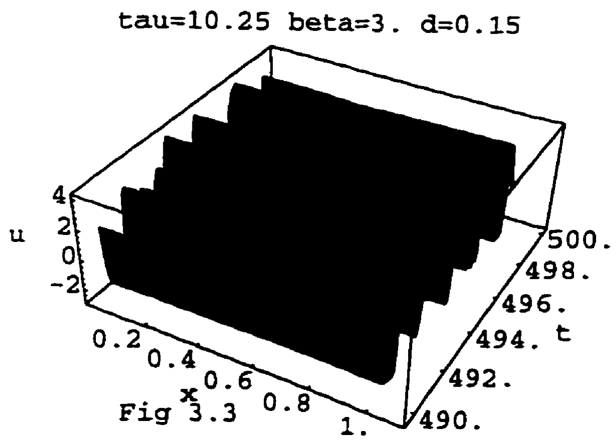

Fig 3.3

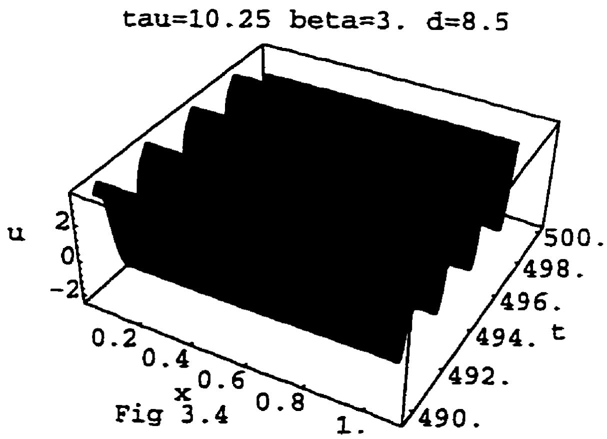

Fig 3.4

REMARK. Basically, the lower–upper solution method rests on the maximum principle of parabolic inequality. To my understanding, the maximum principal gives rise to a comparison principle, which is also true in parabolic equation with time delay, provided that the time–delay term is a monotone function in some sense. Therefore the approach of this chapter can also be applied to our general type diffusive delayed equation. We can also study the oscillation behavior in a similar manner, since these oscillation analyses of nonlinear diffusive delayed equations are dominated by the oscillation behavior of the linearized counterparts.

Combining Theorems 4.2 and 2.5, one may see a gap of the parametric range since there is no assertion on the case of $e < \frac{p}{\delta} < e^2$. To fill out this gap, we may need an alternative approach which will be introduced in the next chapter. By a slight modification, we can still conclude the global attractivity of a positive equilibrium in this case.

It is important to realize that there are still many problems unsolved for Neumann boundary value problems. Besides the complicated dynamics resulting from time delays, interactions between diffusion and time delays are believed to produce more complicated dynamics. But this is far from clear nowadays. Hale (1986) claimed that the solutions of systems of delayed reaction-diffusion equations with Neumann boundary conditions are asymptotic to the solutions of delayed differential equations if the diffusivity is large (see also, Conway, Hoff and Smoller (1978)). By no means does this imply the insignificance of the research on the dynamics of Neumann boundary value problems. How the diffusion impacts on the structure of the chaotic attractors, for example, is a huge interesting project. As observed in our numerical simulation, spatial pattern is no longer simple when the diffusion rate is small. Further research is needed to in this area.

CHAPTER IV

DIRICHLET BOUNDARY VALUE PROBLEMS

4.1 INTRODUCTION

We have studied some dynamics of functional partial differential equations with Neumann boundary condition. In particular, criteria for the global attractivity of the nonnegative equilibria were obtained. In addition, the existence and stability of periodic solution were studied by using a Hopf bifurcation analysis. In this chapter, we will switch our attention to Dirichlet boundary value problems. We still choose the diffusive Nicholson's blowflies equation as the representative of our general type delayed reaction–diffusion equations. More specifically, by rescaling the temporal and spatial variable, we will consider the diffusive Nicholson's blowflies equation as follows:

$$
\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} = d\Delta u(t,x) - \tau u(t,x) + \beta\tau u(t-1,x)e^{-u(t-1,x)}, && \text{in } D\\
&u(t,x) = 0, && \text{on } \Gamma\\
&u(\theta,x) = u_0(\theta,x) \geq 0, && \text{in } D_1
\end{aligned}
$$

where, $x \in \Omega \subset \mathbb{R}^n$, Ω is a bounded domain with a smooth boundary $\partial\Omega$, $(t,x) \in D \equiv (0,\infty) \times \Omega$, $\Gamma \equiv (0,\infty) \times \partial\Omega$, and $D_1 \equiv [-1,0] \times \bar{\Omega}$.

Apparently, there are only a handful of papers treating the long time behavior of solutions for a reaction diffusion equation with delay under Dirichlet boundary conditions. Among those, the nonlinear term containing the delay is often assumed to satisfy a monotonicity or quasi-monotonicity condition. Unfortunately, this is not the case here. Friesecke's (1993) results require severe restrictions on the delay due to the use of a Lyapunov function for a corresponding reaction-diffusion equation (without delay). Inoue, Miyakawa, and Yoshida's (1977) approach, on the other hand, can only give local attractivity. Although convergence results could be found for a large number of semilinear parabolic Volterra integro–differential equations (c.f. Engler (1981), Schiaffina and Tesei (1981), Heard and Rankin (1988), Yamada (1993), and the references therein), those approaches cannot be applied to our equation either. One should also mention Cooke and Huang (1992), who investigated the global dynamics of the generalized diffusive Hutchinson's equation with Dirichlet boundary conditions. But the idea in their paper is essentially similar to that of Yamada (1993). In this chapter, we will develop a new approach to studying the global attractivity of the positive steady state for a reaction diffusion equation with time delays under Dirichlet boundary conditions. Roughly speaking, the idea is to divide the spatial domain according to the information given by the positive steady state and treat the subdomains separately. Our approach should be applicable to other Dirichlet problems, although our analysis is specialized to the diffusive Nicholson's blowflies equation.

The rest of this chapter is organized as follows. In section 2, we give some preliminary results on the solutions of the diffusive Nicholson's blowflies equation, followed by existence and uniqueness of the positive steady state. The global attractivity of the zero solution is presented in section 3. In section 4, the local stability of the positive steady state is studied by analyzing the spectrum of the associated linear operator, a procedure used in Green and Stech (1981) and Huang (1991). Finally, in section 5, we discuss the global attractivity of the positive steady state. Here, a new approach is introduced and a better criterion is obtained along this approach than that via the theory of monotone semiflow. At the end of this paper, we also improve the attractivity results in the sense of $C^1(\Omega)$ by using an interpolation inequality (the so-called Nirenberg–Gagliardo inequality) and an a priori estimate.

4.2 PRELIMINARIES

We consider the diffusive Nicholson's blowflies equation

$$\frac{\partial u(t,x)}{\partial t} = d\Delta u(t,x) - \tau u(t,x) + \beta\tau u(t-1,x)e^{-u(t-1,x)}, \tag{2.1}$$

where $(t,x) \in D \equiv (0,\infty) \times \Omega$, with (homogeneous) Dirichlet boundary condition

$$u = 0, \qquad \text{on } \Gamma \equiv (0,\infty) \times \partial\Omega \tag{2.2}$$

and initial condition

$$u(\theta, x) = u_0(\theta, x) \geq 0, \qquad \text{in } D_1 \equiv [-1, 0] \times \overline{\Omega}, \tag{2.3}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. Here β, τ, d are positive constants. The steady states ϕ of (2.1)–(2.2) satisfy:

$$\begin{aligned} d\Delta\phi(x) - \tau\phi(x) + \beta\tau\phi(x)e^{-\phi(x)} &= 0, \qquad \text{for } x \in \Omega \\ \phi(x) &= 0, \qquad \text{for } x \in \partial\Omega \end{aligned} \tag{2.4}$$

Let $n < p < \infty$ and put $X = L^p(\Omega)$. Let $\mathcal{C} := C([-1, 0]; X)$ and define the operator $A : D(A) \to X$ by

$$(Au)(x) = -d\Delta u(t, x) + \tau u(t, x),$$

$$D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

It is well-known that $-A$ generates an analytic, compact semigroup $T(t)$ $(t \geq 0)$ on X. For any $\alpha > 0$, we define

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At}\, dt$$

and let $A^\alpha = (A^{-\alpha})^{-1}$. Let $X_1 = D(A)$ and $X_\alpha = D(A^\alpha)$, where $\frac{1}{2} + \frac{n}{2p} < \alpha < 1$, and equip these spaces with their corresponding graph norms. Then $X_1 \subset X_\alpha \subset C^1(\bar{\Omega})$. Furthermore,

$$||A^\alpha T(t)|| \leq \frac{K_1}{t^\alpha} e^{-\omega t} \tag{2.5}$$

for some positive constants K_1 and ω. For details, we refer to Pazy (1983, p.243), Henry (1980, p.39) and Friedman (1976, p.160). Moreover, let us define $F : \mathcal{C} \to X$

by

$$[F(u_0)](x) = \beta\tau u_0(-1, x)e^{-u_0(-1,x)}.$$

Then (2.1)–(2.3) can be written in an integral form (the variation of constants formula)

$$u(t) = T(t)u_0(0) + \int_0^t T(t-s)F(u_s)\,ds. \tag{2.6}$$

It is clear that F is Lipschitz continuous and hence the existence and the uniqueness of a solution of (2.6) (called "mild solution" of (2.1)–(2.3)) follow from Travis and Webb (1974, 1978). Furthermore, global continuation of the solution of (2.6) is due to the following proposition.

PROPOSITION 2.1. *Assume that there exist locally integrable functions k_1 and k_2 such that $|F(t, u_0)| \leq k_1(t)|u_0| + k_2(t)$ for $u_0 \in \mathcal{C}$ and $t \geq 0$. Then equation (2.6) admits global solutions.*

PROOF. See Wu (1996, p.49–50).

One should also note that according to Fitzgibbon (1978) and Martin and Smith (1990), every mild solution is a classical solution of (2.1)–(2.3) for $t > 1$ since $T(t)$ $(t \geq 0)$ is analytic. Furthermore, one has :

PROPOSITION 2.2. *Let $u(t)$ be a solution of (2.6) with $u_0(0,\cdot) \in L^p(\Omega)$. Then there exists a constant K independent of t such that*

$$||u(t,\cdot)||_{C^{1+\mu}(\Omega)} \leq K \tag{2.7}$$

for all $t \geq 1$, where $0 < \mu < 1$.

PROOF. Multiplying (2.6) by A^α and using (2.5), one has

$$\begin{aligned}
&||A^\alpha u(t,\cdot)||_{L^p(\Omega)} \\
&\leq ||A^\alpha T(t)u_0(0,\cdot)||_{L^p(\Omega)} \\
&\qquad + \beta\tau \int_0^t ||A^\alpha T(t-s)u(s-1,\cdot)e^{-u(s-1,\cdot)}||_{L^p(\Omega)}\, ds \\
&\leq \frac{K_1 e^{-\omega t}}{t^\alpha}||u_0(0,\cdot)||_{L^p(\Omega)} \\
&\qquad + \beta\tau \int_0^t \frac{K_1 e^{-\omega(t-s)}}{(t-s)^\alpha}||u(s-1,\cdot)e^{-u(s-1,\cdot)}||_{L^p(\Omega)}\, ds \\
&\leq \frac{K_1 e^{-\omega t}}{t^\alpha}||u_0(0,\cdot)||_{L^p(\Omega)} + K_1\beta\tau e^{-1}|\Omega|^{\frac{1}{p}}\omega^{\alpha-1}\Gamma(1-\alpha).
\end{aligned} \tag{2.8}$$

Now recall (c.f. Amann (1978)) that for $p > n$ and $0 < \mu < 1 - \frac{n}{p}$, there exists a constant K_2 independent of u and t such that

$$||u(t,\cdot)||_{C^{1+\mu}(\Omega)} \leq K_2||A^\alpha u(t,\cdot)||_{L^p(\Omega)} \tag{2.9}$$

for all $u \in X_\alpha$, where, $\frac{1}{2} + \frac{\mu}{2} + \frac{n}{2p} < \alpha < 1$. Therefore, one obtains from (2.8) a constant K_3 independent of t such that

$$||A^\alpha u(t,\cdot)||_{L^p(\Omega)} \leq K_3 \quad \text{for all} \quad t \geq 1. \tag{2.10}$$

Thus, (2.7) follows by substituting (2.10) into (2.9) with $K = K_2K_3$. This completes the proof.

The following result is due to Hess (1977).

LEMMA 2.3. *Consider the Dirichlet problem*

$$\mathcal{L}u + h(x,u)u = 0 \qquad \text{in } \Omega\ , \tag{2.11}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{2.12}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ *is a bounded domain with smooth boundary* $\partial\Omega$. $\mathcal{L} = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$, *with smooth real-valued coefficient functions* $a_{ij} = a_{ji}$, *is a uniformly elliptic, formally self-adjoint linear differential operator. Assume* $h : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ *is continuous. Suppose there exists a constant* $M > 0$ *such that*

(i) $h(x,u) \geq 0$ *for all* $u \geq M$, $x \in \overline{\Omega}$;

(ii) $h(x,u)$ *is strictly increasing in* u *for* $u \in [0, M]$, *for all* $x \in \overline{\Omega}$.

Then (2.11)–(2.12) admits at most one non-trivial, non-negative solution u. *If, in addition,*

(iii) $h(x,0) < -\lambda_1$, *for all* $x \in \overline{\Omega}$, *where* λ_1 *is the principal eigenvalue of* $\mathcal{L}$ *with (homogeneous) Dirichlet boundary condition,*

then (2.11)–(2.12) has a unique positive solution.

PROOF. See Hess's (1977) theorem and the remark following it.

Using Lemma 2.3, we have following existence and uniqueness result on the positive solution of (2.4).

COROLLARY 2.4. *The boundary value problem (2.4) possesses a unique positive solution if and only if*

$$(\beta - 1)\tau > d\lambda_1, \tag{2.13}$$

where λ_1 is the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

PROOF. Let $\mathcal{L} = -\Delta$, $h(x,u) = \frac{\tau}{d}(1 - \beta e^{-u})$ and $M = \ln \beta$. Then (i)–(ii) in Lemma 2.3 are satisfied. Suppose (2.13) holds. The existence and uniqueness of a positive solution for (2.4) follows immediately from Lemma 2.3. Conversely, suppose $\phi(x)$ is a positive solution of (2.4). Then ϕ is unique according to Lemma 2.3. Multiplying (2.4) by $\phi(x)$ and integrating over Ω, we obtain (using the variational characterization of λ_1, i.e. Poincaré's inequality)

$$\lambda_1 ||\phi||^2_{L^2(\Omega)} \leq \frac{\tau}{d}\left(-||\phi||^2_{L^2(\Omega)} + \beta ||\phi e^{-\phi/2}||^2_{L^2(\Omega)}\right) < \frac{\tau}{d}(\beta - 1)||\phi||^2_{L^2(\Omega)}.$$

This implies (2.13).

REMARK 2.5. One easily shows, by means of the maximum principle, that $||\phi||_\infty \leq \ln \beta$ for any positive solution ϕ of (2.4). Indeed, suppose there exists $x^* \in \Omega$ such that $\ln \beta < \phi(x^*) = \max\{\phi(x) : x \in \Omega\}$. Then $\Delta\phi(x^*) \leq 0$ but $1 - \beta e^{-\phi(x^*)} > 0$, which is a contradiction.

4.3 GLOBAL ATTRACTIVITY OF THE ZERO SOLUTION

In this section, we will consider the global attractivity of the trivial solution by using the method of steps. First we have

LEMMA 3.1. *Suppose $y(t) \geq 0$ satisfies the differential inequality*

$$\dot{y}(t) \leq -\alpha y(t) + \gamma y(t-1).$$

If $\alpha > \gamma \geq 0$, then $\lim_{t\to\infty} y(t) = 0$.

PROOF. Let $M = \max\{y(t) : t \in [-1, 0]\}$ and $f_0(t) = 1$ for $t \in [-1, 0]$. For $t \in [0, 1]$, we have

$$\dot{y}(t) \leq -\alpha y(t) + \gamma f_0(t-1)M$$

which gives $y(t) \leq f_1(t)M$ for $t \in [0, 1]$, where $f_1(t) = e^{-\alpha t} + \frac{\gamma}{\alpha}(1 - e^{-\alpha t})$ for $t \in [0, 1]$. Similarly, for $t \in [1, 2]$ we have

$$\dot{y}(t) \leq -\alpha y(t) + \gamma f_1(t-1)M$$

and

$$y(t) \leq f_2(t)M,$$

where

$$f_2(t) = e^{\alpha(1-t)} f_1(1) + \gamma \int_1^t f_1(s-1) e^{\alpha(s-t)}\, ds.$$

Inductively, we get

$$y(t) \le f_{k+1}(t)M \qquad \text{for } t \in [k, k+1], \tag{3.1}$$

where

$$f_{k+1}(t) = e^{\alpha(k-t)} f_k(k) + \gamma \int_k^t f_k(s-1) e^{\alpha(s-t)}\, ds, \qquad t \in [k, k+1]. \tag{3.2}$$

Note that $f_{k+1}(t) > 0$ on $[k, k+1]$ and $f_{k+1}(k) = f_k(k)$. In order to show $y(t) \to 0$ as $t \to \infty$, it suffices to show that $\max_{t \in [k,k+1]} f_{k+1}(t) \to 0$ as $k \to \infty$.

Now consider the recurrence relation (3.2), where $f_0(t) = 1$ for $t \in [-1, 0]$. We will show by induction that $f_k(t)$ is decreasing on $[k-1, k]$ and $-\alpha f_k(k) + \gamma f_k(k-1) < 0$, for $k = 0, 1, \ldots$. Clearly $f_0(t) = 1$ is decreasing and $-\alpha f_0(0) + \gamma f_0(-1) = -\alpha + \gamma < 0$. Assume $f_k(t)$ is decreasing for $t \in [k-1, k]$ and $-\alpha f_k(k) + \gamma f_k(k-1) < 0$. Then for $t \in [k, k+1]$,

$$\begin{aligned} f'_{k+1}(t) &\le -\alpha e^{\alpha(k-t)} f_k(k) + \gamma f_{k-1}(t-1) - \gamma \alpha f_k(t-1) \int_k^t e^{\alpha(s-t)}\, ds \\ &\le (-\alpha f_k(k) + \gamma f_k(t-1)) e^{\alpha(k-t)} \le (-\alpha f_k(k) + \gamma f_k(k-1)) e^{\alpha(k-t)} \le 0. \end{aligned}$$

Moreover,

$$\begin{aligned} & -\alpha f_{k+1}(k+1) + \gamma f_{k+1}(k) \\ &= -\alpha \left[e^{-\alpha} f_k(k) + \gamma \int_k^{k+1} f_k(s-1) e^{\alpha(s-k-1)} ds \right] + \gamma f_{k+1}(k) \\ &\le -\alpha e^{-\alpha} f_k(k) - \gamma f_k(k)(1 - e^{-\alpha}) + \gamma f_{k+1}(k) \\ &= (-\alpha + \gamma) e^{-\alpha} f_k(k) < 0. \end{aligned}$$

Therefore $f_{k+1}(t)$ is decreasing for $t \in [k, k+1]$ and hence by (3.2)

$$y(t) \le f_k(k)M \qquad \text{for } t \in [k, k+1].$$

Next, we will show that $f_k(k) \to 0$ as $k \to \infty$. Let $a_k = f_k(k)$. Then

$$a_k = f_k(k) \le f_k(k-1) = f_{k-1}(k-1) = a_{k-1},$$

so that $\{a_k\}$ is a monotone decreasing sequence. Since $a_k > 0$ for $k \ge 1$, the limit $\lim_{k\to\infty} a_k = A$ exists. According to (3.2),

$$a_{k+1} \le e^{-\alpha} a_k + \frac{\gamma}{\alpha} a_{k-1}(1 - e^{-\alpha})$$

which implies

$$A \le e^{-\alpha} A + \frac{\gamma}{\alpha} A(1 - e^{-\alpha}).$$

Since $A \ge 0$ and

$$1 > e^{-\alpha} + \frac{\gamma}{\alpha}(1 - e^{-\alpha}),$$

therefore $A = 0$. The proof is complete.

We have the following theorem.

THEOREM 3.2. *Suppose*

$$(\beta - 1)\tau < d\lambda_1. \tag{3.3}$$

Then the solutions of (2.1)–(2.2) satisfy $||u(t,\cdot)||_{L^2(\Omega)} \to 0$ *as* $t \to \infty$.

PROOF. First we multiply (2.1) by $u(t,x)$ and integrate it over Ω. Using integration by parts, the Poincaré inequality and the Hölder inequality, one obtains

$$\frac{d}{dt}||u(t,\cdot)||_{L^2(\Omega)} \le -(d\lambda_1 + \tau)||u(t,\cdot)||_{L^2(\Omega)} + \beta\tau||u(t-1,\cdot)||_{L^2(\Omega)}.$$

The conclusion then follows immediately from Lemma 3.1. The proof is complete.

4.4 LOCAL STABILITY OF THE POSITIVE STEADY STATE

From now on, we assume that (2.13) holds. Hence there exists a unique positive steady state $\phi(x)$ according to Corollary 2.4. Linearizing (2.1) about this steady state, we get

$$\begin{aligned} &\frac{\partial v(t,x)}{\partial t} = d\Delta v(t,x) - \tau v(t,x) + \beta\tau e^{-\phi(x)}[1-\phi(x)]v(t-1,x) && \text{in } D \\ &v(t,x) = 0 && \text{on } \Gamma. \end{aligned} \tag{4.1}$$

The corresponding eigenvalue problem is

$$\begin{aligned} -d\Delta\psi + \left(\tau + \lambda - \beta\tau e^{-\phi(x)}[1-\phi(x)]e^{-\lambda}\right)\psi = 0 \qquad &\text{in } \Omega \\ \psi = 0 \qquad &\text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

The following lemma is an analogue of the Sturm comparison theorem in one dimension.

LEMMA 4.1. *Let*

$$-d\Delta\psi + P(x)\psi = 0 \qquad \textit{in} \quad \Omega$$

$$\psi = 0 \qquad \textit{on} \quad \partial\Omega,$$

and

$$-d\Delta\phi + Q(x)\phi = 0 \qquad \textit{in } \Omega$$

$$\phi = 0 \qquad \textit{on } \partial\Omega.$$

Suppose $\phi > 0$ *in* Ω *and* $P(x) > Q(x)$ *in* Ω. *Then* $\psi \equiv 0$.

PROOF. Suppose $\Omega_+ = \psi^{-1}(0,\infty)$ is non-empty and let Ω_1 be a connected component of Ω_+. Multiplying the first differential equation by ϕ and the second by ψ, subtracting and integrating over Ω_1, we get

$$-d\int_{\partial\Omega_1}\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) + \int_{\Omega_1}(P-Q)\phi\psi = 0.$$

This contradicts the fact that $\psi = 0$ and $\frac{\partial\psi}{\partial n} \leq 0$ on $\partial\Omega_1$, and hence the proof is complete.

Let us compare

$$-d\Delta\psi + \left(\tau - \beta\tau e^{-\phi(x)}[1-\phi(x)]\right)\psi = 0 \qquad \text{in } \Omega$$

$$\psi = 0 \qquad \text{on } \partial\Omega,$$

with

$$-d\Delta\phi + \left(\tau - \beta\tau e^{-\phi(x)}\right)\phi = 0 \qquad \text{in } \Omega$$

$$\phi = 0 \qquad \text{on } \partial\Omega.$$

We introduce the notation

$$\tilde{\Delta}(\lambda,\psi) := d\Delta\psi + \left(-\tau - \lambda + \beta\tau e^{-\phi(x)}[1-\phi(x)]e^{-\lambda}\right)\psi.$$

Since $\tau - \beta\tau e^{-\phi(x)}\,[1-\phi(x)] > \tau - \beta\tau e^{-\phi(x)}$, by Lemma 4.1, we have, $0 \notin \sigma(\tilde{\Delta})$, where

$$\sigma(\tilde{\Delta}) := \left\{\lambda \in \mathbb{C} : \text{ there exists } \psi \not\equiv 0 \text{ with } \psi = 0 \text{ on } \partial\Omega \text{ such that } \tilde{\Delta}(\lambda,\psi) = 0\right\}.$$

Let $\mathcal{L} := d\Delta - \tau + \beta\tau e^{-\phi(x)}$. Since $\mathcal{L}$ is (formally) self-adjoint, the eigenvalues of $\mathcal{L}$ are real. Since

$$-\tau + \beta\tau e^{-\phi(x)} > -\lambda - \tau + \beta\tau e^{-\phi(x)},$$

it follows from Lemma 4.1 that all the eigenvalues of $\mathcal{L}$ are non-positive. Therefore $(\mathcal{L}\psi,\psi) \le 0$ for all $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$. Let ψ be a solution of (4.2). Multiplying (4.2) by $\overline{\psi}$ and integrating the result over Ω, we get

$$-(\mathcal{L}\psi,\psi) + \int_\Omega \left(\lambda - \beta\tau e^{-\phi(x)}[1-\phi(x)]e^{-\lambda} + \beta\tau e^{-\phi(x)}\right)|\psi|^2 = 0. \qquad (4.3)$$

THEOREM 4.2. *Suppose $1 < \beta \le e^2$. Then all the eigenvalues of (4.2) have negative real parts.*

PROOF. Let $\lambda = a + bi$ and let ψ be a non-trivial solution of (4.2). Then

(4.3) can be rewritten as

$$-(\mathcal{L}\psi,\psi)+\int_{\Omega}\left(a-\beta\tau e^{-\phi(x)}[1-\phi(x)]e^{-a}\cos b+\beta\tau e^{-\phi(x)}\right)|\psi|^2=0, \qquad (4.4)$$

and

$$\int_{\Omega}\left(b+\beta\tau e^{-\phi(x)}[1-\phi(x)]e^{-a}\sin b\right)|\psi|^2=0. \qquad (4.5)$$

Note that $|1-\phi(x)|\le 1$ for all $x\in\bar{\Omega}$, since $0\le\phi(x)\le\ln\beta\le 2$, according to Remark 2.5. We now show that $a\le 0$. Suppose $a>0$. Then

$$\begin{aligned}
&a-\beta\tau e^{-\phi(x)}[1-\phi(x)]e^{-a}\cos b+\beta\tau e^{-\phi(x)}\\
&\ge a-\beta\tau e^{-\phi(x)}|[1-\phi(x)]e^{-a}\cos b|+\beta\tau e^{-\phi(x)}\\
&\ge a-\beta\tau e^{-\phi(x)}+\beta\tau e^{-\phi(x)}=a>0.
\end{aligned}$$

This contradicts (4.4) since $-(\mathcal{L}\psi,\psi)\ge 0$. Next we will show that $a\ne 0$. Suppose $a=0$. Then $b\ne 0$, since $0\notin\sigma(\tilde{\Delta})$. Equation (4.5) implies that b cannot be an integer multiple of π and hence $|\cos b|<1$. Moreover, by (4.4)

$$\begin{aligned}
0&=-(\mathcal{L}\psi,\psi)+\int_{\Omega}\left(-\beta\tau e^{-\phi(x)}[1-\phi(x)]\cos b+\beta\tau e^{-\phi(x)}\right)|\psi|^2\\
&\ge\beta\tau\,(1-|\cos b|)\int_{\Omega}e^{-\phi(x)}|\psi|^2>0,
\end{aligned}$$

which is a contradiction. This completes the proof.

It follows from Theorem 4.2 that the positive steady state is locally stable without any restriction on the time delay in the case where $1<\beta\le e^2$. Thus, the time delay is *harmless* in this case. When $\beta>e^2$, however, the local stability of the positive steady state can be guaranteed only for small delay. We have the following theorem.

THEOREM 4.3. *Suppose $\beta > e^2$. Then all the eigenvalues of (4.2) have negative real parts provided $\tau \in [0, \tau_s]$, where*

$$\tau_s := \frac{\pi - \arccos\frac{1}{\ln\beta - 1}}{\beta\sqrt{e^{-1}(\ln\beta - 2)}}. \tag{4.6}$$

PROOF. Let $\lambda = a + bi$ $(b \geq 0)$ be an eigenvalue of (4.2) with a corresponding eigenfunction ψ such that $||\psi||_{L^2(\Omega)} = 1$. Suppose $a \geq 0$. There are two possibilities to consider.

(i) $\cos b \geq 0$. By (4.4),

$$0 = -(\mathcal{L}\psi, \psi) + \int_\Omega \left[(a + \beta\tau e^{-\phi}\phi e^{-a}\cos b) + \beta\tau e^{-\phi}(1 - e^{-a}\cos b)\right]|\psi|^2 > 0,$$

which is a contradiction.

(ii) $\cos b < 0$ and $\sin b \leq 0$. By (4.5), $\int_\Omega e^{-\phi}(1-\phi)|\psi|^2 > 0$ so that by (4.4)

$$0 = -(\mathcal{L}\psi, \psi) + \int_\Omega \left(a + \beta\tau e^{-\phi}\right)|\psi|^2 - \beta\tau\cos b e^{-a}\int_\Omega e^{-\phi}(1-\phi)|\psi|^2 > 0,$$

which is also a contradiction.

Hence $\cos b < 0$ and $\sin b > 0$. Let $I_1(\psi) = -\int_\Omega e^{-\phi(x)}\left(1 - \phi(x)\right)|\psi|^2$. By (4.5) and the fact that b is a second quadrant angle, we have $I_1(\psi) > 0$. Let

$I_2(\psi) = \int_\Omega e^{-\phi(x)}|\psi|^2 > 0$. Then by (4.4), we have

$$0 < \frac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)} \leq 1.$$

Also, by (4.4)

$$\cos b \leq -\frac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)},$$

which leads to

$$b \geq \pi - \arccos\frac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)}.$$

Since $x \mapsto \frac{x}{\sin x}$ is increasing for $\frac{\pi}{2} < x < \pi$, therefore we have

$$\frac{b}{\sin b} \geq \frac{\pi - \arccos\frac{a+\beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)}}{\sqrt{1 - \left(\frac{a+\beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)}\right)^2}}. \tag{4.7}$$

It follows immediately from (4.7) and (4.5) that

$$\sqrt{\left(\beta\tau e^{-a} I_1(\psi)\right)^2 - \left(a + \beta\tau I_2(\psi)\right)^2} + \arccos\frac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)} \geq \pi.$$

However, the following Lemma 4.4 shows that this is impossible. Hence $a < 0$ and the proof is complete.

LEMMA 4.4. *Suppose $\beta > e^2$ and $\tau \in [0, \tau_s]$. Let $a \geq 0$ and $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$, with $||\psi||_{L^2(\Omega)} = 1$ and $a + \beta\tau I_2(\psi) \leq \beta\tau e^{-a} I_1(\psi)$. Then $F(\psi, a) < \pi$,* *where,*

$$F(\psi, a) := \sqrt{(\beta\tau e^{-a} I_1(\psi))^2 - (a + \beta\tau I_2(\psi))^2} + \arccos\frac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)}.$$

PROOF. Note that $I_2(\psi) \leq I_1(\psi)$. By differentiating F with respect to a, it is easy to show that $F(\psi, a)$ is decreasing in a. Therefore,

$$F(\psi, a) \leq F(\psi, 0) = \beta\tau\sqrt{I_1^2(\psi) - I_2^2(\psi)} + \arccos\frac{I_2(\psi)}{I_1(\psi)}.$$

Now, $\phi(x) \leq \ln\beta$ and $I_1 > 0$, and therefore

$$\frac{I_2(\psi)}{I_1(\psi)} = \frac{\int_\Omega e^{-\phi}|\psi|^2}{\int_\Omega e^{-\phi}(\phi - 1)|\psi|^2} \geq \frac{\int_\Omega e^{-\phi}|\psi|^2}{(\ln\beta - 1)\int_\Omega e^{-\phi}|\psi|^2} = \frac{1}{\ln\beta - 1}.$$

Also

$$\begin{aligned} I_1^2(\psi) - I_2^2(\psi) &= (I_1(\psi) + I_2(\psi))(I_1(\psi) - I_2(\psi)) \\ &\leq \left(\int_\Omega e^{-\phi}\phi|\psi|^2\right)\left(\int_\Omega e^{-\phi}(\phi - 2)|\psi|^2\right). \end{aligned}$$

Thus

$$F(\psi, a) \leq \beta\tau\sqrt{e^{-1}(\ln\beta - 2)} + \arccos\frac{1}{\ln\beta - 1} < \pi,$$

since $0 \leq \int_\Omega e^{-\phi}\phi|\psi|^2 \leq \int_\Omega e^{-1}|\psi|^2 = e^{-1}$ and $\tau \leq \tau_s$. The proof is complete.

4.5 GLOBAL ATTRACTIVITY OF THE POSITIVE STEADY STATE

In this section, we will consider the global dynamics of the diffusive blowflies equation. For the case $1 < \beta < e$, the well-known monotone method is applicable. We will develop a new approach to handle the case where $e \leq \beta < e^2$. The first lemma below provides an appropriate bound for the solutions to (2.1)–(2.2) as $t \to \infty$.

LEMMA 5.1. *Let $u(t,x)$ be the solution of (2.1)-(2.3). Then $u(t,x) \geq 0$ for all $x \in \bar{\Omega}$ and $t > 0$. Moreover, $u(t,x) > 0$ for all $x \in \Omega$ and $t > 1$ if $u_0 \neq 0$. Furthermore,* $\limsup_{t\to\infty} u(t,x) \leq \beta e^{-1}$.

PROOF. It is easy to show that $u(t,x) \geq 0$ for all $x \in \bar{\Omega}$ and $t > 0$. Since $u_0 \neq 0$, we have

$$\{t \geq 0, \quad u(t,x) = 0, \forall x \in \Omega\} \not\supseteq [0,1].$$

Therefore there exists $t_0 \in [0,1)$ such that for any given $t > t_0$, we can find $x \in \Omega$ satisfying $u(t,x) \neq 0$. Moreover, according to the minimum principle and the strong minimum principle (c.f. Protter and Weinberger (1984)), we have $u(t,x) > 0$ for $(t,x) \in (t_0,\infty) \times \Omega$, and $\frac{\partial u}{\partial n}|_{\partial\Omega} < 0$ for $t > t_0$. Let $w(t,x) = u(t,x) - \beta e^{-1}$. Then

$$\frac{\partial w}{\partial t} \leq d\Delta w - \tau w.$$

Therefore w is a lower solution of the parabolic equation

$$\frac{\partial v}{\partial t} = d\Delta v - \tau v$$

together with Dirichlet boundary conditions and an initial condition which dominate those of w. By the comparison theorem, we have

$$w(t,x) \leq v(t,x).$$

It follows from Friedman (1964, p.158 Theorem 1) that $\lim_{t\to\infty} v(t,x) = 0$ uniformly in Ω. Consequently, $\limsup_{t\to\infty} w(t,x) \leq \lim_{t\to\infty} v(t,x) = 0$. This completes the proof.

Subsequently, one has the following convergence theorem whose proof will be carried out using the monotone method which is originally used by Sattinger (1972) for reaction diffusion equations (without time delay). Of course, a modification is needed to apply this method to the time delayed reaction diffusion equations.

THEOREM 5.2. *Suppose $1 < \beta < e$. Then the solutions of (2.1)-(2.2) converge to the positive solution of (2.4).*

PROOF. Lemma 5.1 implies that for sufficiently large t, $0 < u(t,x) \leq 1$ for all $x \in \Omega$. On the other hand, since the function $u \mapsto ue^{-u}$ is increasing for $0 \leq u \leq 1$, the monotone method can be applied. Consider the eigenvalue problem in $H_0^1(\Omega) \cap H^2(\Omega)$:

$$d\Delta\phi - \tau\phi + \beta\tau\phi = \lambda\phi$$

which has a positive solution (λ^*, ϕ^*) since $(\beta - 1)\tau > d\lambda_1$. Then for ϵ small enough such that $e^{-\epsilon\phi^*} > 1 - \frac{\lambda^*}{\beta\tau}$, $\epsilon\phi^*$ is a lower solution of (2.4). Let $\underline{u}(t, x)$ be the solution of (2.1)–(2.2) with initial condition $\epsilon\phi^*$. Claim: $\frac{\partial \underline{u}}{\partial t} \geq 0$. Consider the set $S = \{t \geq 0 \ : \ \frac{\partial \underline{u}}{\partial t} \geq 0, \forall x \in \Omega\}$. Clearly, $0 \in S$ since

$$\begin{aligned}\lim_{t\to 0+} \frac{\partial \underline{u}}{\partial t} &= d\Delta(\epsilon\phi^*) - \tau(\epsilon\phi^*) + \beta\tau(\epsilon\phi^*)e^{-(\epsilon\phi^*)} \\ &> 0.\end{aligned}$$

We will show $(0,1) \subset S$. For $t \in (0,1)$, let $w_h(t,x) = \underline{u}(t+h, x) - \underline{u}(t, x)$, where h is sufficiently small such that $t + h \in (0,1]$ and $\underline{u}(h,x) - \underline{u}(0,x) \geq 0$. Then we have

$$\begin{aligned}\frac{\partial w_h}{\partial t} &= d\Delta w_h - \tau w_h + \beta\tau u(t+h-1)e^{-u(t+h-1)} - \beta\tau u(t-1)e^{-u(t-1)} \\ &= d\Delta w_h - \tau w_h,\end{aligned}$$

and

$$w_h(0,x) = \underline{u}(h,x) - \underline{u}(0,x) \geq 0.$$

The maximum principle implies that $w_h(t,x) \geq 0$ and hence $\frac{\partial \underline{u}}{\partial t} \geq 0$. Therefore $[0,1) \subset S$ holds. Moreover since S is a closed set, $[0,1] \subset S$ holds as well. Noting that the nonlinear term (delay term) is a monotone increasing function for

$0 < u \le 1$, we obtain by induction that $[0, n] \subset S$ for any integer $n \ge 0$. Hence $[0, \infty) = S$, that is $\frac{\partial \underline{u}}{\partial t} \ge 0$ for all $t \ge 0$. Therefore $\underline{u}(t, x) \to \phi(x)$ as $t \to \infty$. Similarly, we can show that $\overline{\phi} = 1$ is an upper solution of (2.4). Let $\overline{u}(t, x)$ be the solution of (2.1)–(2.2) with initial data $\overline{\phi}$. Then we use the same approach as above, with a slight modification if necessary, to obtain $\frac{\partial \overline{u}}{\partial t} \le 0$. Hence $\overline{u}(t, x) \to \phi(x)$ as $t \to \infty$. This completes the proof.

Next, we will consider the case $e \le \beta < e^2$. To prove the convergence theorem in this case, we propose a new approach. We expect that this new method is applicable to other non–monotone Dirichlet boundary problems as well. Our idea is as follows.

First we decompose the space Ω into two parts, i.e.,

$$\Omega = \{x \in \Omega, \quad \phi(x) \le 1\} \cup \{x \in \Omega, \quad \phi(x) > 1\},$$

where $\phi(x)$ is the positive solution of (2.4). Let $u(t, x)$ be a positive solution of (2.1)–(2.2). We will prove that $u(t, x) \to \phi(x)$ for $x \in \{x \in \Omega, \quad \phi(x) \le 1\}$. This can be done by the monotone method together with an extension trick (see Lemma 5.6). As for $x \in \{x \in \Omega, \quad \phi(x) > 1\}$, we will show that the difference $u(t, x) - \phi(x)$ is either a decreasing oscillating function dominated by the boundary value of the function itself (see Lemma 5.10 and Corollary 5.11) or an eventually positive (negative) function (see Lemmas 5.7 and 5.9). In either case, it follows that $u(t,x) \to \phi(x)$. Based on Lemma 5.5, Lemma 5.8 is an auxiliary result to Lemmas 5.9 and 5.10. Also, Lemma 5.4 is used in the proofs of Lemma 5.7 and 5.9. Combination of Lemmas 5.4–5.11 gives rise to the following global attractivity result.

THEOREM 5.3. *Suppose $e \le \beta < e^2$. Then the solutions of (2.1)–(2.2) converge to the positive solution of (2.4).*

PROOF. The proof will be carried out in a number of lemmas.

LEMMA 5.4. *Consider*

$$\frac{dy(t)}{dt} \le r(t) - cy(t), \tag{5.1}$$

where $c > 0$, $y(t) \ge 0$, and $r(t) \ge 0$. Suppose $r(t) \to 0$ as $t \to \infty$. Then $y(t) \to 0$ as $t \to \infty$.

PROOF. The proof is straightforward.

Lemma 5.4 and the following Lemma 5.5 are helpful in the proof of Lemmas 5.7–5.9. They are also the bases of our approach to spatial decomposition.

LEMMA 5.5. *Assume $e < \beta \le e^2$. Let $u(t,x)$ be a solution of (2.1)–(2.2) and let Ω_1 be an open subset of Ω satisfying $\overline{\Omega_1} \subset \Omega$. Suppose that there ex-*

ists $T_0 \geq 1$, *for all* $t > T_0$, *there exists* $(\xi(t), \eta(t)) \in [t-1, t] \times \Omega_1$, *such that* $u(\xi(t), \eta(t)) = \min_{(\xi,x)\in[t-1,t]\times\overline{\Omega_1}} u(\xi, x)$. *Then there exists* $T_c \geq T_0$ *sufficiently large so that* $u(t, x) \geq 1$ *for* $(t, x) \in [T_c, \infty) \times \overline{\Omega_1}$.

PROOF. We define

$$g(\epsilon) = \beta(\beta e^{-1} + \epsilon) - (1+\epsilon)e^{\beta e^{-1}+\epsilon}.$$

Note that $g(0) = \beta^2 e^{-1} - e^{\beta e^{-1}} > 0$ for $e < \beta \leq e^2$. Then there exists $\epsilon_1 > 0$, such that $g(\epsilon) \geq 0$ for all $0 \leq \epsilon \leq \epsilon_1$. Hence,

$$\beta(\beta e^{-1} + \epsilon)e^{-(\beta e^{-1}+\epsilon)} \geq 1 + \epsilon, \qquad \text{for } 0 \leq \epsilon \leq \epsilon_1.$$

Now for any $0 < \epsilon \leq \epsilon_1$, by Lemma 5.1, there exists $T_1 \geq T_0$, such that

$$u(t-1, x) \leq \beta e^{-1} + \epsilon, \qquad \text{for all } t \geq T_1 \text{ and } x \in \overline{\Omega}. \tag{5.2}$$

To complete our proof, we divide our discussion into three parts.

Part A. Suppose that for any given $t \geq T_1$, $\xi(t) > t - 1$. Then, we have $\frac{\partial u}{\partial t}(\xi(t), \eta(t)) \leq 0$ and $\Delta u(\xi(t), \eta(t)) \geq 0$. By (2.1), this implies

$$u(\xi(t), \eta(t)) \geq \beta u(\xi(t) - 1, \eta(t))e^{-u(\xi(t)-1, \eta(t))}. \tag{5.3}$$

We will complete Part A by discussing the following two cases.

Case 1. There exists $T_2 \geq T_1$ such that $u(\xi(T_2) - 1, \eta(T_2)) \geq 1$. Since $u \mapsto ue^{-u}$

is decreasing for $u \geq 1$, we use (5.2) and (5.3) to get

$$u(\xi(T_2), \eta(T_2)) \geq \beta(\beta e^{-1} + \epsilon)e^{-(\beta e^{-1}+\epsilon)}$$

$$\geq 1 + \epsilon > 1.$$

Consequently, $u(\xi, x) \geq u(\xi(T_2), \eta(T_2)) \geq 1$ for all $(\xi, x) \in [T_2 - 1, T_2] \times \overline{\Omega_1}$. Therefore, by induction, we conclude that $u(t, x) \geq 1$ for all $(t, x) \in [T_2, \infty) \times \overline{\Omega_1}$.

Case 2. $u(\xi(t) - 1, \eta(t)) < 1$ for all $t \geq T_1$. Denote

$$m(t) := \min_{(\xi,x)\in[t-1,t]\times\overline{\Omega_1}} u(\xi, x).$$

Then by (5.3) we have

$$\begin{aligned} m(t) &= u(\xi(t), \eta(t)) \\ &\geq \beta u(\xi(t) - 1, \eta(t))e^{-u(\xi(t)-1,\eta(t))} \\ &> \beta u(\xi(t) - 1, \eta(t))e^{-1} \\ &> u(\xi(t) - 1, \eta(t)) \geq m(t - 1). \end{aligned}$$

for all $t \geq T_1$. Next we show that the function $m(t)$ is monotone increasing for $t \geq T_1$. For $s \geq T_1$, suppose that $t - 1 \leq s \leq t$. Firstly, if $t - 1 \leq \xi(s)$ and $\xi(t) \leq s$, then clearly $m(s) = m(t)$. Secondly, if $s - 1 < \xi(s) \leq t - 1$ and $t - 1 < \xi(t) \leq s$, one concludes that $m(s) \leq m(t)$ since $u(\xi(s), \eta(s))$ is the minimum of $u(\xi, x)$ on

$[s-1, s] \times \overline{\Omega_1}$. Thirdly, if $s < \xi(t) \leq t$, then $s-1 < \xi(t)-1 \leq t-1 \leq s$ and hence

$$\begin{aligned} m(t) &= u(\xi(t), \eta(t)) \\ &\geq \beta u(\xi(t)-1, \eta(t)) e^{-u(\xi(t)-1, \eta(t))} \\ &\geq \beta u(\xi(s), \eta(s)) e^{-u(\xi(s), \eta(s))} \\ &> \beta u(\xi(s), \eta(s)) e^{-1} \\ &> u(\xi(s), \eta(s)) = m(s). \end{aligned}$$

On the other hand, suppose $T_1 \leq s < t-1$. Then there exists an integer $l \geq 1$ such that $s \in [t-l-1, \quad t-l]$, so that we have

$$m(s) \leq m(t-l) \leq m(t).$$

Therefore we conclude that $m(t)$ is monotone increasing for $t \geq T_1$. Let

$$m_0 := \lim_{t \to \infty} m(t) > 0.$$

We will show $m_0 > 1$. In fact, since

$$\begin{aligned} m(t) &= u(\xi(t), \eta(t)) \\ &\geq \beta u(\xi(t)-1, \eta(t)) e^{-u(\xi(t)-1, \eta(t))} \\ &\geq \beta m(t-1) e^{-m(t-1)}, \end{aligned}$$

we take the limits as $t \to \infty$ to obtain

$$m_0 \geq \beta m_0 e^{-m_0}.$$

This implies $e^{m_0} \geq \beta$ and hence $m_0 > 1$ since $\beta > e$. Therefore, there exists $T_2 > T_1$ such that $m(T_2) \geq 1$. Then one concludes that, by repeating Case 1, $u(t,x) \geq 1$ for all $(t,x) \in [T_2, \infty) \times \overline{\Omega_1}$. This completes the proof of Part A.

Part B. Suppose that for any given $t \geq T_1$, $\xi(t) = t - 1$. Let $m(t)$ be defined as in Part A. Clearly, $m(t)$ is monotone increasing for $t \geq T_1$. Now, for $0 < |h| < 1$, we have

$$\begin{aligned}
&\frac{m(t+1+h) - m(t+1)}{h} \\
&= \frac{u(t+h, \eta(t+1+h)) - u(t, \eta(t+1))}{h} \\
&= \frac{u(t+h, \eta(t+1+h)) - u(t+h, \eta(t+1))}{h} + \frac{u(t+h, \eta(t+1)) - u(t, \eta(t+1))}{h}.
\end{aligned}$$

Noting that $u(t+h, \eta(t+1+h)) \leq u(t+h, \eta(t+1))$, we obtain

$$\frac{m(t+1+h) - m(t+1)}{h} \leq \frac{u(t+h, \eta(t+1)) - u(t, \eta(t+1))}{h}, \qquad \text{for } 0 < h < 1.$$

and

$$\frac{m(t+1+h) - m(t+1)}{h} \geq \frac{u(t+h, \eta(t+1)) - u(t, \eta(t+1))}{h}, \qquad \text{for } -1 < h < 0.$$

Therefore

$$D^- m(t+1) \geq D_- m(t+1) \geq \frac{\partial u(t, \eta(t+1))}{\partial t} \geq D^+ m(t+1) \geq D_+ m(t+1),$$

where D^-, D_-, D^+, D_+ are the Dini derivatives. Note that monotone function is differentiable almost everywhere. Therefore, we have

$$\frac{dm(t+1)}{dt} = \frac{\partial u(t, \eta(t+1))}{\partial t}, \quad \text{a.e. for } t \geq T_1. \tag{5.4}$$

Next, we will show that for any $0 < \epsilon \leq \epsilon_1$, there exists a sequence $\{t_k\}$ satisfying

$$\begin{aligned}
&t_k \geq T_1, \quad 0 < t_{k+1} - t_k < 1, && \text{for all } k \geq 1; \\
&t_k \to \infty, \quad \text{as } k \to \infty; && \text{and} \\
&0 \leq \frac{dm(t_k+1)}{dt} < \epsilon, && \text{for all } k \geq 1.
\end{aligned} \tag{5.5}$$

If this is not the case, then one can find $0 < \epsilon_0 \leq \epsilon_1$, and a sequence of intervals $\{I_k := (a_k, b_k)\}_{k=1}^{\infty}$, on which $\dfrac{dm(t+1)}{dt} \geq \epsilon_0$, where $a_k < b_k < a_{k+1}$, $b_k \to \infty$ as $k \to \infty$, and moreover, $|I_k| \geq 1$. Therefore, for any $k \geq 1$, we have

$$m(b_k+1) - m(a_1+1) \geq \int_{a_1}^{b_k} \frac{dm(t+1)}{dt} \geq \sum_{l=1}^{k} \int_{I_l} \frac{dm(t+1)}{dt} \geq k\epsilon_0.$$

This contradicts the boundedness of the function $m(t)$. Hence the aforementioned sequence $\{t_k\}$ exists. Now, for $k \geq 1$, since $u(t_k, \eta(t_k+1))$ is the minimum of $u(\xi, x)$ on $[t_k, t_k+1] \times \overline{\Omega_1}$ and $\eta(t_k+1) \in \Omega_1$, we use (2.1) and (5.4) to get,

$$\frac{dm(t_k+1)}{dt} \geq -\tau m(t_k+1) + \beta\tau u(t_k - 1, \eta(t_k+1))e^{-u(t_k-1,\eta(t_k+1))}. \tag{5.6}$$

Using (5.5) and (5.6) instead of (5.3), one can follow in a similar manner to the proof of Part A to complete the proof of Part B.

Part C. This is the complement of Parts A and B. Suppose that there exists an increasing sequence $\{t_k\}$, where $t_k \geq T_1$ for all $k \geq 1$ and $t_k \to \infty$ as $n \to \infty$, such that $\xi(t_k) > t_k - 1$ and $\eta(t_k) \in \Omega_1$. Therefore, we have

$$m(t_k) \geq \beta u(\xi(t_k) - 1, \eta(t_k))e^{-u(\xi(t_k)-1,\eta(t_k))}, \quad \text{for all } k \geq 1. \tag{5.7}$$

Without loss of generality, we assume $t_{k+1} - t_k > 1$.

Claim: There exists $T_2 \geq t_1$, such that $m(T_2) \geq 1$.

Using the arguments similar to Case 1 of Part A, one can show that, if there exists $k_0 \geq 1$, satisfying $u(\xi(t_{k_0}) - 1, \eta(t_{k_0})) \geq 1$, then $m(t_{k_0}) > 1$. Therefore, the claim is true for this case.

Next, we assume that $m(t) < 1$ for all $t \geq t_1$, and that $u(\xi(t_k) - 1, \eta(t_k)) < 1$ for all $k \geq 1$. We show that $\{m(t_k)\}_{k=1}^{\infty}$ is a monotone increasing sequence. For any $k \geq 1$, we choose an integer $l \geq 2$, such that $t_k \in [t_{k+1} - l, \quad t_{k+1} - l + 1]$. According to (5.7), we have

$$\begin{aligned} m(t_{k+1}) &\geq \beta u(\xi(t_{k+1}) - 1, \eta(t_{k+1}))e^{-u(\xi(t_{k+1})-1, \eta(t_{k+1}))} \\ &\geq \beta m(t_{k+1} - 1)e^{-m(t_{k+1}-1)} \qquad (5.8) \\ &\geq m(t_{k+1} - 1) \end{aligned}$$

We now consider $[t_{k+1} - 2, \quad t_{k+1} - 1] \times \overline{\Omega_1}$. Clearly, if $\xi(t_{k+1} - 1) = t_{k+1} - 2$, we have

$$m(t_{k+1} - 1) \geq m(t_{k+1} - 2) \qquad (5.9)$$

On the other hand, suppose $\xi(t_{k+1} - 1) > t_{k+1} - 2$. According to (5.7), we have

$$m(t_{k+1} - 1) \geq \beta u(\xi(t_{k+1} - 1) - 1, \eta(t_{k+1} - 1))e^{-u(\xi(t_{k+1}-1)-1, \eta(t_{k+1}-1))},$$

Clearly, $u(\xi(t_{k+1}-1)-1, \eta(t_{k+1}-1)) < 1$. Otherwise, we can get $m(t_{k+1}-1) > 1$ by following the same arguments as Case 1 of Part A. This contradicts our assumption.

Now, using the same discussion as Case 2 of Part A, we obtain (5.9) as well. Invoking the above arguments $(l-2)$ times, we eventually get

$$m(t_{k+1}-1) \geq m(t_{k+1}-l+1). \tag{5.10}$$

Subclaim: $m(t_{k+1}-l+1) \geq m(t_k)$.

Clearly, on $[t_{k+1}-l, \quad t_{k+1}-l+1] \times \overline{\Omega_1}$, this is true if $\xi(t_{k+1}-l+1) \in [t_{k+1}-l, t_k]$. Now we consider the case where $\xi(t_{k+1}-l+1) > t_k$. According to (5.7), we have

$$\begin{aligned} & m(t_{k+1}-l+1) \\ & \geq \beta u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1)) e^{-u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1))}. \end{aligned} \tag{5.11}$$

Obviously, $u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1)) < 1$. Otherwise we use the same discussion as Case 1 of Part A to get $m(t_{k+1}-l+1) > 1$. This contradicts our assumption. Noting that $\xi(t_{k+1}-l+1) \in (t_k, t_{k+1}-l+1]$, we have

$$m(t_k) \leq u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1)) < 1. \tag{5.12}$$

Since $u \mapsto ue^{-u}$ is monotone increasing for $u \leq 1$, we use (5.11) and (5.12) to obtain

$$\begin{aligned} & m(t_{k+1}-l+1) \\ & \geq \beta u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1)) e^{-u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1))} \\ & \geq \beta m(t_k) e^{-m(t_k)} \\ & \geq \beta e^{-1} m(t_k) \geq m(t_k). \end{aligned} \tag{5.13}$$

Hence, the subclaim holds.

Now we combine (5.8), (5.10), and (5.13) to get $m(t_{k+1}) \geq m(t_k)$, and moreover

$$m(t_{k+1}) \geq \beta m(t_k) e^{-m(t_k)}.$$

Then we take the limit as $k \to \infty$ to obtain

$$m_0 \geq \beta m_0 e^{-m_0},$$

where $m_0 := \lim_{k\to\infty} m(t_k) > 0$. Hence $m_0 > 1$ since $\beta > e$. Therefore, there exists $k_0 \geq 1$, such that $m(t_{k_0}) \geq 1$. Again, this contradicts our assumption. The proof of the Claim is complete.

Finally, one can easily show $u(t,x) \geq 1$ for $(t,x) \in [T_2, \infty) \times \overline{\Omega_1}$. For that we just need to consider $[T_2, T_2+1] \times \overline{\Omega_1}$. Obviously, if $\xi(T_2+1) = T_2$, we have $m(T_2+1) \geq m(T_2) \geq 1$. On the other hand, if $\xi(T_2+1) > T_2$, then we use the same discussion as Case 1 of Part A to get $m(T_2+1) > 1$. This completes the proof of Part C.

We introduce the following notations

$$\Omega^1_\infty = \{x \in \Omega, \quad \phi(x) < 1\},$$

$$\tilde{\Omega}^1_\infty = \{x \in \Omega, \quad \phi(x) > 1\},$$

$$\Omega^1_t(u) = \{x \in \Omega, \quad u(t,x) < 1\}.$$

If $\beta = e$, it follows from Remark 2.5 that $\tilde{\Omega}^1_\infty$ is empty. Therefore, for this case, the global attractivity of the positive steady state can be concluded by the following lemma.

LEMMA 5.6. *Let $u(t,x)$ be a solution of (2.1)–(2.2). Then, for $x \in \overline{\Omega^1_\infty}$, $u(t,x) \to \phi(x)$ (pointwise) as $t \to \infty$.*

PROOF. Without loss of generality, we assume that $u_0(\theta,x) > 0$ for all $x \in \Omega$ and $\theta \in [-1,0]$. We also assume that $\frac{\partial u_0}{\partial n}|_{\partial\Omega} < 0$ for all $\theta \in [-1,0]$, and that $u(t,x)$ satisfies (2.7) for all $t \geq 0$. Let $\underline{u}(t,x)$ be the solution of (2.1)–(2.2) with the initial condition $\epsilon\phi^*(x)$, where ϵ is chosen small enough such that $u_0(\theta,x) \geq \epsilon\phi^*(x)$ for all $x \in \bar{\Omega}$ and $\theta \in [-1,0]$, and $\phi^*(x)$ is defined as Theorem 5.2. Then one can use the same arguments as the proof of Theorem 5.2 to show $\underline{u}(t,x) \leq \phi(x)$ and $\frac{\partial \underline{u}}{\partial t} \geq 0$ for all $x \in \overline{\Omega^1_\infty}$ and $t > 0$. Therefore, we have that

$$\lim_{t\to\infty} \underline{u}(t,x) = \phi(x) < 1, \qquad \text{for all } x \in \overline{\Omega^1_\infty}.$$

Let $\overline{u}(t,x)$ be the solution of (2.1)–(2.2) with the initial condition $\zeta\phi(x)$, where $\zeta > 1$ is large enough so that $\zeta\phi(x) \geq u_0(\theta,x)$ for all $x \in \overline{\Omega}$ and $\theta \in [-1,0]$. Let

$$\Omega^1_{-1} = \{x \in \Omega, \quad \zeta\phi(x) < 1\}.$$

As before, we can show that $\frac{\partial \overline{u}}{\partial t} \leq 0$ for all $x \in \Omega^1_{-1}$ and hence

$$\lim_{t\to\infty} \overline{u}(t,x) = \phi(x), \qquad \text{for } x \in \Omega^1_{-1}.$$

We denote

$$\Omega_0^1(u) = \bigcap_{t \geq 0} \Omega_t^1(u).$$

It is easy to see that, according to (2.7) and the (homogeneous) Dirichlet boundary conditions, $\Omega_0^1(u)$ is a nonempty open set. Moreover, $\Omega_0^1(u) \cap \Omega_\infty^1$ is nonempty and $u(t,x) \geq \underline{u}(t,x)$ for all $x \in \Omega_0^1(u) \cap \Omega_\infty^1$. Hence

$$\liminf_{t\to\infty} u(t,x) \geq \lim_{t\to\infty} \underline{u}(t,x) = \phi(x), \quad \text{for all } x \in \Omega_0^1(u) \cap \Omega_\infty^1.$$

On the other hand, for $x \in \Omega_0^1(u) \cap \Omega_\infty^1 \cap \Omega_{-1}^1$, one has $\overline{u}(t,x) \geq u(t,x)$. This leads to

$$\limsup_{t\to\infty} u(t,x) \leq \lim_{t\to\infty} \overline{u}(t,x) = \phi(x), \quad \text{for all } x \in \Omega_0^1(u) \cap \Omega_\infty^1 \cap \Omega_{-1}^1.$$

Therefore,

$$\lim_{t\to\infty} u(t,x) = \phi(x), \quad \text{for all } x \in \Omega_0^1(u) \cap \Omega_\infty^1 \cap \Omega_{-1}^1.$$

Next, we extend the region of convergence to the entire $\overline{\Omega_\infty^1}$. We denote

$$S_0 := \Omega_0^1(u) \cap \Omega_\infty^1 \cap \Omega_{-1}^1.$$

Suppose $S_0 \subsetneqq \Omega_\infty^1$. Let Ω_ϵ be an open subset of Ω_∞^1 such that

$$S_0 \subset \Omega_\epsilon \subset \Omega_\infty^1.$$

For any given $\delta > 0$, we define a subset of $\overline{S_0}$ as follows:

$$S_\delta(\partial\Omega) := \{x \in \overline{S_0} \ : \ \mathrm{dist}(x, \partial\Omega) < \delta\},$$

where $\mathrm{dist}(x, \partial\Omega)$ means the distance from x to the boundary of Ω. Let δ be chosen small enough such that $S_\delta(\partial\Omega) \subsetneqq \overline{S_0}$. It is clear that $S_\delta(\partial\Omega) \supset \partial\Omega$. We denote

$$\overline{K} = \max_{x \in \overline{\Omega_e}} \phi(x), \qquad \underline{K} = \min_{x \in \overline{\Omega_e} \setminus S_\delta(\partial\Omega)} \phi(x).$$

Clearly, $0 < \underline{K} \leq \overline{K} < 1$. Now for any given $0 < \epsilon < 1 - \overline{K}$, one can use the compactness of $\Omega_s := \overline{S_0} \setminus S_\delta(\partial\Omega)$ and the continuity of $\phi(x)$ to find $T_1 \geq 0$, only depending on ϵ, such that

$$u(t, x) \leq (1 + \epsilon)\phi(x), \quad \text{for all } x \in \Omega_s \text{ and } t \geq T_1. \tag{5.14}$$

Indeed, since $\lim_{t \to \infty} u(t, x) = \phi(x)$ for all $x \in \overline{S_0}$, we conclude that, for the above chosen ϵ and any $\tilde{x} \in \Omega_s$, there exists $\tilde{t}(\tilde{x}, \epsilon) \geq 0$, such that

$$|u(t, \tilde{x}) - \phi(\tilde{x})| < \frac{\underline{K}\epsilon}{3}, \quad \text{for all } t \geq \tilde{t}(\tilde{x}, \epsilon). \tag{5.15}$$

Moreover, since $\phi(x)$ is continuous and since $u(t, x)$ satisfies (2.7), we can find an open neighborhood $\mathcal{N}(\tilde{x}, \epsilon)$ of $\tilde{x}$ such that

$$|\phi(x) - \phi(\tilde{x})| < \frac{\underline{K}\epsilon}{3}, \tag{5.16}$$

and

$$|u(t,x)-u(t,\tilde{x})| = \left|\sum_{j=1}^{n}(x^{(j)}-\tilde{x}^{(j)})\frac{\partial u(t,\eta^{(j)})}{\partial x^{(j)}}\right| < \frac{\underline{K}\epsilon}{3}, \tag{5.17}$$

for all $x \in \mathcal{N}(\tilde{x},\epsilon)$, where $\eta^{(j)}$, the j^{th} component of η, is an intermediate value between $x^{(j)}$ and $\tilde{x}^{(j)}$. Since $\bigcup_{\tilde{x}\in\Omega_s}\mathcal{N}(\tilde{x},\epsilon) \supset \Omega_s$ and that Ω_s is compact, there exist $\mathcal{N}(\tilde{x}_i,\epsilon)$, $\quad i = 1,2,\cdots,l < \infty$, such that $\bigcup_{i=1}^{l}\mathcal{N}(\tilde{x}_i,\epsilon) \supset \Omega_s$, and for each $\tilde{x}_i$, $\quad i = 1,2,\cdots,l$, (5.15)–(5.17) hold. Then for any $x \in \Omega_s$, there exists $1 \le i_0 \le l$ such that $x \in \mathcal{N}(\tilde{x}_{i_0},\epsilon)$. Let

$$T_1 = \max_{1\le i\le l}\{\tilde{t}(\tilde{x}_i,\epsilon)\}.$$

Then, for all $t \ge T_1$, we have

$$\begin{aligned}
&u(t,x)-\phi(x)\\
&\le |u(t,x)-u(t,\tilde{x}_{i_0})| + |u(t,\tilde{x}_{i_0})-\phi(\tilde{x}_{i_0})| + |\phi(\tilde{x}_{i_0})-\phi(x)|\\
&\le \underline{K}\epsilon.
\end{aligned}$$

Note $\underline{K} \le \phi(x)$ for $x \in \Omega_s$. This implies (5.14).

For $B_s := \partial\Omega_s \cap (\Omega_\infty^1 \backslash S_0)$, we can follow the same discussion as above to choose finite number of open balls, $B(\tilde{x}_{k,\partial})$, $\quad k = 1,2,\cdots,l'$, satisfying

$$\bigcup_{\tilde{x}_{k,\partial}\in B_s} B(\tilde{x}_{k,\partial}) \supset B_s.$$

We denote

$$S_1 := \left(\bigcup_{k=1}^{l'} B(\tilde{x}_{k,\partial}) \bigcap \Omega_e\right) \bigcup S_0.$$

Clearly, $S_0 \subsetneqq S_1$. Furthermore, for all $t \geq T_1$ and $x \in S_1 \backslash S_0$, we have

$$u(t,x) \leq (1+\epsilon)\phi(x).$$

Noting that $0 < \epsilon < 1 - \overline{K}$ and that $\phi(x) < 1$ for $x \in S_1$, we get

$$\epsilon\phi(x) < \epsilon < 1 - \overline{K} < 1 - \phi(x).$$

Now, we choose $v(x) = (1+\epsilon)\phi(x)$. Then, for all $x \in S_1$, $v(x)$ satisfies the following properties

$$u(t,x) \leq v(x) \leq 1, \quad \text{for all } t \in [T_1, T_1+1], \text{ and}$$

$$d\Delta v(x) - \tau v(x) + \beta\tau v(x)e^{-v(x)} \leq 0.$$

By redefining $\overline{u}(t,x)$ with the initial function $v(x)$ for $t \in [T_1, T_1+1]$, we can show as before that $\lim_{t\to\infty} u(t,x) = \phi(x)$ for all $x \in S_1$. We can keep repeating the above extension to obtain a sequence of open sets $\{S_k\}, k = 1, 2, \cdots$, satisfying

$$S_k \subset S_{k+1} \subset \Omega^1_\infty, \quad \text{for all } k \geq 0.$$

Obviously, $\lim_{k\to\infty} S_k = \Omega^1_\infty$. Therefore, we have $\lim_{t\to\infty} u(t,x) = \phi(x)$ for all $x \in \Omega^1_\infty$. The convergence of $u(t,x)$ to $\phi(x)$ on $\overline{\Omega^1_\infty}$ follows immediately from the continuity of $u(t,x)$ and $\phi(x)$. This completes the proof.

LEMMA 5.7. *Suppose $e < \beta \leq e^2$. If eventually $u(t,x) \geq \phi(x)$ for $x \in \tilde{\Omega}^1_\infty$, then $u(t,x) \to \phi(x)$ in $L^2(\Omega)$ as $t \to \infty$.*

PROOF. Recall that

$$\frac{\partial(u-\phi)}{\partial t} = d\Delta(u-\phi) - \tau(u-\phi) + \beta\tau\left[u(t-1)e^{-u(t-1)} - \phi e^{-\phi}\right]. \tag{5.18}$$

Multiplying (5.18) by $(u-\phi)$ and taking integral over Ω, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u-\phi||^2_{L^2(\Omega)} \le& -(d\lambda_1+\tau)||u-\phi||^2_{L^2(\Omega)} \\ &+ \beta\tau\int_\Omega \left(u(t-1)e^{-u(t-1)} - \phi e^{-\phi}\right)(u-\phi)dx\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u-\phi||^2_{L^2(\Omega)} \le& -(d\lambda_1+\tau)||u-\phi||^2_{L^2(\Omega)} \\ &+ \beta\tau\int_{\Omega^1_\infty} (u(t-1)e^{-u(t-1)} - \phi e^{-\phi})(u-\phi)dx \\ &+ \beta\tau\int_{\tilde{\Omega}^1_\infty} (u(t-1)e^{-u(t-1)} - \phi e^{-\phi})(u-\phi)dx \\ \le& -(d\lambda_1+\tau)||u-\phi||^2_{L^2(\Omega)} \\ &+ \beta\tau\int_{\overline{\Omega^1_\infty}} (u(t-1)e^{-u(t-1)} - \phi e^{-\phi})(u-\phi)dx.\end{aligned}$$

Then we apply Lemma 5.4 to get the conclusion. This completes the proof.

The following lemma implies that for any given $\epsilon > 0$, there exists $\tilde{T}$ such that $u(t,x)$, a solution of (2.1)–(2.2), is bounded below from $1-\epsilon$ for all $t \ge \tilde{T}$ and $x \in \overline{\tilde{\Omega}^1_\infty}$.

LEMMA 5.8. *Suppose $e < \beta \le e^2$ and let $u(t,x)$ is a solution of (2.1)–(2.2). Then*

$$\liminf_{t\to\infty} u(t,x) \ge 1, \tag{5.19}$$

uniformly for $x \in \overline{\tilde{\Omega}^1_\infty}$.

PROOF. We denote

$$m(t) := \min_{(\xi,x)\in[t-1,t]\times\overline{\tilde{\Omega}^1_\infty}} u(\xi,x),$$

and

$$m_\partial(t) := \min_{(\xi,x)\in[t-1,t]\times\partial\tilde{\Omega}^1_\infty} u(\xi,x).$$

Clearly, if there exists $T_0 > 1$ such that

$$m(t) < m_\partial(t),$$

for all $t \ge T_0$, then, by Lemma 5.5, we have $u(t,x) \ge 1$ for all $x \in \overline{\tilde{\Omega}^1_\infty}$ and all sufficiently large t. Hence, (5.19) holds. Next, we consider the case where there exists an increasing sequence $\{t_k\}_{k=1}^\infty$ satisfying

$$\begin{aligned} &1 \le t_k < t_{k+1}, && \text{for all } k \ge 1;\\ &t_k \to \infty, && \text{as } k \to \infty;\\ &m(t_k) = m_\partial(t_k), && \text{for all } k \ge 1. \end{aligned}$$

By Lemma 5.6, $\lim_{t\to\infty} u(t,x) = \phi(x) = 1$ for all $x \in \partial\tilde{\Omega}^1_\infty$. Hence, for any sufficiently small $\epsilon > 0$, there exists $k_0 \geq 1$ such that

$$m_\partial(t) > 1 - \epsilon, \quad \text{for all } t \geq t_{k_0}. \tag{5.20}$$

Claim: For any integer $l \geq 1$, we have

$$m(t_{k_0} + l) \geq \min\{m_\partial(t_{k_0} + l), m_\partial(t_{k_0} + l - 1), \cdots, m_\partial(t_{k_0}), 1\}. \tag{5.21}$$

Proof of the Claim. Clearly, (5.21) holds if $m(t_{k_0} + l) = m_\partial(t_{k_0} + l)$. Now suppose $m(t_{k_0} + l) < m_\partial(t_{k_0} + l)$. On $[t_{k_0} + l - 1, t_{k_0} + l] \times \tilde{\Omega}^1_\infty$, if the minimum of $u(\xi, x)$ is obtained at $t_{k_0} + l - 1$, then, $m(t_{k_0} + l) \geq m(t_{k_0} + l - 1)$. On the other hand, suppose the minimum of $u(\xi, x)$ is obtained in $(t_{k_0} + l - 1, t_{k_0} + l] \times \tilde{\Omega}^1_\infty$. We follow the proof of Part A of Lemma 5.5 to obtain

$$m(t_{k_0} + l) \geq \min\{m(t_{k_0} + l - 1), 1\}.$$

Note that $m(t_{k_0}) = m_\partial(t_{k_0})$. We get (5.21) after invoking the same procedure as above for at most $(l - 1)$ times. This completes the proof of the Claim.

For any $t \geq t_{k_0}$, we find an integer $l \geq 1$ such that $t_{k_0} + l - 1 \leq t \leq t_{k_0} + l$. From (5.20) and (5.21), we get

$$\begin{aligned} u(t, x) &\geq m(t_{k_0} + l) \\ &\geq \min\{m_\partial(t_{k_0} + l), m_\partial(t_{k_0} + l - 1), \cdots, m_\partial(t_{k_0}), 1\} \\ &\geq 1 - \epsilon. \end{aligned}$$

Therefore,

$$\liminf_{t\to\infty} u(t,x) \geq 1-\epsilon,$$

uniformly for $x \in \overline{\tilde{\Omega}^1_\infty}$. Since ϵ is arbitrary, (5.19) follows. This completes the proof.

With the help of Lemma 5.8, we are ready to prove the following lemma.

LEMMA 5.9. *Suppose* $e < \beta \leq e^2$. *If eventually* $u(t,x) \leq \phi(x)$ *for* $x \in \tilde{\Omega}^1_\infty$, *then* $u(t,x) \to \phi(x)$ *in* $L^2(\Omega)$ *as* $t \to \infty$.

PROOF. Using the calculation similar to the proof of Lemma 5.7, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||u-\phi||^2_{L^2(\Omega)} \leq & -(d\lambda_1+\tau)||u-\phi||^2_{L^2(\Omega)} \\
& + \beta\tau\int_{\Omega^1_\infty}(u(t-1)e^{-u(t-1)}-\phi e^{-\phi})(u-\phi)dx \\
& + \beta\tau\int_{\tilde{\Omega}^1_\infty}(u(t-1)e^{-u(t-1)}-\phi e^{-\phi})(u-\phi)dx \\
\leq & -(d\lambda_1+\tau)||u-\phi||^2_{L^2(\Omega)} \\
& + \beta\tau\int_{\tilde{\Omega}^1_\infty\cap\Omega^1_{t-1}(u)}|u(t-1)e^{-u(t-1)}-\phi e^{-\phi}||u-\phi|dx \\
& + \beta\tau\int_{\overline{\Omega^1_\infty}}(u(t-1)e^{-u(t-1)}-\phi e^{-\phi})(u-\phi)dx.
\end{aligned}$$

Noticing that from Lemma 5.8, $\liminf_{t\to\infty} u(t,x) \geq 1$ uniformly for $x \in \tilde{\Omega}^1_\infty$, and that $u(t,x)$ satisfies (2.7), we have

$$\lim_{t\to\infty}\int_{\tilde{\Omega}^1_\infty\cap\Omega^1_{t-1}(u)}|u(t-1)e^{-u(t-1)}-\phi e^{-\phi}||u-\phi|dx=0.$$

Now we apply Lemma 5.4 to obtain the conclusion. The proof is complete.

LEMMA 5.10. *Suppose $e<\beta\le e^2$ and there exist $T_0\ge 1$, $c_0>0$ such that*

$$M(t)=\max_{(\xi,x)\in[t-1,t]\times\overline{\tilde{\Omega}^1_\infty}}|u(\xi,x)-\phi(x)|=|u(t_0,x_0)-\phi(x_0)|\ge c_0,$$

where $(t_0,x_0)\in[t-1,t]\times\tilde{\Omega}^1_\infty$ and $t\ge T_0$. Then, there exists $T_1\ge T_0$ such that $M(t)$ is monotone decreasing for $t\ge T_1$.

PROOF. According to Lemma 5.8, for any given $0<\epsilon<\frac{c_0}{2e^2}$, we can find $T_1\ge T_0$, such that $u(t-1,x)+\epsilon\ge 1$ for all $t\ge T_1$ and $x\in\overline{\tilde{\Omega}^1_\infty}$. In the rest of the proof, we assume $t\ge T_1$. We consider $[t-1,t]\times\overline{\tilde{\Omega}^1_\infty}$. Clearly, if the maximum of $|u(\xi,x)-\phi(x)|$ is obtained at $t-1$, then we have

$$M(t)\le M(t-1). \tag{5.22}$$

On the other hand, suppose $(t_0,x_0)\in(t-1,t]\times\tilde{\Omega}^1_\infty$. We will show (5.22) still holds. Suppose

$$M(t)=\max_{(\xi,x)\in[t-1,t]\times\overline{\tilde{\Omega}^1_\infty}}|u(\xi,x)-\phi(x)|=u(t_0,x_0)-\phi(x_0)$$

is the positive maximum in $(t-1,t] \times \tilde{\Omega}^1_\infty$, where $t_0 \in (t-1,t]$ and $x_0 \in \tilde{\Omega}^1_\infty$. Then, we have $\frac{\partial(u-\phi)}{\partial t} \geq 0$ and $d\Delta(u-\phi) \leq 0$ at (t_0, x_0). By (5.18), this leads to

$$u(t_0,x_0) - \phi(x_0) \leq \beta \left[u(t_0-1,x_0)e^{-u(t_0-1,x_0)} - \phi(x_0)e^{-\phi(x_0)}\right]. \tag{5.23}$$

Clearly, $u(t_0-1,x_0) < \phi(x_0)$. Otherwise, since $x \mapsto xe^{-x}$ is decreasing for $x \geq 1$ we have

$$u(t_0,x_0) - \phi(x_0) \leq 0,$$

which is a contradiction. Moreover, there exists $\xi \in [u(t_0-1,x_0), \phi(x_0)]$ such that

$$\begin{aligned} & u(t_0-1,x_0)e^{-u(t_0-1,x_0)} - \phi(x_0)e^{-\phi(x_0)} \\ &= (1-\xi)e^{-\xi}\left(u(t_0-1,x_0) - \phi(x_0)\right) \\ &\leq e^{-2}\left(\phi(x_0) - u(t_0-1,x_0)\right), \end{aligned} \tag{5.24}$$

Substituting (5.24) into (5.23) gives rise to

$$\begin{aligned} M(t) &= u(t_0,x_0) - \phi(x_0) \\ &\leq \beta e^{-2}(\phi(x_0) - u(t_0-1,x_0)) \\ &\leq \beta e^{-2} M(t-1) \leq M(t-1). \end{aligned} \tag{5.25}$$

On the other hand, suppose

$$M(t) = \max_{(\xi,x)\in[t-1,t]\times\overline{\tilde{\Omega}^1_\infty}} |u(\xi,x) - \phi(x)| = -[u(t_0,x_0) - \phi(x_0)],$$

i.e. $u(t_0,x_0) - \phi(x_0)$ is the negative minimum in $(t-1,t] \times \tilde{\Omega}^1_\infty$, where $t_0 \in (t-1,t]$ and $x_0 \in \tilde{\Omega}^1_\infty$. We divide our discussion into two cases.

Case 1. $u(t_0-1,x_0)>\phi(x_0)$. We use the same arguments as above to get

$$u(t_0,x_0)-\phi(x_0)\geq -\beta e^{-2}\left(u(t_0-1,x_0)-\phi(x_0)\right). \tag{5.26}$$

Case 2. $u(t_0-1,x_0)<\phi(x_0)$. Note that $u(t_0-1,x_0)+\epsilon\geq 1$. Therefore we obtain

$$\begin{aligned}
&u(t_0,x_0)-\phi(x_0)\\
&\geq \beta\left[(u(t_0-1,x_0)+\epsilon)e^{-(u(t_0-1,x_0)+\epsilon)}-(\phi(x_0)+\epsilon)e^{-(\phi(x_0)+\epsilon)}\right]\\
&\quad+\beta\left[u(t_0-1,x_0)e^{-u(t_0-1,x_0)}-(u(t_0-1,x_0)+\epsilon)e^{-(u(t_0-1,x_0)+\epsilon)}\right]\\
&\quad-\beta\left[\phi(x_0)e^{-\phi(x_0)}-(\phi(x_0)+\epsilon)e^{-(\phi(x_0)+\epsilon)}\right]\\
&\geq -2\beta\epsilon\\
&\geq -\beta e^{-2}M(t-1).
\end{aligned} \tag{5.27}$$

We combine (5.26)–(5.27) to obtain

$$M(t)\leq \beta e^{-2}M(t-1)\leq M(t-1). \tag{5.28}$$

Now for any $s\geq T_1$ and $t-1\leq s\leq t$, if $t_0\in(s-1,s]$ and $t_1\in(s-1,s]$, then

$$|u(t_0,x_0)-\phi(x_0)|\leq M(s),$$

and

$$|u(t_1,x_1)-\phi(x_1)|\leq M(s),$$

which imply $M(t)\leq M(s)$. If $t_0\notin(s-1,s]$ or $t_1\notin(s-1,s]$, then $(t_0-1)\in(s-1,s]$ or $(t_1-1)\in(s-1,s]$, we still get the same conclusion. Now if $T_1\leq s<(t-1)$, then there exists a positive integer $l \geq 1$ such that $s \in [t-l-1, t-l]$. According to (5.20), (5.25), and (5.28), we have

$$M(s) \geq M(t-k) \geq M(t).$$

This completes the proof.

COROLLARY 5.11. *Suppose $e < \beta < e^2$. If $u(t,x) - \phi(x)$ oscillates for $x \in \tilde{\Omega}^1_\infty$, and $t \geq 1$, then $u(t,x) \to \phi(x)$ in $C(\tilde{\Omega}^1_\infty)$ as $t \to \infty$.*

PROOF. Let

$$M(t) := |u(\xi(t), \eta(t)) - \phi(\eta(t))| = \max_{(\xi,x)\in[t-1,t]\times\overline{\tilde{\Omega}^1_\infty}} |u(\xi,x) - \phi(x)|,$$

and

$$M_\partial(t) := \max_{(\xi,x)\in[t-1,t]\times\partial\tilde{\Omega}^1_\infty} |u(\xi,x) - \phi(x)|.$$

Claim: For any sufficiently small $\epsilon > 0$, there exists t_ϵ such that

$$M(t_\epsilon) < \epsilon.$$

Proof of the Claim. Suppose the Claim fails, i.e. there exist $\epsilon_0 > 0$ and $T_0 > 1$ such that

$$M(t) \geq \epsilon_0, \quad \text{for all } t \geq T_0. \tag{5.29}$$

We will show that there exists an increasing sequence $\{t_k\}_{k=1}^{\infty}$ satisfying

$$T_0 \leq t_k \leq t_{k+1}, \quad \text{for all } k \geq 1;$$

$$t_k \to \infty, \qquad \text{as } k \to \infty; \text{ and}$$

$$M(t_k) = M_{\partial}(t_k), \quad \text{for all } k \geq 1.$$

Indeed, if this is not the case, i.e. for all $t \geq T_0$, the maximum of $|u(\xi, x) - \phi(x)|$ on $[t-1, t] \times \overline{\tilde{\Omega}^1_{\infty}}$ is obtained in $[t-1, t] \times \tilde{\Omega}^1_{\infty}$, then, according to Lemma 5.10, $M(t)$ is monotone decreasing for $t \geq T_1$. We denote $M_0 := \lim_{t\to\infty} M(t)$. We will discuss two cases to show $M_0 = 0$.

Case 1. Suppose that $\xi(t) = t - 1$ and $\eta(t) \in \tilde{\Omega}^1_{\infty}$, for all $t \geq T_1$. For any $0 < \epsilon < \min\{\tau(1 - \beta e^{-2})\epsilon_0, \frac{\epsilon_0}{2e^2}\}$, we use the arguments similar to Part B of the proof of Lemma 5.5 to find a sequence $\{\tilde{t}_k\}_{k=1}^{\infty}$ such that

$$\begin{aligned} &\tilde{t}_k \geq T_1, \quad 0 < \tilde{t}_{k+1} - \tilde{t}_k < 1, && \text{for all } k \geq 1; \\ &\tilde{t}_k \to \infty \quad \text{as } k \to \infty; && \text{and} \\ &-\epsilon < \frac{dM(\tilde{t}_k + 1)}{dt} \leq 0, && \text{for all } k \geq 1. \end{aligned} \tag{5.30}$$

For any $k \geq 1$, we consider

$$M(\tilde{t}_k + 1) = u(\tilde{t}_k, \eta(\tilde{t}_k + 1)) - \phi(\eta(\tilde{t}_k + 1)),$$

i.e. $u(\tilde{t}_k, \eta(\tilde{t}_k + 1)) - \phi(\eta(\tilde{t}_k + 1))$ is the positive maximum. Without loss of generality, we may assume, by using the continuity of $u(t, x)$ and $\phi(x)$, that there exists a sequence $\{h_j\}_{j=1}^{\infty}$, satisfying $0 < |h_j| < 1$ and $h_j \to 0$ as $j \to \infty$, such that

$$M(\tilde{t}_k + 1 + h_j) = u(\tilde{t}_k + h_j, \eta(\tilde{t}_k + 1 + h_j)) - \phi(\eta(\tilde{t}_k + 1 + h_j)).$$

Now we follow Part B of the proof of Lemma 5.5 to obtain

$$\frac{dM(\tilde{t}_k+1)}{dt} = \frac{\partial u(\tilde{t}_k, \eta(\tilde{t}_k+1))}{\partial t}. \tag{5.31}$$

On the other hand, suppose

$$M(\tilde{t}_k+1) = -[u(\tilde{t}_k, \eta(\tilde{t}_k+1)) - \phi(\eta(\tilde{t}_k+1))],$$

i.e. $u(\tilde{t}_k, \eta(\tilde{t}_k+1)) - \phi(\eta(\tilde{t}_k+1))$ is the negative minimum. Proceeding as before, we get

$$\frac{dM(\tilde{t}_k+1)}{dt} = -\frac{\partial u(\tilde{t}_k, \eta(\tilde{t}_k+1))}{\partial t}. \tag{5.32}$$

Using (5.30)–(5.32) and following a similar argument as in the proof of Lemma 5.10, we obtain

$$\tau M(\tilde{t}_k+1) \le \epsilon + \tau\beta e^{-2} M(\tilde{t}_k), \tag{5.33}$$

Therefore, we take the limit as $k \to \infty$ to get

$$\tau M_0 \le \epsilon + \tau\beta e^{-2} M_0,$$

that is

$$M_0 \le \frac{\epsilon}{\tau(1-\beta e^{-2})}.$$

This implies $M_0 = 0$ since ϵ can be arbitrarily small.

Case 2. Next we assume that there exists an increasing sequence, still denoted by $\{\tilde{t}_k\}_{k=1}^{\infty}$, such that on $[\tilde{t}_k, \tilde{t}_k+1] \times \overline{\tilde{\Omega}^1_\infty}$, we have $\xi(\tilde{t}_k+1) > \tilde{t}_k$ and $\eta(\tilde{t}_k+1) \in \tilde{\Omega}^1_\infty$. In this case, we follow the proof of Lemma 5.10 to get

$$M(\tilde{t}_k + 1) \leq \beta e^{-2} M(\tilde{t}_k).$$

Then we take the limit as $k \to \infty$ to obtain

$$M_0 \leq \beta e^{-2} M_0,$$

which implies $M_0 = 0$ since $\beta e^{-2} < 1$.

Therefore, the aforementioned sequence $\{t_k\}$ exists. By Lemma 5.6,

$$\lim_{t \to \infty} u(t,x) = \phi(x), \quad \text{for } x \in \partial\tilde{\Omega}^1_\infty.$$

Hence, for any $0 < \epsilon < \epsilon_0$, there exists $k_0 \geq 1$ such that

$$M_\partial(t) < \epsilon, \quad \text{for all } t \geq t_{k_0}. \tag{5.34}$$

Next, we show

$$M(t_{k_0}+1) \leq \max\{M_\partial(t_{k_0}+1), M_\partial(t_{k_0})\}. \tag{5.35}$$

Clearly, (5.35) is true if $M(t_{k_0}+1) = M_\partial(t_{k_0}+1)$. Now suppose $M(t_{k_0}+1) > M_\partial(t_{k_0}+1)$. On $[t_{k_0}, t_{k_0}+1] \times \tilde{\Omega}^1_\infty$, if $\xi(t_{k_0}+1) = t_{k_0}$, then, $M(t_{k_0}+1) \leq M_\partial(t_{k_0})$. On the other hand, suppose $\xi(t_{k_0}+1) > t_{k_0}$. We follow the proof of Lemma 5.10

to obtain $M(t_{k_0}+1) \leq M(t_{k_0}) = M_\partial(t_{k_0})$. Therefore (5.35) is true. We combine (5.34) and (5.35) to get $M(t_{k_0}+1) \leq \epsilon < \epsilon_0$. This contradicts (5.29) as well. The proof of the Claim is complete.

Now that the claim holds, we conclude $\lim_{t\to\infty} u(t,x) = \phi(x)$ according to the local asymptotic stability of the positive steady state (Theorem 4.2). This completes the proof.

So far, we have shown the global attractivity in the sense of $L^2(\Omega)$. Next, we claim that the convergence theorems can be enhanced by using an a priori estimate and an interpolation inequality. More precisely, we have the following theorem.

THEOREM 5.12. *Let $u(t,x)$ be a solution of (2.1)–(2.2) and $U(x)$ be the corresponding steady state, i.e. the zero solution or the positive steady state $\phi(x)$. Then, there exists a constant K, independent of time t, such that*

$$||u(t,\cdot) - U(\cdot)||_{C^1(\Omega)} \leq K||u(t,\cdot) - U(\cdot)||^{1-\alpha}_{L^2(\Omega)}, \qquad \text{for all } t > 1,$$

where $0 < \alpha < 1$ is a constant decided by (5.41).

PROOF. Throughout the proof, we use K to denote various constants independent of t. For any $n < p < \infty$, let operator $A : D(A) \to L^p(\Omega)$ be defined as in section 2. Clearly, A^{-1} is bounded in $L^p(\Omega)$. Therefore we have

$$||u(t,\cdot) - U(\cdot)||_{L^p(\Omega)} \leq K||A[u(t,\cdot) - U(\cdot)]||_{L^p(\Omega)}, \tag{5.36}$$

for some positive constant K. Now since $u(t,x)$ is a solution of (2.1)–(2.2) with $u(0,\cdot) \in L^p(\Omega)$ we have $u(t,\cdot) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for $t > 1$. Using (5.36) and an a priori estimate (c.f. Pazy (1983, p.242)), we get, for $t > 1$,

$$\begin{aligned} &||u(t,\cdot) - U(\cdot)||_{W^{2,p}(\Omega)} \\ &\le K(||A[u(t,\cdot) - U(\cdot)]||_{L^p(\Omega)} + ||u(t,\cdot) - U(\cdot)||_{L^p(\Omega)}) \\ &\le K||A[u(t,\cdot) - U(\cdot)]||_{L^p(\Omega)}. \end{aligned} \tag{5.37}$$

Following an argument similar to that in section 2, we have

$$||A[u(t,\cdot) - U(\cdot)]||_{L^p(\Omega)} \le K, \qquad \text{for } t > 1. \tag{5.38}$$

Hence combining (5.37) and (5.38), we obtain

$$||u(t,\cdot) - U(\cdot)||_{W^{2,p}(\Omega)} \le K. \tag{5.39}$$

Now using Theorem 10.1 in Friedman (1976, p.27), we have

$$\begin{aligned} &||u(t,\cdot) - U(\cdot)||_{C^1(\Omega)} \\ &\le K||u(t,\cdot) - U(\cdot)||^{\alpha}_{W^{2,p}(\Omega)}||u(t,\cdot) - U(\cdot)||^{1-\alpha}_{L^2(\Omega)}, \end{aligned} \tag{5.40}$$

where $p > n$, $0 < \alpha < 1$, and moreover, p and α satisfy

$$-1 = (\frac{n}{p} - 2)\alpha + (1-\alpha)\frac{n}{2}. \tag{5.41}$$

Substituting (5.39) into (5.40) gives rise to our conclusion. This completes the proof.

REMARK. As we mentioned, the same approach with a slight modification can be applied to our general type of delayed reaction diffusion equations. Oscillation analysis can be carried out by following the very similar procedure as in Chapter 3. Unfortunately, Hopf bifurcation analysis for the Dirichlet boundary problem is far from easy and computer aid may be required. We will discuss this in a separate chapter.

CHAPTER V

NUMERICAL INVESTIGATION AND HOPF BIFURCATION

5.1 INTRODUCTION

We continue studying Dirichlet boundary problems of the diffusive Nicholson's blowflies equation. For simplicity, we confine the spatial variable to be in one dimension. More precisely, we will consider the modified equation as follows:

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u}{\partial x^2}(t,x) - \tau u(t,x) + \beta\tau u(t-1,x)e^{-u(t-1,x)}, \tag{1.1}$$

$$u(t,0) = u(t,1) = 0, \tag{1.2}$$

$$u(\theta,x) = u_0(\theta,x).$$

where $x \in (0,1)$, $\quad t > 0$, and $\theta \in [-1,0]$. Then the steady state of this system will be the two-point boundary problem

$$\begin{aligned} d\phi_{xx} - \tau\phi + \beta\tau\phi e^{-\phi} = 0, \\ \phi(0) = \phi(1) = 0. \end{aligned} \tag{1.3}$$

As we have shown in chapter 4, the boundary value problem (1.3) has a unique positive solution if and only if

$$(\beta - 1)\tau > d\lambda_1, \tag{1.4}$$

where λ_1 is the principle eigenvalue of $-\frac{\partial^2}{\partial x^2}$ with a (homogeneous) two-point boundary condition. One also observes that $||\phi||_\infty \leq \ln \beta$ for any positive solution ϕ of (1.3).

Our motivation to study (1.3) numerically derives from the studies of Hopf bifurcation. Recall that the linearized equation of (1.1)–(1.2) about the positive steady state is

$$\begin{aligned} \frac{\partial v(t,x)}{\partial t} &= d\frac{\partial^2 v}{\partial x^2}(t,x) - \tau v(t,x) + \beta\tau e^{-\phi(x)}(1-\phi(x))v(t-1,x), \\ v(t,0) &= v(t,1) = 0. \end{aligned} \tag{1.5}$$

The corresponding eigenvalue problem is

$$\begin{aligned} -d\psi_{xx} + \left(\tau + \lambda - \beta\tau e^{-\phi(x)}(1-\phi(x))e^{-\lambda}\right)\psi &= 0, \\ \psi(0) = \psi(1) &= 0. \end{aligned} \tag{1.6}$$

Numerical investigation of Hopf bifurcation requires us to compute the steady state first.

Throughout this chapter, we assume inequality (1.4). Under such assumption, system (1.3) has two solutions. There are several papers concerning the numerical solutions to nonlinear two-point boundary value problems with multiple solutions. Jin's (1992) paper is the only one that can be found with such a title. One should also mention Allgower (1975), and Allgower and McCormick (1978) in this subject.

In this chapter, we propose a new approach which proves to be applicable to system (1.3). We will not compare our approach with others, even though we had tried with other methods to solve our problem.

The rest of this chapter is organized as follows. In section 2, we present the numerical methods applied to equations (1.3). Based on this approach, some numerical simulations are provided in section 3. Following in section 4 is the outlined idea of proving the existence of pure imaginary eigenvalues of the eigenvalue problem (1.6). Necessary conditions are also obtained in this section.

5.2 METHODS

In this section we will describe the approach as follows. Note that $\phi(x)$ is a smooth function and that $\phi(0) = \phi(1) = 0$, there exists at least one point $x_0 \in (0,1)$ such that $\phi'(x_0) = 0$ and $m := \phi(x_0) = \max_{x\in[0,1]} \phi(x)$. Multiplying $-\phi_x$ in (1.3) and then integrating, we obtain

$$-\frac{d}{2}\phi_x^2 + \tau \int_m^{\phi(x)} w\, dw - \beta\tau \int_m^{\phi(x)} we^{-w}\, dw = 0.$$

Evaluate the integrals to get

$$\frac{d}{2}\phi_x^2 = \frac{\tau}{2}(\phi^2 - m^2) + \beta\tau\left[(\phi+1)e^{-\phi} - (m+1)e^{-m}\right]. \tag{2.1}$$

Rewrite (2.1) into

$$\frac{d\phi}{dx} = \pm\sqrt{\frac{\tau}{d}\left[\phi^2 + 2\beta(\phi+1)e^{-\phi}\right] - \frac{\tau}{d}\left[m^2 + 2\beta(m+1)e^{-m}\right]}. \tag{2.2}$$

Let us suppose x_0 is a maximal point such that positive sign is chosen in (2.2) for $x \in [0, x_0)$. Define

$$f(w) := \frac{\tau}{d}\left[w^2 + 2\beta(w+1)e^{-w}\right]. \tag{2.3}$$

Then, solving (2.2) over $[0, x_0]$, we have

$$\int_0^{\phi(x)} \frac{1}{\sqrt{f(w) - f(m)}}\, dw = x. \tag{2.4}$$

Now let $x = x_0$ and note that $\phi(x_0) = m$ we have from (2.4)

$$\int_0^{m} \frac{1}{\sqrt{f(w) - f(m)}}\, dw = x_0.$$

Making the substitution $w = mt$ we then write the above equation as

$$\int_0^{1} \frac{1}{\sqrt{g(m,t)}}\, dt = x_0, \tag{2.5}$$

where,

$$g(s,t) := \frac{\tau}{d}\left[(t^2 - 1) + \frac{2\beta(st+1)e^{-st} - 2\beta(s+1)e^{-s}}{s^2}\right]. \tag{2.6}$$

Next, we will show that the steady state ϕ has only one maximal point. In fact, let us consider the function $F(s)$ defined by

$$F(s) := \int_0^{1} \frac{1}{\sqrt{g(s,t)}}\, dt. \tag{2.7}$$

Note that

$$\int_0^1 \frac{1}{\sqrt{g(s,t)}}\,dt = \int_0^1 \frac{1}{(1-t)^{\frac{1}{2}}}\sqrt{\frac{1-t}{g(s,t)}}\,dt, \tag{2.8}$$

and that by Taylor's expansion

$$\frac{g(s,t)}{1-t} = \frac{\tau}{d}\left[2(\beta e^{-s}-1) + (\beta(s-1)e^{-s}+1)(1-t) + 2\beta R(s,t)\right], \tag{2.9}$$

where,

$$R(s,t) := -\frac{1}{2}se^{-s}(1-t)^2 + s(s+1)e^{-s}(1-t)^2\sum_{k=0}^{\infty}\frac{s^k(1-t)^k}{(k+3)!}$$
$$- s^2e^{-s}(1-t)^3\sum_{k=0}^{\infty}\frac{s^k(1-t)^k}{(k+3)!}.$$

Clearly, for $0 \le t \le 1$ and $0 \le s \le \ln\beta$, we have

$$|R(s,t)| \le K_1(1-t)^2 + (\ln\beta)^2(1-t)^3, \tag{2.10}$$

where, $K_1 = \frac{1}{2}e^{-1} + \ln\beta(\ln\beta+1)$. Hence, for any $0 \le s_0 < \ln\beta$, there exists a constant K, depending on s_0, such that $\frac{1-t}{g(s,t)} \le K$ for all $t \in [0,1]$ and $0 \le s \le s_0$. Therefore, the improper integral $\int_0^1 \frac{1}{\sqrt{g(s,t)}}\,dt$ is uniformly convergent for $0 \le s \le s_0$. This implies that the function $F(s)$ is continuous for $0 \le s < \ln\beta$. By the same token, we can show that the improper integral $\int_0^1 \frac{\frac{\partial g(s,t)}{\partial s}}{\sqrt{g^3(s,t)}}\,dt$ is also uniformly convergent for $0 \le s \le s_0$, where again, $s_0 \in [0, \ln\beta)$. Hence by Leibniz's rule,

$$F'(s) = -\frac{1}{2}\int_0^1 \frac{\frac{\partial g(s,t)}{\partial s}}{\sqrt{g^3(s,t)}}\,dt.$$

Moreover, we have the following.

PROPOSITION 2.1. *Let $g(s,t)$ be defined by (2.6). Then*

$$\frac{\partial g(s,t)}{\partial s} \leq 0,$$

for any $t \in [0,1]$.

PROOF. In fact, by direct calculation one has

$$\frac{\partial g(s,t)}{\partial s} = \frac{2\beta\tau}{d}\frac{T(s,t)}{s^3},$$

where $T(s,t) := \left[-(st+1)^2 - 1\right] e^{-st} + \left[(s+1)^2 + 1\right] e^{-s}$. So it is sufficient to show that $T(s,t) \leq 0$ for any $t \in [0,1]$. Note that $T(s,1) = 0$ and that $\frac{\partial T(s,t)}{\partial t} = s^3 t^2 e^{-st} \geq 0$, and so we have $T(s,t) \leq 0$ for $t \in [0,1]$. The proof is complete.

Thus, $F(s)$ is an increasing and continuously differentiable function. Furthermore, one can show the following.

THEOREM 2.2. *Let $F(s)$ be defined by (2.7). Then $F(s)$ is increasing and continuously differentiable for $s \in [0, \ln\beta)$. Moreover,*

$$\lim_{s \to \ln\beta} F(s) = +\infty. \tag{2.11}$$

PROOF. We just need to show that (2.11) holds. In practice, for any $M > 0$, we choose

$$s_1 = \max\left\{0, \quad \ln\beta - \ln\left(1 + \frac{e^{-\frac{M}{K_2}}}{2}\right)\right\},$$

where, $K_2 = \sqrt{\frac{d}{\tau}}\sqrt{\frac{1}{\beta+2+2\beta(K_1+(\ln\beta)^2)}}$. Let $\delta = 1 - 2(\beta e^{-s_1} - 1)$. Then, $0 \le s_1 < \ln\beta$ and $0 < \delta < 1$. Now for any $s_1 \le s < \ln\beta$, since $F(s)$ is monotone increasing, we obtain by using (2.8)–(2.10)

$$
\begin{aligned}
F(s) &\ge F(s_1) \\
&= \int_0^1 \frac{1}{\sqrt{g(s_1,t)}}\,dt \\
&= \int_0^1 \frac{1}{(1-t)^{\frac{1}{2}}}\sqrt{\frac{1-t}{g(s_1,t)}}\,dt \\
&\ge \sqrt{\frac{d}{\tau}}\int_0^\delta \frac{1}{\sqrt{1-t}\sqrt{[|\beta(s_1-1)e^{-s_1}+1|+1+2\beta(K_1+(\ln\beta)^2)]\,(1-t)}}\,dt \\
&\ge K_2\int_0^\delta \frac{1}{(1-t)}\,dt \\
&= K_2\ln\left(\frac{1}{1-\delta}\right) \ge M.
\end{aligned}
$$

Therefore $F(s) \to +\infty$ as $s \to \ln\beta$. This completes the proof.

The above theorem together with equation (2.5) rules out that the steady state has more than one maximal point. If on the contrary, there are two maximal points, say x_0 and x_1, then it is easy to show that $\phi(x_0) = \phi(x_1) = \ln\beta$. This is impossible according to equations (2.5) and (2.11). Therefore, we have the following corollary.

COROLLARY 2.3. *The steady state has only one maximal point.*

So far, we have made it clear that there is a unique maximum satisfying equation (2.5). Let us still denote by x_0 the unique maximal point. Then, $x_0 = \frac{1}{2}$ (c.f. Allgower (1975)). Therefore according to equation (2.5), one can solve $F(s) = \frac{1}{2}$ for the maximum value m. Newton iteration is applied here and the convergence of this method is guaranteed since $F(s)$ is monotone increasing for $0 \leq s < \ln\beta$. An initial guess is chosen slightly less than $\ln\beta$. Note that $\ln\beta$ cannot be an initial guess.

Since the expression of $F(s)$ is an integration, one needs to evaluate the integral before engaging in Newton iteration. Note that $g(s,1) = 0$ and hence $t = 1$ is a singular point for the integration. Therefore we choose a formula of Gauss type (c.f. Davis and Rabinowitz (1984), p.179):

$$\int_0^1 \frac{h(x)}{(1-x)^{\frac{1}{2}}}\, dx = \sum_{k=1}^{n} \omega_k h(x_k) + \frac{2^{4n+1}\left[(2n)!\right]^3}{(4n+1)\left[(4n)!\right]^2} h^{(2n)}(\xi).$$

Here, $0 \leq \xi \leq 1$, $x_k = 1 - \xi_k^2$, ξ_k is the k^{th} positive zero of the Legendre polynomial $P_{2n}(x)$ and ω_k^{2n} is the weight corresponding to ξ_k in the rule G_{2n}, i.e. a 2n-point interpolation by Gauss rule (c.f. Davis and Rabinowitz (1984), p.97). To apply this formula we need to rewrite $F(s)$ into (2.8). By using the same scheme as above, one can also compute

$$F'(s) = -\frac{1}{2}\int_0^1 \frac{1}{(1-t)^{\frac{1}{2}}} \sqrt{\frac{1-t}{g^3(s,t)}}\, \frac{\partial g(s,t)}{\partial s}\, dt,$$

where,

$$\lim_{t\to 1^-} \sqrt{\frac{1-t}{g^3(s,t)}}\, \frac{\partial g(s,t)}{\partial s} = \sqrt{\left[\frac{d}{2\tau(\beta e^{-s}-1)}\right]^3} \left(-\frac{2\beta\tau}{d}e^{-s}\right),$$

is finite for $0 \leq s < \ln\beta$.

Now for any $x \in (0, x_0)$, let $\alpha = \phi(x)$. Viewed as a function of α with fixed

m, equation (2.4) can be rewritten as

$$G(\alpha) := \int_0^1 \frac{\alpha\, dt}{\sqrt{f(\alpha t) - f(m)}} = x.$$

One can also show that for $0 < \alpha < m, \quad G'(\alpha) > 0$. Therefore again, Newton iteration will be applied. In order to get an ideal initial guess of α, we usually start with $x < x_0$ near x_0 and hence α is closer to m. On the other hand, since $\alpha < m$, the integral has no singular point for $t \in [0, 1]$. Thus we choose Simpson's formula (c.f. Davis and Rabinowitz (1984), p.57–p.58) to evaluate the integral.

5.3 NUMERICAL RESULTS

We have proposed an approach to solve two-point boundary problem (1.3). Although there are many numerical methods in treating two-point boundary problems together with a well–developed computer solver (see for instance, Allgower and McCormick (1978), Cash (1986, 1988), Cash and Wright (1990, 1991, 1995), Duvallet (1990), Jacobs (1990), Jankowski (1991), Kalaba and Spingarn (1977), Watson and Scott (1987)), however, very few of them are successful in solving our problems. The difficulties lie in that two solutions (the zero solution and the positive solution) exist in equation (1.3). We had tried solver TWPBVP, which is a Fortran program based on the mono-implicit Runge-Kutta formula and an adaptive mesh refinement for solving two-point boundary problems (see Cash (1986, 1988), Cash and Wright (1990, 1991, 1995), and the references therein for details). Unfortunately, when applied to our equation, this software always ends up with the zero solution, no matter what ranges of parameters are chosen. We also applied other methods, for example more straightforward methods like a difference scheme, but the results are still unsatisfactory. Surprisingly, the proposed methods in section 2 always work. One should mention the well-known shooting method, which is expected to be applicable to our problem. It might be a good idea to compare the shooting method with the proposed methods in section 2. But for the time being, we just present some numerical examples based on our methods without comparing with any other methods. Fig 5.1 and Fig 5.2 illustrate that, for larger β and τ, the top of the curves looks like a flat roof. As β and τ decrease, the flat roof gradually disappears (see Fig 5.3 and Fig 5.4). Nevertheless, in each of the figures, there is only one maximum contained within the interval.

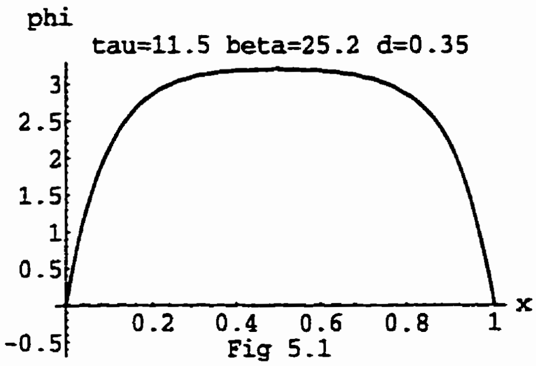

Fig 5.1

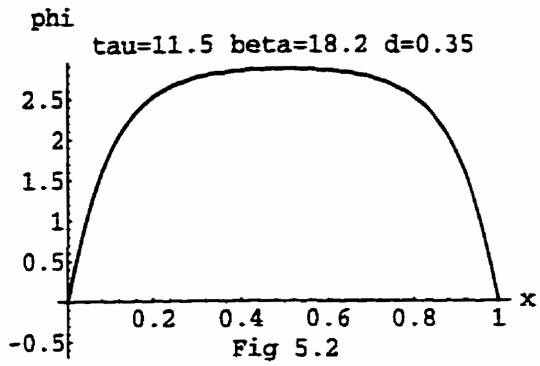

Fig 5.2

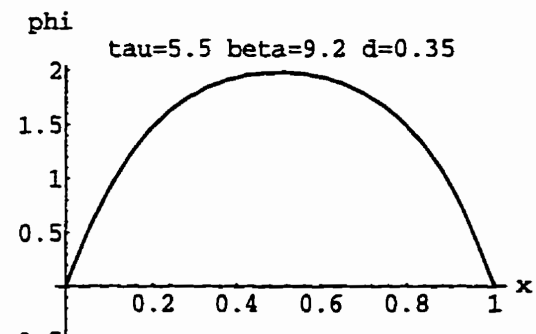

Fig 5.3

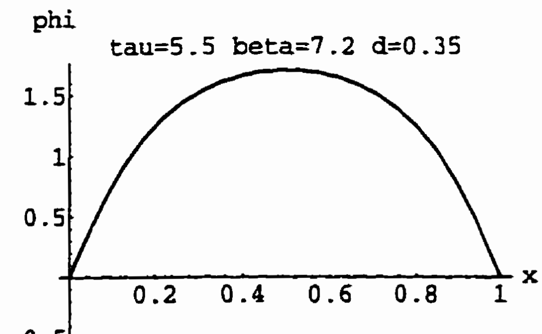

Fig 5.4

5.4 DISCUSSION ON HOPF BIFURCATION

As we mentioned in section 1, our motivation to study the numerical simulation of a positive steady state is that we desire to determine periodic solutions bifurcating from that positive steady state. There are very few papers dealing with the Hopf bifurcation analysis for Dirichlet boundary problems. For the diffusive Hutchinson equation with Dirichlet boundary conditions, Busenberg and Huang (1996) prove the existence of a Hopf bifurcation by using perturbation methods together with the implicit function theorem. Unfortunately, their approach can not be applied to our case. In what follows, a new idea is presented to solve the eigenvalue problem.

Recall equations (1.6), the characteristic equation of the linearized equation about a positive steady state is,

$$\begin{aligned} -d\psi_{xx} + \left(\tau + \lambda - \beta\tau e^{-\phi(x)}(1-\phi(x))e^{-\lambda}\right)\psi = 0, \\ \psi(0) = \psi(1) = 0. \end{aligned} \tag{4.1}$$

For any $\lambda \in \mathbb{C}$, let $G(x,y,\lambda)$ be defined by

$$G(x,y,\lambda) = \begin{cases} \frac{\sin[(-\tau-\lambda)^{\frac{1}{2}}x]\sin[(-\tau-\lambda)^{\frac{1}{2}}(1-y)]}{(-\tau-\lambda)^{\frac{1}{2}}\sin(-\tau-\lambda)^{\frac{1}{2}}} & \text{for } 0 \le x \le y \le 1, \\ \frac{\sin[(-\tau-\lambda)^{\frac{1}{2}}y]\sin[(-\tau-\lambda)^{\frac{1}{2}}(1-x)]}{(-\tau-\lambda)^{\frac{1}{2}}\sin(-\tau-\lambda)^{\frac{1}{2}}} & \text{for } 0 \le y \le x \le 1. \end{cases} \tag{4.2}$$

The function G is known as *Green's function* of equations (4.1) (see Coddington and Levinson (1955), p.192) and equations (4.1) can be rewritten as

$$\psi(x) = \beta\tau e^{-\lambda}\int_0^1 G(x,y,\lambda)e^{-\phi(y)}(1-\phi(y))\psi(y)\,dy. \tag{4.3}$$

We are interested in finding non-zero $\psi(x)$ with $\psi(0) = \psi(1) = 0$, such that (4.3) possesses pure imaginary eigenvalue $\lambda = bi$, where $b \in \mathbb{R}^+$. Let's consider (4.3) together with the constraint

$$||\psi||^2_{L^2(0,1)} := \int_0^1 |\psi|^2 = 1. \tag{4.4}$$

We intend to look for b and ψ such that $(b, \psi) \neq 0$, i.e. $b \neq 0$ and $0 \neq \psi \in H^1_0(0,1) \cap H^2(0,1)$. Here is the idea. Let

$$U_0 := \left\{\psi \in H^1_0(0,1) \cap H^2(0,1), \quad ||\psi||_{L^2(0,1)} = 1\right\}.$$

Choose $\psi_0 \in U_0$, and then find $b_0 \in [\frac{\pi}{2}, \pi]$, such that $\psi_1 \in U_0$, where ψ_1 is defined by (4.3),

$$\psi_1(x) = \beta\tau e^{-b_0 i} \int_0^1 G(x, y, b_0 i) e^{-\phi(y)}(1 - \phi(y))\psi_0(y)\, dy.$$

Following this procedure, one can produce sequences $\{\psi_n\}$ and $\{b_n\}$, where $\psi_n \in U_0$ and $b_n \in [\frac{\pi}{2}, \pi]$ for each nonnegative integer n. Applying the weak compact theorem of bounded sequence in reflexive Banach space (see Zeidler (1990), p.255), we can pick up convergent subsequences and still use $\{\psi_n\}$ and $\{b_n\}$ for simplicity, such that $\psi_n \to \psi$ weakly and $b_n \to b$ for some $\psi \in U_0$ and $b \in [\frac{\pi}{2}, \pi]$. Moreover in our case, one can show that this weak convergence results in strong convergence. Therefore (b, ψ) is the nontrivial solution to (4.3). Obviously, the key step of realizing this idea is to prove the existence of $\{b_n\}$. We will accomplish this task in the near future.

Before we end our discussion, let's derive necessary conditions for the existence of non-zero solutions of equations (4.3)–(4.4). That is equivalent to finding necessary conditions for non-zero solutions of

$$\beta\tau\int_0^1 e^{-\phi}(1-\phi)|\psi|^2 + \frac{b}{\sin b} = 0, \tag{4.5}$$

$$d||\psi_x||^2_{L^2(0,1)} + \tau + b\cot b = 0, \tag{4.6}$$

for $\psi \in H_0^1(0,1)\cap H^2(0,1)$. Notice that $||\psi_x||^2_{L^2(0,1)} \geq \lambda_1 ||\psi||^2_{L^2(0,1)} = \lambda_1$. Then equation (4.6) implies

$$0 \leq \tau \leq -b\cot b - d\lambda_1. \tag{4.7}$$

On the other hand, equation (4.5) gives,

$$\begin{aligned}
\frac{b}{\sin b} &= \beta\tau\int_0^1 e^{-\phi}(\phi-1)|\psi|^2 \\
&= \beta\tau\left[\int_{\Omega_1^\infty} e^{-\phi}(\phi-1)|\psi|^2 + \int_{\tilde{\Omega}_1^\infty} e^{-\phi}(\phi-1)|\psi|^2\right] \\
&\leq \beta\tau\int_{\tilde{\Omega}_1^\infty} e^{-\phi}(\phi-1)|\psi|^2 \\
&\leq \beta\tau e^{-2}\int_{\tilde{\Omega}_1^\infty} |\psi|^2 \\
&\leq \beta\tau e^{-2},
\end{aligned}$$

where, $\Omega_1^\infty := \{x \in (0,1), \quad \phi(x) < 1\}$ and $\tilde{\Omega}_1^\infty$ is the complement of Ω_1^∞ in $(0,1)$. Therefore we have

$$\tau \geq \frac{1}{\beta e^{-2}}\frac{b}{\sin b}. \tag{4.8}$$

A combination of (4.7) and (4.8) gives

$$\frac{1}{\beta e^{-2}} \frac{b}{\sin b} \leq \tau \leq -b \cot b - d\lambda_1. \tag{4.9}$$

Here we perceive that $\frac{b}{\sin b} > 0$ and that $\cot b < 0$ (see chapter 4 for details). Now we can conclude following.

THEOREM 4.1. *Necessary conditions in order that equation (4.3) has non-trivial solutions are that $\beta e^{-2} > 1$ and $\tau \geq \tau_c$, where,*

$$\tau_c = -b_c \cot b_c - d\lambda_1, \tag{4.10}$$

and b_c is the unique solution of

$$\frac{b}{\beta e^{-2} \sin b} = -b \cot b - d\lambda_1 \qquad \textit{for } b \in \left[\frac{\pi}{2}, \pi\right). \tag{4.11}$$

PROOF. Let's consider two functions

$$f(b) := \frac{b}{\beta e^{-2} \sin b},$$

and

$$g(b) := -b \cot b - d\lambda_1.$$

Both functions are monotone increasing for $b \in [\frac{\pi}{2}, \pi)$. Notice that

$$h(b) := f(b) - g(b) = \left(\frac{1}{\beta e^{-2}} + \cos b\right) \frac{b}{\sin b} + d\lambda_1.$$

The existence of nontrivial solutions of (4.3) implies there exists $b \in [\frac{\pi}{2}, \pi)$, such that $f(b) < g(b)$. This implies $\frac{1}{\beta e^{-2}} + \cos b < 0$. In other words, $\beta e^{-2} > 1$. Consequently, there always exists $b_c \in [\frac{\pi}{2}, \pi)$, such that $f(b_c) - g(b_c) = 0$, because

$$f(\frac{\pi}{2}) - g(\frac{\pi}{2}) > 0,$$

and

$$\lim_{b \to \pi^-} (f(b) - g(b)) = -\infty.$$

Moreover, we can show that b_c is the unique root of $h(b) = 0$ in $[\frac{\pi}{2}, \pi)$. In fact, without loss of generality, we suppose b_c is the smallest one of the zeros of $h(b)$ in $[\frac{\pi}{2}, \pi)$, that is, if there is another b_1, such that $h(b_1) = 0$, then $b_c \leq b_1$. Now, since $h(b_c) = 0$, we have

$$\cos b_c = -\frac{1}{\beta e^{-2}} - \frac{\sin b_c}{b_c} d\lambda_1 < -\frac{1}{\beta e^{-2}}.$$

Claim: $h(b)$ is monotone decreasing for $b \in [\pi - \arccos \frac{1}{\beta e^{-2}}, \pi)$. In fact, since $\frac{1}{\beta e^{-2}} + \cos b < 0$ and $\frac{\sin b - b \cos b}{\sin^2 b} > 0$ for $b \in [\pi - \arccos \frac{1}{\beta e^{-2}}, \pi)$, we have,

$$h'(b) = -b + \left(\frac{1}{\beta e^{-2}} + \cos b \right) \frac{\sin b - b \cos b}{\sin^2 b} < 0.$$

Therefore, $h(b) < 0$ for $b \in [b_c, \pi)$. This implies that b_c is the unique zero of $h(b)$ in $[\frac{\pi}{2}, \pi)$ and also $f(b) < g(b)$ for $b \in [b_c, \pi)$. Furthermore, since both $f(b)$ and $g(b)$

are increasing, (4.8) implies that

$$\tau \geq \frac{b}{\beta e^{-2} \sin b} \geq \frac{b_c}{\beta e^{-2} \sin b_c} = \tau_c.$$

This completes the proof.

REMARK. This section gives a brief description of solving eigenvalue problems. Our idea also provides a procedure for a computer to search for pure imaginary eigenvalues. Hopefully, with aid of a computer, one can investigate Hopf bifurcations. Fortran program BIFDD developed by Hassard (1986) might be applicable to locate Hopf bifurcation points and to analyze their stabilities, provided that relatively accurate parameters, such as τ and bi are available to run the program.

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APPENDIX

SUMMARY OF NICHOLSON'S EXPERIMENTS AND MODELS

A.1 NICHOLSON'S BLOWFLIES EXPERIMENTS

In this section, we will summarize Nicholson's experiments on the Australian sheep blowfly *Lucilia cuprina* (see Nicholson (1954) for details).

EXPERIMENT ONE: Intraspecific Competition

The details of this experiment are described in Nicholson (1948). In the following, we sketch the general ideas and conclusions. In this experiment, Nicholson uses a number of glass tubes, each of which contains a different number of freshly hatched *Lucilia cuprina* larvae, with one gram of homogenized bullock's brain as food. Such series of cultures are replicated many times. The number of emerging adults is then plotted against the number of larvae from which they have been derived. Let us now look at the figure (Fig A.1) adopted from Nicholson (1954).

This figure shows that, above a certain critical density, a further increase in larval density causes not only an increase in the percentage mortality but also an actual reduction in the number of adults produced from the gram of food

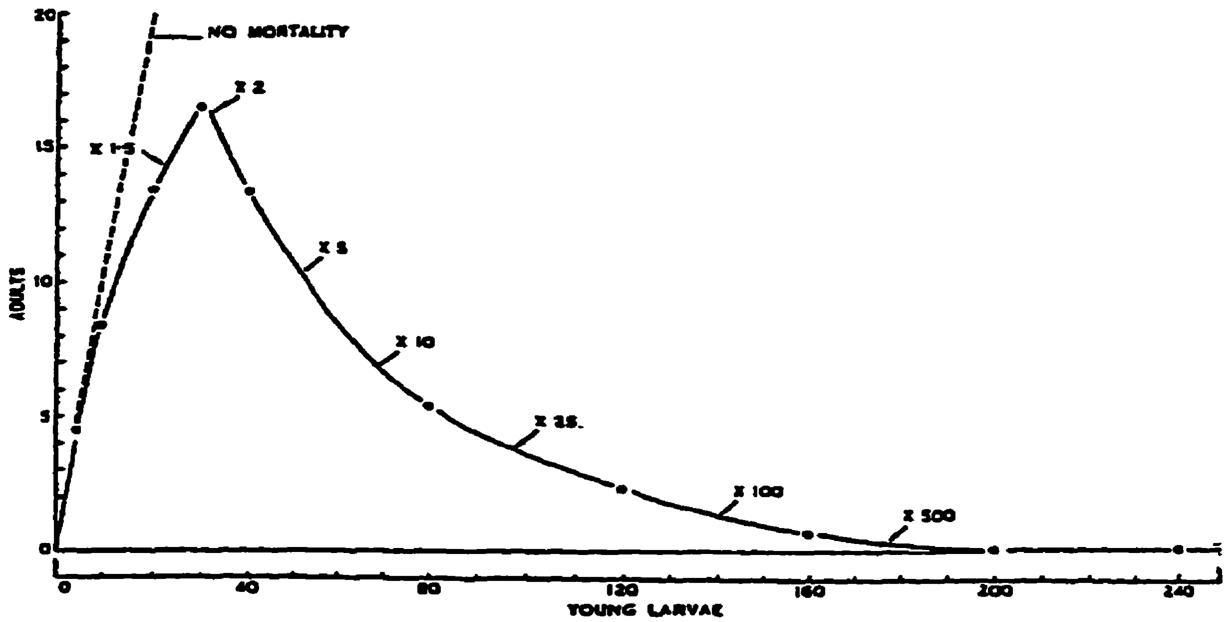

FIG A.1. Graph from Nicholson (1954, p.18)

consumed. This is because increasing quantities of food are consumed by the increasing number of larvae which fail to mature. Starting at a low density, the tendency is for a population to grow progressively as long as the percentage of offspring destroyed by competition equals the percentage which is surplus to that necessary for the replacement of mature animals when they die. Consequently, the greater the power of increase (i.e. the ratio of offspring to parents in the absence of competition effects) the smaller is the number of adults produced from a given quantity of food at equilibrium. This is clearly shown in the figure, in which the points indicated by number preceded by the multiplication sign show the equilibrium levels of larvae and adults at each power of increase represented by these numbers. Inspection shows that any departure of density from the equilibrium level for any given power of increase (which is assumed to remain constant) leads to an oscillation about this level; unless the power of increase is less than 2 (in the example given), when an asymptotic approach to this level results.

This experiment demonstrates the important part played by the wide scatter in the properties of animals upon the reaction of populations to depletion of their requisites. Were there no such scatter of properties and opportunities, increasing density would produce no mortality until the point is passed at which the amount of food obtained by each individual falls below that necessary for the production of viable pupae.

EXPERIMENT TWO: Mechanism of Balance

In this experiment, a population is maintained under as nearly constant conditions as possible. The culture room is held at $25°C$, water and sugar for the adults, and also larval food (to which the adults does not have access), being provided in excess of requirements at all times. The governing requisite is ground liver, which is available to the adults alone, and each day 0.5g of this is placed in the breeding cage. The results of the experiment are graphed (Fig A.2) as follows.

From this figure, Nicholson reads that the outstanding characteristic of the culture is the maintenance of violent and fairly regular oscillations in the density of the adult population. It is also observed that significant egg generation occurs only when the adult population is very low. At higher densities competition amongst

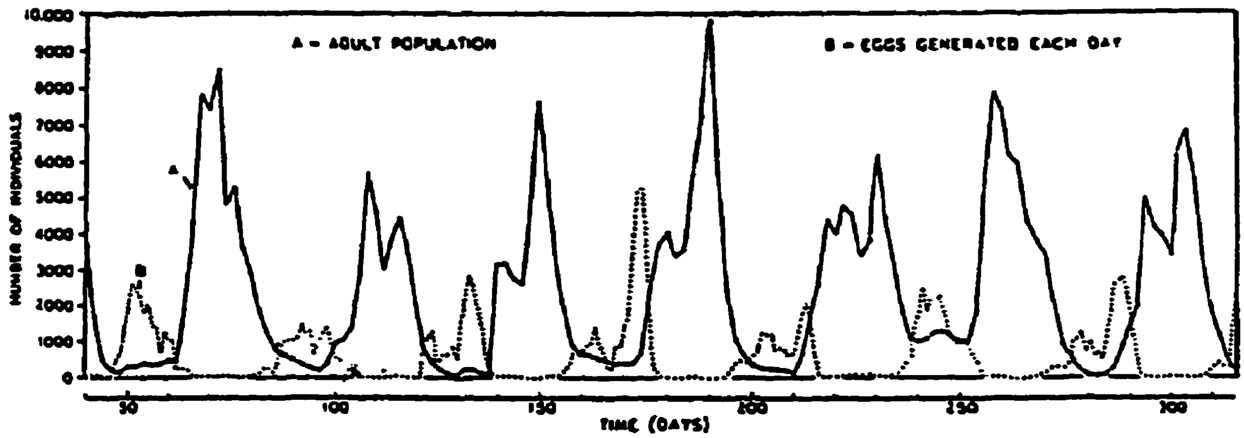

FIG A.2. Graph from Nicholson (1954, p.21)

the adults for the ground liver is so severe that few or no individuals secure sufficient food to enable them to develop eggs. Normal mortality, therefore, cause the population to dwindle until the consequent reduced severity of competition permits some individuals to secure adequate liver and so to lay eggs. The eggs then generated in due time give rise to new adults, which lead to a rapid increase in the adult population, and the resultant overcrowding causes virtual cessation of egg production. A new cycle of oscillation then begins.

The system of balance is often highly oscillatory, simply because animals commonly take a significant time to grow up, so causing a time lag between stimulus and reaction. During this lag period the stimulus continues to generate more and more reaction, and this continues to come into operation for a similar lag period after reaction has removed the stimulus. As Nicholson explains, if increased acquisition of food are to cause fully mature adults to come into being immediately (instead of merely initiating the subsequent production of eggs and the still later development of adults) this prompt reaction will cause the system to be non-oscillatory. This is because reaction will cause the population first to approach, and then to maintain the equilibrium density of the species under the prevailing conditions, this being the density at which production of offspring precisely compensates for the loss of adults by death; for any departure from this level would immediately bring compensating reaction into play, and this will cease as soon as the equilibrium density is attained again. This is the balancing mechanism which holds population density in general relation to the prevailing conditions.

To show that the general density level has relation to the environment, Nicholson does many other experiments. By adjusting supplies of ground liver, he finds that average density is almost precisely proportional to the supply of the governing requisite. Furthermore, by a series of experiments testing other governing factors, Nicholson concludes that the governing reaction does not merely operate to oppose any departures of population from its equilibrium density, but also enables populations to adjust themselves to withstand very great environmental stresses (particularly when their inherent reproductive capacity is high), and to maintain themselves in a state of balance under widely different environmental conditions. Moreover, the reduction in density which adverse factors produced as a primary effect is always opposed by compensatory reaction, being lessened, or even converted into an increase in density, when the population adjusts itself to the continued operation of the adverse factor.

A.2 BLOWFLIES MODELLING

Based on Nicholson's experimental data, several models have been proposed so as to fit these data. For example, May (1976) simulate one of Nicholson's experiments using a form of the Nicholson-Bailey equation with time delay,

$$\frac{dN(t)}{dt} = rN\left[1 - \frac{N(t-T_d)}{K}\right]$$

(see also Readshaw and Cuff (1980) for an alternative modelling). Here, we are only interested in presenting the mathematical model developed by Gurney, Blythe, and Nisbet (1980). Their model agrees with Nicholson's data better than that of May (1976) and Readshaw and Cuff (1980). They start with the Malthusian law

$$\frac{dN}{dt} = R - D, \tag{A.2.1}$$

where R is the rate of recruitment to the adult population and D the total adult death rate. Usually, $D = \delta N$ where δ is a constant independent of N. Before depicting R further, they assume the following.

(H1) The rate at which eggs are produced depends only on the current size of the adult population.

(H2) All eggs which develop into sexually mature adults take exactly T_d time units to do so.

(H3) The probability of a given egg maturing into a viable adult depends only on the number of competitors of the same age.

These assumptions imply that the rate of recruitment at time t can only be a function of the size of the adult population at time $t - T_d$, i.e.

$$R = R(N(t - T_d))$$

Moreover, they explain the appropriate choice of an algebraic form of $R(N)$. According to Nicholson's experiments, egg to adult survival may reasonably be expected to be density independent, so that the rate of adult recruitment at time t will be directly proportional to the rate at which eggs were being laid at time $t - T_d$. Secondly, it seems to suppose that, in the presence of excess food, the total rate at which eggs are produced by a population of N adults will be directly proportional to N. However, when food is supplied at a limited rate, intraspecific competition will clearly act to reduce the average per capital fecundity of the members of large populations to well below its physiological saturation value. Therefore, any plausible functional form for $R(N)$ must go to zero as N becomes either very large or very small. In addition, it seems likely that most recruitment curves will display a single maximum (see Fig A.1) at an intermediate population whose size is determined by the available resources. They therefore chose to represent $R(N)$ by a simple function which displays all these properties

$$R(N) = PNe^{-N/N_0}$$

where, P is the maximum possible *per capita* egg production rate (corrected for egg to adult survival) and N_0 is the population size at which the population as a whole achieves maximum reproductive success.

Equation (A.2.1) therefore becomes into

$$\frac{dN(t)}{dt} = PN(t - T_d)e^{-N(t-T_d)/N_0} - \delta N(t) \tag{A.2.2}$$

This model is in fact conceals a wealth of complexity as well as provides a clear understanding of Nicholson's observation. Comparing the following graph (Fig A.3) based on the numerical simulation of the equation (A.2.2) with the experimental results (Fig A.2), one sees that this model provides a satisfying qualitative fit to Nicholson's blowflies data.

A.3 COLLECTION OF MATHEMATICAL RESULTS

In this section, we will collect, to the extent of availability, those mathematical results about the Nicholson's blowflies equation in the form of

$$\begin{aligned} \dot{N}(t) &= -\delta N(t) + PN(t-\tau)e^{-aN(t-\tau)}, && \text{for } t > 0, \\ N(\theta) &= N_0(\theta), && \text{for } \theta \in [-\tau, 0]. \end{aligned} \tag{A.3.1}$$

There are three categories of theoretical results in this equation: attractivity (or stability); oscillation; and periodicity. It is well-known that periodic solutions of delayed differential equation are well considered by mathematicians in the studies of dynamical systems, but there are no specific contribution to (A.3.1). Kulenovic and Ladas (1987) give a sufficient condition for which the solutions of (A.3.1)

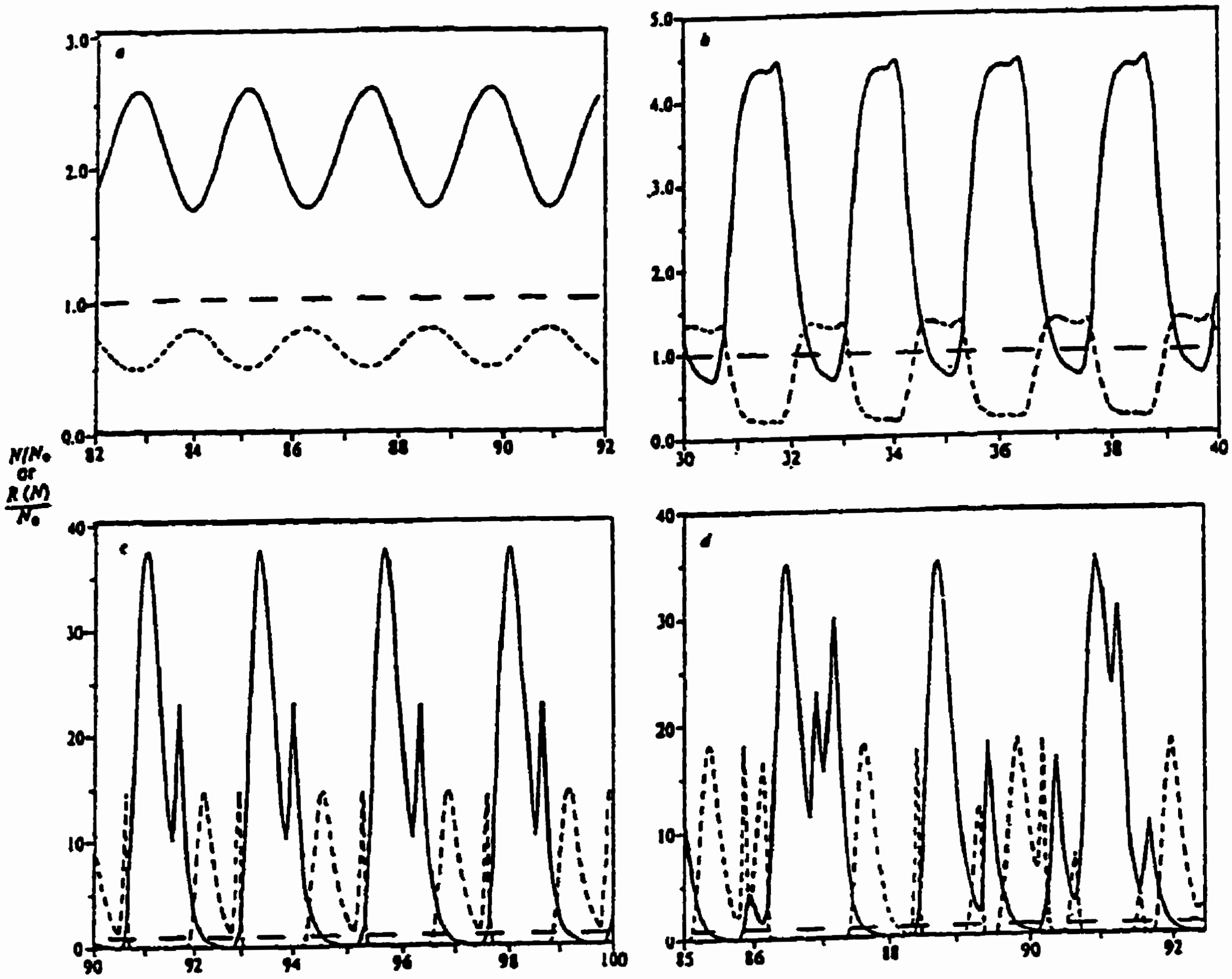

FIG A.3. Graph from Gurney *et al* (1980, p.21).

oscillate. More precisely, they obtain the following.

THEOREM A.3.1. *Assume that*

$$\frac{P}{\delta} > e, \tag{A.3.2}$$

and that

$$\delta\tau e^{\delta\tau}\left[\ln\left(\frac{P}{\delta}\right) - 1\right] > e^{-1}. \tag{A.3.3}$$

Then the solution $N(t)$ of (A.3.1) oscillates about its positive equilibrium state $N^* := \frac{1}{a}\ln\left(\frac{P}{\delta}\right)$. Furthermore, If (A.3.2) is replaced by the condition

$$\frac{P}{\delta} > e^2, \tag{A.3.4}$$

then $N(t)$ oscillates about N^* if and only if (A.3.3) holds.

Global attractivities of the nonnegative equilibria have been attracting more researchers who provide varieties of conditions when this simple global dynamics takes place. The following theorem is due to So and Yu (1994), giving a condition under which the zero solution is a global attractor.

THEOREM A.3.2. *Assume*

$$0 < \frac{P}{\delta} \leq 1, \tag{A.3.5}$$

then the solution $N(t)$ of (A.3.1) tends to zero as $t \to \infty$.

As So and Yu (1994) point out, when

$$\frac{P}{\delta} > 1, \tag{A.3.6}$$

then the zero solution no longer attracts any nontrivial solution. Instead, the positive equilibrium should be considered.

THEOREM A.3.3. *Assume that (A.3.6) holds and that*

$$(e^{\delta\tau} - 1)(\frac{P}{\delta} - 1) \leq 1. \tag{A.3.7}$$

Let $N_0 \in C([-\tau 0];\mathbb{R}^+)$ *with* $N_0(0) > 0$ *and let* $N(t)$ *be the unique solution of equation (A.3.1). Then* $\lim_{t\to\infty} N(t) = N^*$.

THEOREM A.3.4. *Assume that (A.3.6) holds and that*

$$(e^{\delta\tau} - 1)\ln\left(\frac{P}{\delta}\right) \le 1. \tag{A.3.8}$$

Then any non-trivial solution $N(t)$ *of (A.3.1) satisfies* $\lim_{t\to\infty} N(t) = N^*$.

Theorem A.3.3 is due to Kulenovic, Ladas, and Sficas (1992), while Theorem A.3.4 is one of the contributions from So and Yu (1994) . Evidently, condition (A.3.8) is an improvement over condition (A.3.7) since $\ln\left(\frac{P}{\delta}\right) < \frac{P}{\delta} - 1$ for $\frac{P}{\delta} > 1$. Along the same vein, further progress is made recently by Li (1996).

THEOREM A.3.5. *Assume that (A.3.6) holds and that one of the following three conditions is satisfied :*

$$(e^{\delta\tau} - 1)\ln\left(\frac{P}{\delta}\right) \le 1, \tag{A.3.8}$$

$$(e^{\delta\tau} - 1)\ln\left(\frac{P}{\delta}\right) < 1 + \frac{1}{aN^*}, \text{ and } aN^* \ge \frac{\sqrt{5}-1}{2}, \tag{A.3.9}$$

$$(e^{\delta\tau} - 1)\ln\left(\frac{P}{\delta}\right) \le 1 + \frac{1}{aN^*}, \text{ and } aN^* > \frac{\sqrt{1+4\sqrt{3}}-1}{2}. \tag{A.3.10}$$

Then, the positive equilibrium N^* *of (A.3.1) is a global attractor.*

For these three theorems of attractivity, the general idea of the proofs is similar. Distinguished from this approach, geometrical (or topological) methods and monotone method provide us different way. Using the limiting equation theory, Karakostas, Philos, and Sficas (1992) obtain the following theorem.

THEOREM A.3.6. *If the condition*

$$1 < \frac{P}{\delta} < e^2 \tag{A.3.11}$$

is satisfied, then N^* is uniformly stable. Also, if

$$1 < \frac{P}{\delta} < e, \tag{A.3.12}$$

then N^* is uniformly asymptotically stable.

More than that, Kuang (1992) claims that the following conclusion.

THEOREM A.3.7. *Assume that (A.3.11) holds. Then the unique positive steady state of (A.3.1) is absolutely globally asymptotically stable.*

As we know in the case of (A.3.12), equation (A.3.1) is a monotone dynamical system. Smith (1995) therefore concludes as follows:

THEOREM A.3.8. *If $\frac{P}{\delta} < 1$, then the trivial solution of (A.3.1) attracts all other solutions. If $1 < \frac{P}{\delta} \leq e$, the nontrivial equilibrium N^* attracts all nontrivial solutions of (A.3.1).*

In the case of (A.3.2), however, the system (A.3.1) is not quasimonotone any longer. Exponential ordering is introduced by Smith and Thieme (1990). Along this approach, another criterion is followed (see Smith (1995) for the proof).

THEOREM A.3.9. *If (A.3.2), $P\tau < e^2$ and $P\tau < e^{1+\delta\tau}$, then the positive equilibrium attracts all nontrivial solutions of (A.3.1) .*

REMARK. We have presented all the results related to the Nicholson's blowflies equation, from Nicholson's creative experiments to the modelling equation by Gurney *et al*, and then to the mathematical analysis of this equation for the varieties of dynamics by many researchers. Mathematically, the above listed theorems conclude that the zero solution and the positive equilibrium are global attractors in the cases of (A.3.5) and (A.3.11) respectively, without any restriction imposed on the time delay. In the case of (A.3.4), one may ask what is the sharpest condition for τ such that global attractivity of the positive equilibrium is guaranteed? This condition should exist. In fact, it follows from Theorem A.3.3, for example, that the positive steady state is a global attractor at least for small time delay τ. On the other hand, one can claim stable periodic solutions for large τ, following a similar procedure as in chapter 3 in this thesis. Therefore, the positive equilibrium loses its attractivity and hence there exists a critical condition for τ which drives the positive equilibrium to change from global attractivity to local stability or to instability (and then stable periodic solutions arise).