AN EXPLORATION OF GRADE 8 STUDENTS' FRACTION SENSE

by

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ABSTRACT

"Fraction sense" refers to an individual's ability to understand the meaning of fractions; to reason qualitatively about the absolute and relative size of fractions; and to make logical judgments about the reasonableness of calculations with fractions based on one's understanding of fractional numbers and the effect of operations on those numbers. In my research I explored what fraction sense grade 8 students possessed. Specific questions I wished to investigate were:

- What do grade 8 students understand about the meaning of fractions and written fraction notation?
- What understanding do students have for the absolute and relative size of fractions and what methods do they use to determine this relative size?
- How does their fraction sense change with classroom based instruction that stresses fraction sense?

These questions were investigated by interviewing (twice) seven students, of varying ability level, from two grade 8 math classes.

Pre-unit results indicated that students' understanding of fractions was restricted to area models of fractions less than one whole. Students' understanding of the part-whole concept of fractions was deficient. Students tended to treat the numerator and denominator as separate entities and therefore had difficulty coordinating these terms in order to make judgements about fraction size. Also, students lacked appropriate strategies for ordering and comparing fractions.

Post-unit results indicated reasonably good progress in understanding of the part-whole concept of fractions and the ability to model fractions greater than one whole. Students were able to model fractions with a variety of models; however, their primary mental referent continued to be an area model. The use of benchmarks to order fractions showed substantial progress. Nonetheless, many students continued to use inappropriate strategies when ordering fractions. Final results indicated that students would abandon the understanding they had established for fractions when confronted with situations that were unfamiliar or when working with notation not easily compared to benchmarks.

If students are to develop fraction sense they must thoroughly develop mental referents for fractions. These must be thoroughly developed in the early intermediate years and must be mastered before any algorithms with fraction symbols are introduced.

This work is dedicated to my parents, Dianne and Chips Woodward,

for their continuing love, guidance, and support.

And to my partner, Dr. Jeff Rains,

for taking solitaire off the computer and

harassing me until I got this work done.

Mathematical algorithms are like loaded guns--

They are useful and powerful but dangerous in the hands of a novice.

Author Unknown

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CHAPTER I

THE NATURE AND PURPOSE OF THE STUDY

Fraction sense refers to an individual's ability to understand the meaning of fractions, the relative size of fractions, and the reasonableness of calculations with fractions. Individuals with a well-developed sense of fractions have created meaningful, quantitative referents for those fractions. The development of personal referents begins informally in childhood with the sharing of cookies or pizza. At this stage, the individual develops the concept of partitioning a whole or a unit. Later, different visual modes of presentation of fractions are more formally introduced in order to develop a complete conceptual understanding of fractions. In order for the concept of fraction to be fully understood, a selection of models must be presented which can embody the variety of meanings associated with fractions. The individual with fraction sense can easily translate between these different modes and understands the different meanings associated with each (part-whole, quotient, ratio, operator, and measure).

Another hallmark of individuals with fraction sense is their ability to think qualitatively about fractions. These individuals have developed a strong understanding of the meaning of the numerator and the denominator and are able to coordinate this information to make judgments about the size of fractions. They are able to reason qualitatively about the impact of changing the magnitude of the numerator, denominator or both simultaneously. Coupled with this strong conceptual understanding of the meaning of the numerator and denominator is the concept of fraction equivalence.

Individuals who exhibit strong fraction sense will use fraction equivalence and benchmarks, such as 1/2 and 1, to compare the relative size of fractions and will use this information to order fractions and to make predictions about the expected value of operations with fractions.

Rationale for the Study

Nearly all researchers who explore children's understanding and operations with fractions agree with Nancy Mack (1990) when she states, "...that many students' understanding of fractions is characterized by a knowledge of rote procedures, which are often incorrect, rather than by the concepts underlying the procedures" (p.17). Repeatedly, national mathematics assessments have highlighted this same lack of conceptual understanding of fractions and students' reliance on algorithms as their only guide when working with fractions (Kouba et al., 1988; Robitaille, 1990). The National Council of the Teachers of Mathematics (NCTM) in their landmark document Curriculum and Evaluation Standards for School Mathematics (The Standards, 1989) and in the accompanying addenda books, also recognized this problem and the need for improvement when it came to teaching and learning mathematics. These documents stress the need for students to learn conceptually with the aid of manipulatives and cooperative problem solving. The documents emphasize the need for estimation and number sense instruction and de-emphasize the need for practicing tedious paper-and-pencil computations, especially considering technology is so readily available. Students are encouraged to use referents and benchmarks as a way of helping them to develop better conceptual understanding of fractions. Reys et al. (1991) in the addenda book, Developing Number Sense, state that the intuitive understanding of fractions "... is a priority and should precede the study of operating with fractions..." (p. 10).

In my own instructional practice I have been dissatisfied with the results that a traditional approach to teaching fractions had yielded. This traditional approach usually entailed one or two days instruction explaining the meaning of fraction notation using diagrams of pies or sets of circles as presented in the textbook. The algorithm for finding equivalent fractions was presented next and practiced using textbook exercises. Finally, the algorithms for each of the operations was modeled, practiced, and then tested.

Students who could memorize procedures could find the answers to pencil-and-paper calculations; however, they often could not reason about even simple fraction calculations. More often, students would not perform the calculation properly or would confuse the algorithms, which would result in incorrect solutions. When asked if the solutions appeared reasonable, students were unable to judge since they had no feel for the size of the fractions nor did they understand the effect of the operation on those numbers. Their only resort was to repeat the calculation, and if the result turned out the same, then they felt the result was correct.

This lack of fraction sense, however, is understandable since almost no time was spent on developing this understanding or sense of fractions. According to Wearne and Hiebert (1989) many of the difficulties that students experience when dealing with decimal fractions, and presumably common fractions, "... can be traced to an incomplete or nonexistent understanding of the written symbol " (p. 507). Hiebert (1988) states that there are five stages or processes to developing competence with written mathematical symbols and that all five processes must be engaged *in sequence* if success is to be achieved. The first of these five processes is the *connecting* process whereby students connect individual symbols with referents. Hiebert claims that this initial connecting phase is crucial so that students can have the referent always available to the *"mind's eye"*. In this way, students can call up the mental image of the related quantity when they are presented with the written notation. This allows them to mentally look back on the referents so that they can monitor their own actions on the symbols and thereby detect any errors. This creating of *"transparent symbols"* which reveal specific referents is a process which involves "...building bridges between symbols and referents and crossing over them many times" (p. 336). Crossing over them many times suggests that the traditional one or two day lesson on developing the meaning of fraction notation is inadequate. Wearne and Hiebert (1989) agree this is the case "...that students need much more time than is conventionally allotted to develop meaning for written symbols and that if students acquired appropriate meaning for the symbols initially, they could use the meanings to develop procedures to solve a variety of tasks" (p. 507).

Wearne and Hiebert's hypothesis seemed to suggest to me the problem with conventional fraction instruction. Most students had not been given enough time to develop a strong conceptual understanding of the meaning of fractions nor had they developed strong connections between the written symbols and their quantitative referents. The written fraction notation was not transparent and did not bring to mind many referents. As such, the only strategy that most students had when working with fractions was to recall and apply memorized rules to the problem, all the while hoping that they had chosen the correct algorithm and had executed it correctly.

Purpose of the Study

The objective of this research was to study seven grade 8 students' understanding of fractions and fraction notation, and fraction size, and how this understanding could be improved and used to develop their fraction sense. The questions I investigated were:

- What do these grade 8 students understand about the meaning of fractions and written fraction notation?
- What understanding do these students have of the absolute and relative size of fractions and what methods do they use to determine this relative size?
- How does their fraction sense change with classroom based instruction that stresses fraction sense?

My suspicion was that students had little fraction sense and that this accounted for the poor performance on fraction problems. However, I wished to investigate specific areas of their understanding of fractions (or lack thereof) in order to determine which areas needed further instruction and where the obstacles to developing better fraction sense lay.

Organization of the Thesis

I start Chapter II with the framework I used to analyze students' understanding of fractions, fraction notation, and fraction size. I have developed the framework for analyzing students' fraction sense by adapting McIntosh, Reys, and Reys' framework for number sense. In this framework for fraction sense, I have tried to organize and categorize the different skills and behaviours that define one's fraction sense in order to guide and inform my instruction of fractions. The next chapter (Chapter III) presents a review of the literature that addresses students' understanding of the concepts that are crucial to the development of a solid understanding of fractions. These concepts include the process of developing meaning for fractions and fraction notation; the mode of presentation of physical referents: the development of a part-whole schema for fractions; the development of the ability to identify the unit; and finally the process of comparing fractions including the use of benchmarks.

A detailed description of the methodology for the study is presented in Chapter IV. Included in this chapter is a thorough description of the tasks used throughout the pre and post-unit interviews to promote informal or unpracticed responses from the students. A sequential, detailed description of instructional activities is also included in this chapter. The chapter closes with a description of how the data was analyzed and reported.

The data that I collected was used to determine common trends and distinct differences that each of the seven grade 8 students demonstrated in response to the various tasks they were presented with. This analysis of the data forms Chapter V. The data is presented in a number of tables with descriptors of the criteria used for the analysis forming the categories. The analysis of the results concludes with detailed examples of the behaviours exhibited by the seven students that serve to illustrate the findings of my research.

The final chapter (Chapter VI) includes a discussion about my findings and the conclusions that I draw. Discussion of the results is presented according to the framework for fraction sense that I developed. I discuss what I have learned from this research and how it has affected my practice in the classroom. In this chapter I also suggest areas for further research and my suggestions for improving instruction of fractions and fraction sense.

CHAPTER II

FRAMEWORK FOR FRACTION SENSE

In this chapter, I present the framework that I used to analyze each student's fraction sense. I adapted this framework for analyzing fraction sense from the McIntosh, Reys, and Reys (1992) framework for number sense. The literature review that follows in chapter III is based upon the concepts that are raised from my adapted framework of fraction sense.

Framework for Considering Number Sense

In recent years, researchers and curriculum designers have stressed the need for students to develop number sense. The National Council of Teachers of Mathematics (NCTM), in their Curriculum and Evaluation Standards for School Mathematics (Standards) document, list developing number sense as a topic that requires increased attention. Standard 5: Number and Number Relations in the grades 5-8 section of the Standards document states that students should develop number sense for whole numbers, fractions, decimals, integers, and rational numbers (p.87). The Standards Document advocates a need to understand these numbers and their representations and the relationships among them. The need to encounter a variety of representations of the numbers using a variety of models is stressed. However, the Standards Document does not go on to define clearly what number sense is. Indeed, most literature on number sense is somewhat vague in describing exactly what constitutes number sense. A variety of behaviours, however, are described which characterize someone who has number sense. These individuals are said to have an understanding or feel for numbers that allows them to use numbers in a flexible way. They use their understanding of numbers to invent efficient procedures to perform calculations. They have a sense of the effect of operations on numbers and are able to estimate solutions and judge the reasonableness of their answers.

Although these behavioral descriptions are enlightening, they do not give much guidance to educators as to how number sense is acquired nor how lessons and activities should be structured and ordered to help students develop number sense. McIntosh, Reys, and Reys (1992) outline a framework for basic number sense which attempts "...to articulate a structure which clarifies, organizes, and interrelates some of the generally agreed upon components of basic number sense..." (p.4). They stress that the framework does not delineate all possible components of number sense since this sense will grow throughout secondary school and beyond. They also point out that the whole of number sense is probably greater than the sum of its individual parts, and therefore even an exhaustive listing of number sense components would not completely explain an individual's number sense. These conditions aside, the framework is useful in that it helps to identify key components of basic number sense and arranges them according to common themes. The framework for number sense as proposed by McIntosh, Reys, and Reys (1992) is presented in Figure 1.

1. Knowledge and facility with Numbers			
1.1	Sense of orderliness of numbers	1.1.1	Place value
		1.1.2	Relationship between number types
		1.1.3	Ordering numbers within and among number types
1.2	Multiple representations for numbers	1.2.1	Graphical/symbolic
		1.2.2	Equivalent numerical forms (decomposition/recomposition)
		1.2.3	Comparison to benchmarks
1.3	Sense of the relative and absolute magnitude of numbers	1.3.1	Comparing to physical referent
		1.3.2	Comparing to mathematical referent
1.4	System of benchmarks	1.4.1	Mathematical
		1.4.2	Personal
2. Knowledge and facility with OPERATIONS			
2.1	Understanding the effect of operations	2.1.1	Operating on whole numbers
		2.1.2	Operating on fractions/decimals
2.2	Understanding mathematical properties	2.2.1	Commutativity
		2.2.2	Associativity
		2.2.3	Distributivity
		2.2.4	Identities
		2.2.5	Inverses
2.3	Understanding the relationship between operations	2.3.1	Addition/ Multiplication
		2.3.2	Subtraction/ Division
		2.3.3	Addition/ Subtraction
		2.3.4	Multiplication/ Division
3. Applying knowledge of and facility with numbers and operations to COMPUTATIONAL SETTINGS.			
3.1	Understanding the relationship between the problem context and the necessary computation.	3.1.1	Recognize the data as exact or approximate
		3.1.2	Awareness that solutions may be exact or approximate
3.2	Awareness that multiple strategies exist	3.2.1	Ability to create and/or invent strategies
		3.2.2	Ability to apply different strategies
		3.2.3	Ability to select an efficient strategy
3.3	Inclination to utilize an efficient representation and/or method	3.3.1	Facility with various methods (mental/ calculator, paper/pencil)
		3.3.2	Facility choosing efficient number(s)
3.4	Inclination to review data and result for sensibility	3.4.1	Recognize reasonableness of data
		3.4.2	Recognize reasonableness of calculation

Figure 1: McIntosh, Reys, and Reys' framework for considering number sense

My Adapted Framework for Analyzing Students' Fraction Sense

"Number" in the framework presented in Figure 1 refers to whole numbers. I substituted the meaning of "whole number" with that of "fraction" in order to create a framework to guide my analysis of each student's "fraction sense". Fraction sense is similar to number sense in that an individual shows a feel for numbers but in this case, a feel for fractional numbers. It implies an understanding of the concept of fraction and fraction operations and an ability to use this understanding in flexible ways to develop useful strategies for solving problems with fractions. I adapted the framework for basic number sense suggested by McIntosh, Reys, and Reys to produce a framework for basic fraction sense. The adapted framework is presented in Figure 2. For the purpose of this thesis, only the first component, a knowledge and facility with fractions will be considered.

1. Knowledge and facility with FRACTIONS

1.1	Sense of the relative and absolute magnitude of fractions	1.1.1	Comparing to unit/ whole (physical referent)
		1.1.2	Comparing to unit/whole (mathematical referent)
		1.1.3	Relationship of numerator and denominator to unit
1.2	Sense of orderliness of fractions	1.2.1	Relationship of denominator & numerator to fraction size
		1.2.2	Relationship between fractions and alternate equivalents
		1.2.3	Ordering fractions
		1.2.4	Comparison to benchmark referents
1.3	Multiple representations	1.3.1	Graphical/symbolic
		1.3.2	Equivalent numerical forms (equivalent fractions, percents, decimals, de/recomposition)
1.4	System of benchmarks	1.4.1	Mathematical
		1.4.2	Personal

Figure 2: Adapted framework for fraction sense

1.1 Sense of the relative and absolute magnitude of fractions

The ability to recognize that a fraction is a number relative to a defined unit or whole is crucial to understanding fractions. Students must be able to mentally keep track of two pieces of information- size of the partitions of the unit or whole (denominator) and number of the partitions (numerator)- and coordinate both of these simultaneously in order to sense a general size or magnitude for the fraction. A student who is asked to judge how big two-fifths is must have an understanding of the denominations (fifths) as they relate to the whole and then must coordinate this information with the number of those denominations (two) before a single value can be assigned to the number.

This ability to coordinate the meaning of the denominator and numerator as they relate to the defined whole or unit can be achieved by experience with physical referents. In order to develop a generalized concept of fractions it is important that a variety of referents or physical materials be used. Using a variety of referents will present the multiple meanings of fractions to the student. For the purpose of this thesis, the part-whole model of fractions is explored with reference to continuous quantity (area and linear) and discrete quantity (sets). Individuals with a strong sense of fractions are able to translate between these different visual modes without any difficulty. When this occurs, the concept of fraction is generalized and the need for physical referents is reduced. The student is able to compare the value of a given fraction to a more abstract or mathematical concepts of one or half.

1.2 Sense of the orderliness of fractions

The ability to order common fractions is one of the hallmarks of fraction sense. Students with fraction sense can compare fractions, identify which of two fractions is closer to a third number, and identify numbers in between two given fractions. Students who are able to order fractions have developed a quantitative sense of fractions. They have developed an ability to perceive the relative size of fractions. These students have developed an understanding of the inverse relationship between the size of the denominator and the size of the fraction. That is, they understand that as the number of partitions increases (the size of the denominator increases) the size of the fraction decreases.

Students with number sense understand the relative size of the denominators of two or more fractions and are able to coordinate this information with their respective numerators in order to make judgments about the relative size of these fractions. These students would then use this information to order a set of fractions. Comparison of fractions with benchmarks, such as half and one whole, in order to order fractions with unlike denominators is a common strategy among students with fraction sense. The commonly taught procedure of converting to like denominators would not even be considered for these numbers and is often a last resort for students with number sense. These students are more likely to convert fractions to their decimal equivalents before they would consider using such a tedious algorithm.

1.3 Multiple Representations

An important component of number sense is the ability to recognize that numbers appear in many different contexts and can be represented in a variety of symbolic and/or graphical forms. Individuals with number sense recognize these different forms and understand that each form can be manipulated to suit a particular purpose or context (McIntosh, Reys, & Reys, 1992). For example, at times it may be useful to think of three-quarters as 3/4 but at other times the equivalent values of 6/8 or 0.75 or 75% may be more useful given the context. Individuals with number sense see all these forms as related and equivalent and understand by the context of the situation which form is most useful and efficient.

Fractions have multiple representations and meanings. The most common interpretation of fractions is the part-whole meaning, which is explored in this thesis. Other meanings of fractions include the concept of quotient, ratio number, operator, and measure (Behr, Harel, Post, Lesh, 1992, p.298). Individuals with strong fraction sense will be able to relate and use these different meanings in flexible ways. For example, a person with fraction sense will answer the question, "How much pizza will each person get if there are 3 pizzas and 5 people?" by recognizing that 3 ÷ 5 can also be written with the notation "3/5" (quotient concept) which gives the answer that each person will get three-fifths of the pizza (part-whole concept). It is this ability to relate the different meanings of fractions that gives an individual mathematical flexibility and defines his or her fraction sense.

Decomposition and recomposition of a number is a common strategy of individuals with number sense. Decomposing a number involves breaking it down into an equivalent form and then using this decomposed form to solve a problem and then recomposing the form to produce the solution. For example, if you wanted to know how much 3/4 of 32 people is, you might decompose the fraction 3/4 into three- 1/4 units (1/4 + 1/4 + 1/4). Then you would use the decomposed form of 1/4 to determine that 1/4 of 32 is 8. You would then recompose the number three- 1/4 units as three-8 units and get the solution 24 is 3/4 of 32. Decomposition to unit fractions is a common strategy among individuals with fraction sense.

1.4 System of Benchmarks

A system of benchmarks is one the most powerful strategies that an individual with fraction sense has. Numerical benchmarks are usually midpoints such as 1/2 or 50% or endpoints such as 1 or 100%. These benchmarks provide mental referents for judging the size of a fraction or answer to an operation. They are used extensively by students with fraction sense when ordering fractions with unlike denominators. These students make qualitative judgments such as "is it more or less than 1/2 or 1". Often, these judgments are enough to order the set. When they are not, these students would refine his or her analysis of the fractions, perhaps by comparing which fraction is closer to 1/2 or 1. For example, when faced with the two fractions, 2/6 and 3/8, the student would realize that both are below half. The student would then compare the two fractions and notice that each is 1/6 and 1/8 away from 1/2, respectively. S/he would reason that since 1/8 is less than 1/6, 3/8 is less far away, therefore closer, to 1/2 than 2/6, and decide therefore that 3/8 is greater than 2/6. This process demonstrates fraction sense in action. The individual has a strong understanding of the relative size of fractions (e.g. 1/8 vs. 1/6) and of the meaning of equivalence of fractions (e.g. equivalence to 1/2). This understanding would have been gained earlier in the process of learning about fractions. The student has strong referents associated with each fraction that would have been developed through extensive work with a variety of physical models and through the individual's personal experience with fractions in everyday life.

Summary

In my analysis of student fraction sense I worked within the modifications I made to the McIntosh, Reys, and Reys' framework for number sense. My focus was on studying each student's *sense of the relative magnitude of fractions*. In particular, I studied each student's ability to use a physical referent to model a given fraction. I looked for their ability to define the fraction relative to the whole and vice versa. I looked for the way in which the student interpreted the meaning of the numerator and denominator when given physical referents to model the fraction. Inherent in this exploration, was a study of how each student is able to manage the different manipulatives. I observed how each student interpreted the meaning of a given fraction with different physical models, namely continuous models and discrete models. The continuous models involved various area models-pattern blocks, fraction circles, geoboards, and drawings of shapes and length models- Cuisennaire rods and drawings of fraction strips. The discrete models involved sets of objects such as colored blocks and drawings of shapes. My other main focus was to study the students' *sense of orderliness of fractions*. I was most interested in examining whether the students used comparison to benchmarks as a strategy for ordering a set of fractions with unlike denominators. Comparing fractions based on size also involved probing students' understanding about the inverse relationship between the size of the denominator and the size of the fraction.

CHAPTER III

LITERATURE REVIEW

In this chapter, I review literature that is relevant to my study of fraction sense. This literature review is structured according to my adapted framework for fraction sense. In the sections that follow, I present literature that addresses the issues I see as fundamental to the components of fraction sense that I will address in my thesis. These issues include the process of developing meaning for fractions and fraction notation; the mode of presentation of physical referents; the development of a part-whole schema for fractions; the development of the ability to identify the unit; and the process of comparing fractions including the use benchmarks.

A Process for Understanding Mathematical Notation and the Meaning of Fractions

According to Piaget (1977), "...logical and mathematical operations derive from action, and, like physical knowledge, they presuppose experience in the true sense of the word, at least in their initial phases" (p.36). These initial experiences as related to fraction operations would include a child's early experiences with partitioning a whole to create half a cookie or dividing into two equal sized teams. Later in school, these experiences should include work with manipulatives which model the variety of meanings associated with fractions.

These manipulative experiences become the visible referents that result in *transparent symbols* (Hiebert, 1988, p. 336). It is necessary, according to Hiebert, that symbols be transparent so that students can call up from their *mind's eye* the related quantity and reason directly about the quantity to solve the problem. He emphasizes that for students new to the mathematical territory, these cognitive objects or referents are needed in order to support the problem solving process. So, although the goal of mathematics is the generalization of written symbols, the initial goal is to connect written symbols with specific and appropriate quantitative referents.

James Hiebert (1988) outlines a five-step process by which students develop competence with written mathematical symbols. These five major processes are as follows: (1) *connecting* symbols to referents, (2) *developing* symbol manipulation procedures, (3a.) *elaborating* procedures for symbols, (3b.) *routinizing* the procedures for manipulating symbols, and (4) using the symbols and rules for *building* more abstract symbol systems. According to Hiebert (1988), these five processes must be engaged in sequence since "the outcome of an earlier process lays the foundation for mastering the later processes" (p.335). This suggests then that the connecting process must be thoroughly employed before students are moved on to the next processes.

Nancy Mack (1990) supports Hiebert's view. She claims that although students' informal knowledge of fractions is disconnected from their understanding of symbol notation, their informal knowledge can serve as a basis for developing understanding of mathematical symbols "...provided that the connection between the informal knowledge and the fraction symbols is reasonably clear" (Mack, 1990, p.29). She goes on to point out, however, that prior knowledge of procedures (which were often isolated and faulty) frequently interfered with student attempts to give meaning to fractions symbols and procedures. Several other researchers have also found that it is difficult to go back and provide referents for symbols when students have already routinized symbol manipulation rules (Markovits & Sowder, 1994; Wearne & Hiebert, 1989). "Other researchers have noticed and commented on this phenomenon of having more success in teaching unfamiliar content than in reteaching (or replacing) familiar content" (Markovits & Sowder, 1994, p.24).

Although research seems to be mounting that suggests that the connecting process must be thoroughly engaged before a student is moved on to the developing and routinizing processes, it would appear that a traditional unit of instruction on fractions does not correspond with this framework. The process of connecting symbols to quantitative referents is usually introduced briefly in order to move students on to the "important" part of the fraction unit, namely the routinizing of operations with fractions. Many researchers cite this rush to the routinizing stage as the root cause of many students' problems in math (Davis, 1984; Kieren, 1988; Mason 1987). Hiebert agrees and states that "...an important source of students' difficulties in mathematics is a premature emphasis on the formal symbols and rules of mathematics, independent of meaningful referents" (1988, p. 348). He recommends that increased time and attention be given to connecting symbols with referents and that this must precede any other procedures. He believes that this initial outlay of time will result in students who are "... in a good position to recognize the conceptual rationale for symbol manipulations. At the very least, students would not be restricted to memorizing and executing syntactic rules" (Wearne and Hiebert, 1988, p.224).

This hypothesis seems to address the most common complaint about students, namely that their "...understanding of fractions is characterized by a knowledge of rote procedures, which are often incorrect, rather than by the concepts underlying the procedures" (Mack, 1990, p. 17). Hiebert suggests that all one would need to do to correct this situation is engage students in the connecting process for a sufficient period of time. He does admit, however, that developing meaning for the symbols by connecting them to referents is a complex and protracted process. Given the pressure most teachers feel to cover the curriculum, it is no wonder that this step is essentially ignored in order to rush to the routinizing process, which most teachers feel is the ultimate goal of their mathematics program. Unfortunately, research is showing us that, despite all the effort and energy that is being put into the routinizing of fractions and fraction operations, students are not becoming proficient at them. It seems, in fact, that they are generally confused by the multitude of seemingly unconnected rules and procedures that they are asked to learn.

Modes of Presentation of Physical Referents

If students are to connect symbols to referents these referents must be ones that have meaning for the students. Each student creates "meaning" when he or she is given sufficient experience with the referents. When considering fractions, students must be given sufficient experience with a wide variety of referents in order to develop a rich understanding of fractions James Hiebert (1988) warns us that action on a particular referent may capture only some of the features of the symbol rule. He goes on to say that if students only experience one type of referent then this "...single mapping, although appropriate, may be insufficient to develop the full meaning of a rule" (p.339). In other words, the meaning that a student develops for the fraction notation is limited to the referent model with which they have had experience.

Manipulative experience is most often limited to partitioning a whole using an area model (Armstrong and Larson, 1995). Generally students understand fractions to involve cutting up an area and then coloring in or taking away pieces. This narrow concept of fractions is usually created when students are exposed almost exclusively to area models of fractions, such as circular "pies" and "pizzas" or rectangular "cakes". The consequence of this limited understanding of fractions is that students often have difficulty when they are asked to apply their understanding of fractions to discrete models. Indeed, Hiebert (1988) states that "...the success of instructional efforts seems to be restricted to the referents and actions which children have experienced and about which they are knowledgeable" (p.348). It is important, therefore, that students experience a wide variety of models of fractions. These models should include continuous (area and linear) models and discrete (simple and compound set) models. Continuous models are ones in which there is no visual breaks in the model versus discrete models where visual breaks are visible. Students should experience many examples of these various models and also should be guided to see the common characteristics between all these forms so that they can easily translate between each mode.

As was mentioned earlier, each model has limitations and may only capture certain facets of the meaning of fractions. For example, area models, which are the most common referent provided for students are limited in several ways. In order to use an area model, students must be able to coordinate the relationship between two dimensions. However, research tells us that this ability develops slowly. Most students are only able to consider each dimension in sequence and find it difficult to coordinate their thinking about both dimensions until approximately age 12 or 13 (Armstrong and Larson. 1995). Armstrong and Larson (1995) also point out that the partition lines of the area model add another dimension that the student must consider. If a student is not yet able to coordinate all these dimension, then they may find the area model somewhat confusing as a referent for fractions.

Partitioning an area model is designed to illustrate the part-whole concept of fractions. However, Armstrong and Larson (1995) point out that in fact most students use a direct comparison strategy to compare different areas and do not "...deal with the complexity of coordinating all the part-whole relationship conditions" (p.16). The students consider the parts separately from their wholes and simply compare them directly by a strategy Armstrong and Larson (1995) call *Areas of Parts/Decompose-Recompose (AP/DR)*. This strategy involves cutting up one area and reshaping it to fit the other area. It relies on visual information and not the relationship between the parts and the whole. In fact, the whole is of no consequence when this AP/DR strategy is employed. Since the concept of the whole is crucial to fractions, area models that do not force a student to consider the whole are counter-productive to developing an understanding of the relationship of the fractional parts to the whole. Armstrong and Larson (1995) suggest that in order to avoid this tendency to focus on the parts alone, comparison-of-area problems in which the size of the wholes or units differ and where the parts are not congruent or similar in shape need to be included in the intermediate and middle school curriculum.

Another limitation of the area model as a referent for fractions can be found in most student textbooks. In these books, fractions are represented with circles or squares that are partitioned into congruent parts. The static nature of the graphics does not allow the students to manipulate the models. If students are given these manipulative experiences with an area model, they would come to understand that a given area can be partitioned in many different ways. For example, they would come to understand that the condition "fourths" implies "four- equal in quantity" partitions and that these partitions do not necessarily need to be congruent. Armstrong and Larson (1995) suggest that these experiences would lead to the reasoning that "...the area of one-fourth of a region would be equal to one-fourth of another same-sized region whether or not the one-fourths visually looked the same" (p.17).

The limitations of the area model can be overcome by careful consideration of the area models presented to students and by presenting other models, such as linear and set models, to students. Students do not spontaneously translate between these different models therefore it is the job of the teacher to guide students to see the similarities and differences between the different physical modes. Through these comparisons, students will gain more insight into the meaning of fractions. For example, when a student is given a square to partition into fourths, he or she knows that four parts are required and that these parts must be equal in size. However, when presented with a set of 12 blocks, the student must decide what aspect of four to consider- is it four equal groups or three groups of four that determines fourths? This consideration does not arise with an area or linear model and serves to emphasize the point that students must be given experience with many models in order to develop a solid understanding of fractions.

Development of a Part-Whole Schema for Fractions

In order for students to understand the part-whole meaning of fractions they must consider numerous pieces of information. They need to consider the inverse relationship between the number of partitions of the whole and the size of those partitions and then coordinate this information with the number of partitions in order to establish a single value for the fraction. According to Behr, Wachsmuth, Post, and Lesh (1984), the ability of students to understand the compensatory relation between the size and number of equal parts in a partitioned unit is highly variable. "A small percentage of students understand the relationship after only brief instruction. For still others, the relation remains elusive even after they have had ample opportunities to learn and practice" (Behr et al., 1984, p.338). These researchers found that the students who did not fully understand the inverse relationship between the number of partitions and the size of those partitions relied on their whole number schemas to compare and order fractions. This "whole number dominance strategy" occurred even though the denominators were sufficiently small that the students would have had experience ordering them with manipulatives. Nevertheless, responses from most students early in the study were of the type "one-third is less than one-fourth because three is less than four" (Behr et al., 1984, p.328). The researchers did find, however, that the dominance of whole number logic diminished in the face of instruction when the questions were of the non-application variety. Unfortunately, they also noted that instruction appeared to be less effective when students were asked to apply their knowledge to new situations. The researchers suggest that "... even late into instruction a substantial number of children 'back slide' into a whole number-dominance strategy when confronted with problem-solving situations where they must apply their knowledge of the order and equivalence of fractions" (Behr et al., 1984, p. 333). They suggest that the variation in children's ability to understand the compensatory relation of fractions indicates that more instructional time is required to develop this understanding than has been currently allotted in most curricula. The researchers also contend that careful spiraling of the concept is required through several grade levels if the concept is to be solidly attained by students.

Nancy Mack (1990) studied students' informal knowledge of fractions and found that most students understood fractions to involve partitioning units. However, she also found that once the unit was partitioned, many of the students treated the partitions as independent from the whole. That is, they treated each part as if it were a whole number and not a fraction. Many of the students in the Armstrong and Larson study (1995) also thought of the parts as separate from the whole and merely compared the parts directly. Behr et al. (1984) also found similar results. When students were presented with fractions having the same numerators, three fifths of the explanations involved discussion of the denominators only. Although these explanations led to the correct answer, the researchers noted that this "denominator-only" strategy could indicate a lack of awareness that both the numerator and denominator must be considered when judging the order or equivalence of fractions.

The results of these researchers seem to indicate that students appear to focus on the "number of pieces" rather than the "size of the fraction". That is, they seem to treat each part as separate from the unit or whole. Mack noted the consequences of this type of thinking from the students in her study. When she asked which was larger, 4/5 or 5/6, a student in her study replied "They're the same... because there's one piece missing from each- 1/6 missing from 5/6 and there's 1/5 missing from 4/5" (Mack, 1990, p.28). Three other students in Mack's study answered in a similar manner where they focused on the number of missing pieces rather than on the size of the fractions. Markovits and Sowder (1994) also noticed this tendency of students to focus on the number of pieces rather than size of the fractions when they asked students to compare two fractions. When presented with 5/6 and 9/10, a third of the students in the study replied that the numbers were equal in size because each was "one piece away from one..." (p.12).

This focus on the pieces continued when students in Mack's study (1990) were presented with problems involving discrete sets. When presented with six cookies, students were asked to show 2/3 of the cookies. Five of the six students replied that "two cookies" were 2/3 of all the cookies. One student responded that "you want two out of the three" (Mack, 1990, p.29). This student focused on the thirds as three pieces (three cookies) versus three partitions of the whole. He thought of the denominator as entities that were separate from the whole. It is interesting to note that near the end of the instructional sequences, four of the six students could only solve this problem if they thought of the cookies as one big cookie rather than several small cookies. A possible explanation for this finding could be that partitioning an area, such as a circle, into three parts does not require the student to consider the whole, but rather allows him to continue to think of the parts as separate from the whole. However, a compound set (such as 6 cookies) forces the student to consider the partitions in relation to the whole. That is, the student must consider the thirds as three equal partitions of the six cookies (whole) not merely as three parts.

The findings of Armstrong and Larson (1995) would tend to lend support to this view. When students were presented with a variety of rectangular areas and asked to compare the areas with respect to size, most students (regardless of age) used a direct comparison method to solve the problems. That is, they thought of the parts as separate from the wholes and simply compared the parts directly. Armstrong and Larson suggest that this behavior may be a consequence of the graphics presented in most textbooks. They state that "...parts of circular models are easy to compare visually, and therefore, the most efficient strategy is a direct comparison. There is no need to consider the part-whole aspects of the problem. If students' experiences are limited to area models where the parts can be compared visually, students will have no need to develop more sophisticated comparison strategies" (Armstrong & Larson, 1995, p.16). These researchers would seem to suggest, therefore, that the models students are presented with, namely area models, actually encourage students to consider the partitions as separate from the whole. In order to test this hypothesis, Armstrong and Larson (1995) presented students with graphics where the wholes were different sizes. The researchers wanted to see if the students would attend to the size of the wholes and determine that a comparison of the shaded fractional parts was not meaningful since the two wholes differed in size. When the students were presented with two wholes (rectangles) of different size with 3/6 shaded in on both diagrams, students said the shaded portions were equal. This suggested to the researchers that students continued to focus on the parts and ignored the size of the wholes, even when the wholes differed in size. Armstrong and Larson (1995) contend that "comparison of fraction" problems and graphics are usually presented where the wholes are the same size. As a result, they claim "... students do not attend to the size of whole from whence parts come" (Armstrong & Larson, 1995, p.16). These researchers suggest, therefore, that students should be presented with problems that have different size wholes in order to force students' attention to the part-whole, not just part, concept.

Behr et al. (1984) found that students in their study who did have a strong part-whole concept compared same-numerator and same-denominator fractions by a strategy they called "numerator-and-denominator strategy" (p.334). This strategy involved an explanation from the student that referred to both the numerators and denominators. For example, when both fractions had the same numerator the student indicated that the same number of parts was present (numerator) but that the fraction with the larger (or largest) denominator had the smaller (or smallest) sized pieces. The researchers found that the students who employed this strategy seemed to be base their thinking on a mental image of their experience with manipulative aids. In Hiebert's words, these students were bringing mental referents to their mind's eye. These students, according to Behr and his colleagues, had a generalized and abstract concept of fractions. They could call up mental referents that guided their thinking about the compensatory relationship between the number of equal parts of the whole and their size. These students did not need the actual physical manipulatives to act upon but could call up a mental image that guided their thinking. For these students, it appears that their previous work with manipulatives had been connected to the fraction notation such that the symbols were, in Hiebert's terms, transparent symbols. From this research it would seem that the students who had a strong part-whole concept of fractions, also had developed mental referents from the notation which they called to their mind's eye when solving comparison problems.

Armstrong and Larson (1995) found that the use of a part-whole strategy to compare the area of partitioned rectangles was most prevalent amongst older (eighth grade) students and increased significantly when the researchers introduced fractional terms. They found that "...the symbolic representations of the parts to be compared seemed to hold meaning for the eighth-grade students in a way that did not for many of the younger students" (Armstrong & Larson, 1995, p.15). The researchers went on to note that the ability of students to interpret the symbols as part-whole relationships that can also represent quantities of area may be developmental in nature. The researchers also noted, however, that although the grade eight students were more able to interpret the symbolic notations of part-whole relationships than their younger counterparts, they did not apply their knowledge spontaneously. The researchers indicated that "... at the end of middle school, students seldom recognize real-world situations in which they can apply their knowledge of rational numbers" (Armstrong & Larson, 1995, p.17). They concluded their study with several recommendations, including the requirement to give students opportunities to experience the meaning of fractions other than half and more opportunities to connect fractional symbols to fraction terms and models from third grades though middle school. In Hiebert's terms, these students need more experience in the connecting phase if they are to develop an adequate understanding of the part-whole concept of fractions, which according to the researchers is developmental in nature.

The Development of the Ability to Identify the Unit

The central notion to understanding the part-whole concept of fractions is the ability to understand that a defined whole or unit is partitioned in a number of equal parts as defined by the denominator. Although this notion seems obvious, not all students demonstrate an acute awareness for the whole. Armstrong and Larson (1995) found that when students were provided with partitioned rectangles to compare, the majority of the students initially ignored the whole in the part-whole relationship and rather used a direct comparison of the parts. They went on to observe that "even when some of the students used Part-Whole strategies, they ignored the size of the whole when they made their comparison" (Armstrong and Larson, 1995, p.16). To test this hypothesis, Armstrong and Larson (1995) presented students with wholes of different sizes. As the researchers had suspected, these students ignored the difference in the wholes, and gave a part-whole answer as if the wholes were the same. For example, when a student was presented with two wholes (rectangles) of different sizes, the student ignored the difference in the size of the wholes. He went on to explain that the shaded areas of both wholes were the "same, because they're both divided into sixths, and they both have four-sixths of the cake frosted" (Armstrong and Larson, 1995, p.10). This student did not attend directly to the whole even though his answer suggested a part-whole strategy.

Nancy Mack (1990) found that "all students' informal knowledge allowed them to determine the appropriate unit in a real-world problem. However, they had difficulty identifying the unit in situations represented symbolically and concretely" (Mack, 1990, p.22). She found that students treated collections of partitions, when presented concretely and pictorially, as a single unit. One student stated that "fractions are part of a whole....They're always less than one whole" (Mack, 1990, p.22). When another student was presented with a full circle shaded in and a quarter of another shaded, she stated that 5/8 of the circle was shaded. She interpreted the unit to be the entire collection of eight parts. When she was instructed to think of pizzas, the girl replied that one and one-quarter was shaded. Mack contends that when the context of the problem was made clear, the student could correctly identify the unit.

Armstrong and Larson (1995) explain students' lack of awareness or attendance to the whole can be traced, at least in part, to the models of fractions that are presented to students. They state that the circular model is the most common area model used in instruction with fractions. Since the parts are easy to compare directly and since the size of the circles rarely vary between examples, the circular models do not require students to consider the whole and as a result students usually don't. In other words, the models as presented to the students, lead them, albeit unintentionally, to ignore the whole. The whole is a given in the student's mind and need not be contemplated. "This type of presentation has the effect of focusing students' attention on the parts alone when making a comparison, thus discouraging the formation and retention of part-whole relationships in the students' minds" (Armstrong and Larson, 1995, p.17). These researchers continue by asserting that comparison of area problems where the size of the wholes or units differ and where the parts to be compared are not congruent or similar in shape need to be included in the intermediate and middle school curricula in order to draw student attention to the whole in the part-whole relationship.

Process of Comparing Fractions

According to Markovits and Sowder (1994), "the development of rational number sense is highly related to the acquisition of a quantitative notion for fractions" (p.6). Developing a quantitative notion for fractions means coming to understand the magnitude of such numbers. Markovits and Sowder (1994), state that understanding the magnitude of a number within a number domain includes "...the abilities to compare numbers, to identify which of two numbers is closer to a third number, to order numbers, and to find or identify numbers between two given numbers" (p.6). If we consider the framework for fraction sense that I adapted from McIntosh, Reys, and Reys (Figure 2, p. 12), then we can see that these number sense skills of Markovits and Sowder relate to section one (Knowledge and Facility with Fractions) of my adapted framework for fraction sense. In particular, these skills, which are necessary in order to develop a quantitative notion for fractions, are embedded in subsection 1.2 (Sense of orderliness of fractions) and 1.4 (System of benchmarks) of my adapted framework.

Several researchers have studied students' abilities to compare and order fractions. Markovits and Sowder (1994) studied children's abilities to locate fractions between two given fractions and to order a set of fractions. When students used a like denominator and decimal equivalent method to compare fractions, the researchers determined that these responses represented a "rule-based" approach. If the responses used a benchmark strategy then the researchers considered this a "number-sense-based" approach. The researchers pointed out that students who employed the rule-based approach may have had a good understanding of the numbers being compared. However, they felt that since the number-sense-based approach was more efficient, students with an awareness of number size would not use such a tedious algorithm. The researchers found that the use of benchmarks increased on the postinstructional and retentional items and interpreted this to be a manifestation of fractional number sense. They noted as well that comparing fractions with benchmarks introduced some new difficulties for the students. For example, when asked to compare two fractions that were both under 1/2 (3/8 and 7/15), the students needed to determine how far away each fraction was from half, compare the 1/8 and 0.5/15, and then relate the result of the comparison back to the original problem to determine which fraction was closer to half and therefore larger. The researchers point out that it was the final step in the process that gave students the most difficulty.

Behr, Wachsmuth, Post, and Lesh (1984) studied children's thinking when they ordered unequal fractions. These researchers felt that "one measure of children's quantitative notion of rational number is their ability to perceive the relative size of the rational numbers in a pair or a larger set; that is, their ability to determine which of the relations, *is equal to, is less than,* or *is greater than,* holds for a given pair of rational numbers" (p.324). As such, questions posed by the researchers asked the children to decide which of two or three fractions was the lesser or the least and then to explain the reasons for the decision. The fraction questions were grouped as follows: fractions with the same numerators, fractions with the same denominators, and fractions with different numerators and denominators. The researchers found that the children employed a variety of strategies when determining the relative size of a fraction.

Early in instruction, the researchers found that a large number of the responses from the grade four students when asked to compare 1/3 and 1/4 were of the type "one third is less than one fourth because three is less than four" (Behr et al., 1984, p.333). The researchers found that "children's schemas for ordering whole numbers are very strong and, at least during initial instruction in fractions, are overgeneralized..." (Behr et al., 1984, p.333). The researchers also noted that this "...dominance by whole numbers diminishes in the face of instruction" (p.333). However, they warn that this is only for non-application problems and when students are asked to apply their understanding of rational numbers, they have a tendency to "backslide" into whole-number-dominance.

Nancy Mack (1990) and partners Zvia Markovits and Judith Sowder (1994) also detected whole number dominance when students were asked to compare fractions. In all three studies, the fractions were such that students should have had experience ordering them with manipulatives. Behr and colleagues noted that younger children display a great deal of knowledge about the fraction 1/2, 1/3, and 1/4. They propose that the poor performance with ordering the simple fractions could be due to the students being "... overpowered by their knowledge of the ordering of whole numbers" (Behr et al., 1984, p.333)

In later interviews when confronted with same-numerator problems, Behr and colleagues noted that use of the whole-number-dominance strategy had diminished and most of the responses were of the "Numerator and Denominator" or "Denominator Only" variety. They cautioned, however, that three fifths of the explanations employed the denominator only strategy. They felt that "most of the children did not give overt evidence of being aware that both the numerator and denominator must be considered when judging the order or equivalence of two fractions.... (and that) may cause difficulty when the children encounter other types of problems" (Behr et al., 1984, p.334).

By the end of the teaching experiment, most of the students were able to correctly order fractions with the same denominator. However, Behr and his colleagues noted that success was not universal since one-sixth of the explanations incorrectly employed the numerator-and-denominator strategy or some other unspecified strategy. The students who did use the numerator-and-denominator strategy correctly also seemed to be using a mental image of their experiences with manipulative aids according to the researchers. They contrasted this desirable behavior with those students whose explanations required direct action on the physical referents. The researchers felt that although the students' explanations were correct, the reliance on the manipulatives indicated that their thinking was less abstract than their counterparts who could call up a mental image.

An unusual manifestation of the numerator-and-denominator strategy was when students inverted the relation between the numerator and denominator. "Nine thirteenths is less than four thirteenths because 'four pieces are so big, nine pieces would have to be smaller to fit the whole" (Behr et al., 1984, p.330).

Behr and his colleagues found the "reference-point" strategy interesting because it had not been specifically taught in the experimental lessons. They interpreted the use of a third number as a reference point as thought that was abstract and generalized. They stated that "there appears to be a positive relation between thinking based on a reference point and a quantitative understanding of rational numbers" (Behr et al., 1984, p.335). Markovits and Sowder (1994) found on retention tests that more students were using number sense strategies to compare fractions than they had before instruction. They interpreted this to be an indication of improved rational number sense. Mack (1990) also noted that some students used reference points to compare fractions and, like the findings of Behr and his colleagues, these benchmark strategies had not been taught to the students but rather were part of the students' informal knowledge of rational numbers. Mack saw the use of these strategies as a strength and emphasized that students could build upon these informal strategies in order to understand further fraction symbols and procedures.

An incorrect comparison strategy that Mack (1990), Markovits and Sowder (1994), and Peck and Jencks (1981) noted was when students focused on the number of pieces versus the size of the fractions. For example, when asked to compare 4/5 and 5/6 the student replied that "they're the same....because there's one piece missing from each- 1/6 missing from 5/6 and there's 1/5 missing from 4/5" (Mack, 1990, p.28). A similar response is found in Markovits and Sowder's study when students said 5/6 and 9/10 were the same number because they were both "one piece away" from one. Peck and Jencks noted that approximately 20% of the sixth-, seventh-, and ninth-grade students interviewed compared fractions by observing the number of pieces left over. One student correctly sketched diagrams of 2/3 and 3/4 and stated that the fractions were equal because "there are the same number of pieces left over" (Peck & Jencks, 1981, p.344). As was discussed earlier, Armstrong and Larson (1995) contend that this focus on the number of parts versus the size of the fractions is due to a weak understanding of the part-whole relationship of fractions. They feel this weakness could be related, at least in part, to the type of graphics presented to students that do not require this kind of thinking to develop. They feel that more research is needed in this area.

Summary

Research is mounting that suggests that many students lack conceptual understanding of fractions and that their knowledge is characterized by the rote application of memorized procedures. This lack of "fraction sense" is most often characterized by students who incorrectly or inappropriately apply learned algorithms and have no ability to judge the reasonableness of their answers. Many researchers cite the rush to teach algorithms before students have developed a solid conceptual understanding of rational numbers as the difficulty that students are exhibiting when they try to work with fractions. They feel that they need more time than is conventionally allotted to developing meaning for fractions.

If students are to improve their "fraction sense"- if they are to develop a feel for the meaning and size of fractions- then instruction must be structured which promotes this type of thinking. Up until lately, however, "number sense" has been described in vague behavioral terms that did not provide guidance to the educator on how to sequence lessons so that this type of thinking can be encouraged. McIntosh, Reys, and Reys (1992) presented a framework for number sense that identified the key components of number sense and arranged them according to common themes (Figure 1, p. 10). For the purpose of this thesis, I adapted this framework to provide a framework for fraction sense (Figure 2, p. 12) and to act as a guide for lesson sequencing for teaching fraction sense. It is this framework that I use guide my study of each student's sense of the relative magnitude of fractions and sense of orderliness of fractions.

Many components combine to create an understanding of the relative size and order of fractions. These include the process for developing meaning fractions and fraction notation; the mode of presentation of physical referents; the development of a part-whole schema for fractions; the ability to identify the unit or whole; and the process for comparing and ordering fractions. Literature related to these components has been presented in this chapter. The data in this literature served to guide the instructional experiences of the grade 8 students in my math classes and helped to explain the analysis of results in the discussion section (Chapter VI).

CHAPTER IV

METHODOLOGY

In this chapter, I present the methodology that I used to collect the data for my study. This chapter includes a description of individuals who took part in my study and the method that I used to select them; a detailed account of the tasks I used with my subjects both during the pre and post-unit interviews; a description of the interview setting; and a report of the activities that were conducted with my two math 8 classes during our unit of fraction instruction.

Participants

Subjects for my study were solicited from the two math 8 classes that I taught at a middle school (grades 6-8) in a low to middle income suburb of Vancouver. According to the model of a middle school as opposed to a junior high, students remained together as a class for the entire school year for all academic courses (Humanities, French, Math, and Science). This allowed me to teach the same two classes Science 8 as well as Math 8. As such, I worked very closely with all my students for the entire year and felt that I got to know them quite well.

Students in my two math 8 classes were asked to participate in my study. From the volunteers who returned their permission slips, I selected seven students: 4 girls and 3 boys. I selected the students based on their math grades, given by their grade 7 teacher and by me in the first term of grade 8, and on their willingness and ability to articulate their ideas. I wanted to select a cross section of students so I chose 2 "top" students, 2 "average" students, and 3 "below average" students in math. Two students, Debby and Chris (pseudonyms) were considered top students and had received either "A's" or "B's" in math and in most other subjects. They were both confident and very involved in the extracurricular activities in the school. Jacquie and Shari were considered average students and had received mostly "C" grades in math. Jacquie was very social and Shari was more reserved. Both were somewhat hesitant about their ideas and would seek confirmation of their understanding with me during class time. Tammy, Liam, and Jeremy had received mainly below average grades (C-) in math. Both Liam and Jeremy had received learning assistance in math in previous grades and from me during the first half of the school year. Liam was reserved and would ponder his ideas whereas Jeremy would answer quickly and confidently, even if his answers were incorrect. Although Tammy received low grades in math, she was actually quite capable in the subject. However, due to a troubled home-life she would sometimes miss class and generally did not do homework or study. As a result her grades suffered. I chose her however, because I suspected that she had intuitive ability in math and was interested in what she could show me.

Pre-Unit Interview

Setting

Each student took part in a pre-unit, private interview which I videotaped and later transcribed myself. These interviews took place during the school day in a private room in the school. Each interview lasted for approximately 80 minutes each. A variety of manipulatives, including pencils and paper, were available for students to use at all times during the interview.

Pre-Unit Interview Questions

I used the same tasks, manipulatives, and questions for all students. I would, however, expand on the questions that I asked in order to get further clarification from the student if I thought it was needed. The design of the interview tasks was based largely on the literature I reviewed on number sense and fractions. The tasks were all preplanned and were not influenced by the responses that were given. However, as mentioned before, I would deviate from my questions if a response was unclear or curious. A detailed description of each task that I used in this study follows.

A full listing of the tasks that were used with students can be found in Appendix A. In this section I describe only those questions which address each student's ability to sense the relative and absolute magnitude of fractions (section 1.1.1 and 1.1.2, p. 12) and the ability to sense the orderliness of fractions (section 1.2.1; 1.2.3; and 1.2.4, p.12). The data from these questions forms my analysis in chapter IV. I omitted some questions because of the need to narrow the scope of this study.

Section 1.1 Sense of the relative and absolute magnitude of fractions

The ability to recognize that a fraction is a number relative to a defined unit or whole is crucial for a student's fraction sense. For this reason, I have listed it first in my adapted framework. In order to develop fraction sense, students must first come to understand a fraction as a number relative to a defined whole. They need to understand the meaning of the numerator and denominator and consider them in relation to one another and the defined whole in order to develop a quantitative notion for fractions. In the remainder of this section I describe the questions that I asked in the pre-unit interview that addressed each student's understanding of the numerator and denominator and his or her ability to coordinate this information and consider it relative to one whole. Also included in this section are the questions that I asked to determine the models that students used as referents when thinking about fractions.

Task # 1: Interpreting written notation.

The first task (Question 1) involved showing the fraction 4/6 written out on a card and asking the students to say the fraction aloud so I could determine whether they knew the standard form for saying fractions. Students were then asked to explain and/or show with manipulatives what the notation 4/6 meant. I then proceeded to question each student about what the "6" in the fraction tells you and then the "4". My purpose in doing so was to determine what meaning students gave to the numerator and denominator. I also noted what model the student chose to represent the fraction.

Task # 2: Meaning of "fourth".

The second task (question 2) involved having the student separate a geoboard into fourths and explain how s/he knew that it was partitioned properly into fourths. In this task I was interested in finding out if students understood that the fractional name, in this case 'fourths', determined the number of equal size, but not necessarily congruent, pieces. I asked students to provide me with several different arrangements of fourths on their geoboard and asked them if the fourths in one arrangement were equal to the fourths in another arrangement of the same size geoboard. I was interested in this question to determine if students realized that fourths of the same size whole were the same size even though they were not congruent.

If students did not provide an arrangement that had non-congruent pieces, I would make such an arrangement on a geoboard and present it to the student. I asked if what I had made were fourths and asked him/her to explain why or why not. An example of such an arrangement is given in Figure 3.

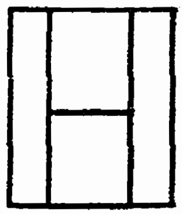

Figure 3. Geoboard example of non-congruent fourths provided by teacher

Students were then shown a variety of examples and non-examples of fourths drawn on a sheet of paper (question 3). Students were asked to look at the figures and determine which ones accurately represented fourths and which ones did not. Students were then asked to explain their thinking for each one. Also, different models of fourths were provided in order to ascertain how generalized each student's understanding of the meaning of "fourth" was. In other words, did students have a generalized idea that "fourth" meant to partition a given unit or whole in to 4 equal sized but not necessarily congruent pieces. Figure 4 illustrates the examples that were shown to students.

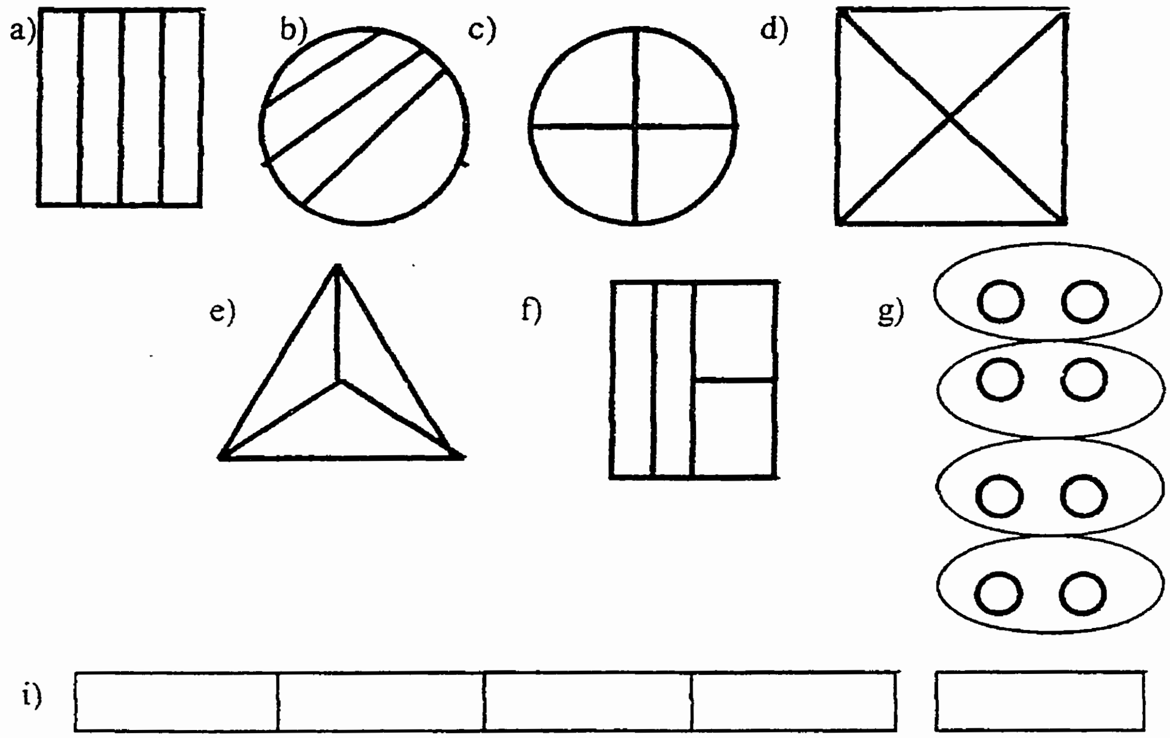

Figure 4: Examples and non-examples of fourths provided to students in the first interview.

Figures 4a, 4c, and 4d all represented continuous or area models partitioned into 4 equal, congruent pieces. Figure 4f represented an area model partitioned into non-congruent fourths. Figures 4b and 4e were also area models but both were non-examples of fourths. Figure 4b was a non-example because the partitions were not equal in size and 4e because they were not partitioned in to 4 parts. Figure 4g was a discrete or set model of fourths. In this question I was interested to determine what aspect of the model students concentrated on to determine if the example was indeed fourths. As a modification of this question, I would introduce another question that rearranged the circles into 2 sets of 4 circles (Figure 5) and asked students if this arrangement represented fourths.

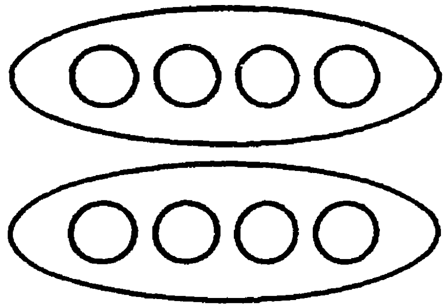

Figure 5: Modification of Figure 4g to produce Figure 4h

The last figure, figure 4i, was a continuous or linear model designed to determine whether or not students could identify the whole and decide if the whole was properly partitioned into fourths. I was trying to determine if students were merely seeing "four" pieces in total versus partitioning the identified whole into four equal pieces to determine fourths. I wanted to see if having "five" congruent pieces would alter each student's ability to explain where the fourths were.

I also analyzed the responses to the diagrams 4a-4i to determine which models students were able to interpret confidently and which ones they were not sure about.

Task # 3: Using different modes to represent fractions

Figures 6 and 7 were the illustrations that I provided to students to specifically determine how they would react to these models of fractions (Questions 4 and 5). Figure 6 is a discrete or compound set model. I asked the student to tell me what part of the set was shaded. I then asked them what they would say if I told them that 4/6 of the set was shaded. I also asked them to explain their thinking. I was interested in this question to determine if each student could group the circles and if so, how they would do so when given the fraction 4/6. Since 4/6 was the fraction given in task # 1, I was interested in comparing each student's response to 4/6 with this model compared to the one s/he used in task # 1.

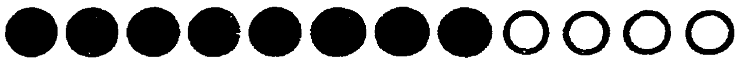

Figure 6: Compound set model to represent 4/6

The illustration in figure 7 is a continuous, linear model that required the student to shade in 2/3 on both 7a and 7b. I was interested to see if students could "unpartition" or ignore the lines in the bar in figure 7a. This task of unpartitioning the bar is similar to the task in figure 6 that required students to group the circles. In figure 7a, students had to group the smaller boxes into one group or double the size of the whole and ignore the lines. Figure 7b was given to students to see which figure they would do first and which they found easier to use as a model.

7a)
7b)

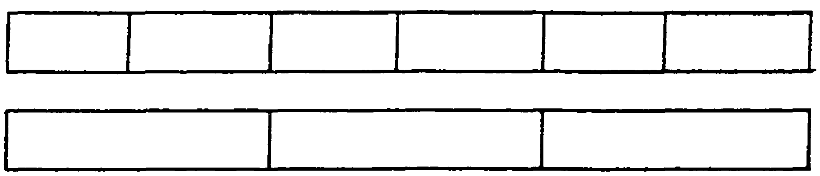

Figure 7: Two continuous (linear) models to represent 2/3

Task # 4: Identifying the whole or unit

The illustrations in figures 8 and 9 were shown to students to determine if they could identify the whole or unit (Questions 6 and 7).

Figure 8: Shaded bar used to identify the whole

Students were shown figure 8 and asked to state a fraction for the part of the bar that was shaded. They were asked to explain how they figured it out. I then asked students if there could be another name for the part that was shaded. I assumed that students would see the entire bar as the whole and state a fraction that was less than one for the shaded portion, so I asked students what they would say if I told them that 1 1/2 was shaded. I was interested if they could explain how figure 8 represented 1 1/2. In particular, I wanted to see if they could identify the whole and what process they would use to do so.

9a)  9b) 

Figure 9: Finding the whole with 4/4 and 4/3

After figure 8 students were given figure 9 and told that figure 9a represented 4/4 and 9b represented 4/3. Students were then asked to identify the whole. I was interested in seeing what each student took as the whole and if s/he could use the information from the given fractions to identify the whole. In particular, I was looking to see if the students could identify 4/4 as equivalent to one whole and therefore represented by the all four boxes. Whereas 4/3 was 3/3 and 1/3, equivalent to one whole and another third. In other words, three boxes represented the whole, one less than the four boxes of the diagram.

Section 1.2 Sense of the orderliness of fractions

Understanding number magnitude, according to Markovits and Sowder, "...encompasses the abilities to compare numbers, to identify which of two numbers is closer to a third number, to order numbers, and to find or identify numbers between two given numbers" (1994, p. 6). If a student is to be able to order fractions and understand fraction equivalence, then he or she has developed an "...understanding of the compensatory relation between the size and number of equal parts in a partitioned unit" (Behr, Wachsmuth, Post, Lesh, 1984, p.338). It is probably through the *connecting* phase of instruction where students develop this understanding. By working with manipulatives, students develop an understanding of the inverse relationship between the size of the denominator and the size of the unit fraction. They may also at this stage develop the relationship between various fractions and the benchmarks of half and one.

In this section, I discuss the questions that I asked students in the first interview that addressed their sense of the orderliness of fractions. I looked for evidence of their understanding of the inverse relationship between the size of the denominator and the size of the unit fraction (section 1.2.1, p.12) and for evidence of strategies to order fractions (section 1.2.3, p.12). In particular, I wanted to see if students used comparison to benchmark referents when ordering fractions with unlike denominators (section 1.2.4, p.12) as a strategy.

Task #5: Ordering unit fractions-inverse relationship (denominator and size).

In this task (question 8) I showed each student a written set of unit fractions (1/5, 1/3, /12, 1/6, 1/4) and asked that the fractions be ordered from smallest to biggest. I provided a piece of paper and pen for this. While performing the task, I asked the student to talk aloud and explain his or her thinking. If the student was unable to perform the task or was unsure about his or her order, I offered the fraction circles so that the answer could be obtained or verified. In this question I was interested to see if students had developed an understanding of the inverse relationship between the size of the denominator and the size of the fraction or if their knowledge of whole numbers interfered with their work with fractions.

Task #6: Ordering non-unit fractions-evidence of benchmark strategy

In the first question of this task (Question 9), each student was asked which fraction was larger- 18/19 or 3/4. This questioned was designed to see if students would use the "convert to like denominator" method generally taught in school to compare the fractions or whether they would use the benchmark of "one" as a strategy. In particular, I noted whether or not each student noticed that each fraction was one "piece" away from one whole and whether or not they compared these unit fractions to determine the larger fraction.

In the second question (Question 10), I asked students to compare 7/15 and 6/10 to the benchmark 1/2. I was interested in the method they would use to determine this relationship to 1/2. The second part of the question involved stating which of the two fractions was larger. I was interested to see if students would use the relationship of the two fractions to 1/2 as a strategy or if they would compare the fractions another way.

Instruction

Setting

Once the interviews were complete I began the unit of instruction on fractions with both math classes. The unit lasted for approximately eight weeks in total, although the section of interest to this study lasted approximately four of those eight weeks. During this time, I began the study of fractions "at the beginning"; that is, I had students work with manipulatives to ensure that they connected the written fractional notation with referents that they could draw upon to guide their future work with fractions. Although this work should have been done long before grade 8, my experience, and indeed that of research into fractions (Hiebert 1988; Mack 1990; Wearne & Hiebert 1989), is that students have had little experience working with manipulatives and developing referents for fractions. As such, I provided approximately 6 one-hour classes to this end. We moved from these connecting exercises to a series of activities that were designed to compare fractions to the benchmarks of one and then half. Next we moved on the exercises that worked on modeling the equivalence algorithm with manipulatives. Most students knew the equivalence algorithm, but could not explain or model what was happening when they multiplied and/or divided the numerator and denominator with the same integer. After this section, I moved the students into strategies which were designed to help them order or compare fractions without using the "like denominator" method. Once these activities were completed, I moved students into operations with fractions. Since this thesis does not involve operations with fractions, I will not be discussing what activities were included in this section of the unit.

Lesson Plans

What follows is an outline of the fraction sense activities, with explanations of each, that were used in both math 8 classes. Each class lasted for approximately 60 minutes.

<u>Lesson # 1:</u> <u>Tige's Treats</u> 1.5 classes

Reference: NCTM Addenda Series: Understanding Rational Numbers

Purpose: To establish that the fractional name tells you how many equal parts to separate the whole into
To establish that "equal" means equal in size or amount not congruent
To establish that all fourths of the same size whole are equal in size, even though they are not congruent
To differentiate between four and fourths
To work with a continuous (area) model as a referent for fractions

Materials: Geoboards, geobands, and geo- dot paper

Explanation: Students worked in small groups (3-4 students) to determine how many ways they could cut a square "cake" (geoboard) into fourths. They had to record all their fourth arrangements on geo-dot paper and answer questions in their math journal. The team with the most different arrangements won a small cake.

During this activity many questions arose which helped us to define fraction. In particular, does separating a given area into 4 (unequal) parts constitute fourths, what does "equal" mean and are non-congruent fourths the same size. A side effect of this work with the area model was learning about conservation; that is, that a half or fourth of a given area is always the same amount, no matter how it is partitioned. Also, students needed to consider how to calculate the area of parts of squares (on the geoboard) to determine the total area of the fourth. A lot of problem solving went on with this activity.

Lesson # 2: Examples and Non-Examples of Fourths 0.5 class

Reference: Elementary School Mathematics John A. Van de Walle p.225

Purpose: To establish the conditions needed for "fourths"- a unit partitioned into 4 equal in size, but not necessarily congruent, parts

Materials: Photocopy of examples and non- examples (Appendix B)

Explanation: Students were given a photocopy of examples and non-examples of fourths. They were asked to explain in their journals which diagrams were and were not examples of fourths and why they thought so. A class discussion of the answers followed individual responses in their math journals.

Lesson # 3: Finding Fractional Parts and Wholes Using Cuisennaire Rods and Pattern Blocks 4 classes

Reference: Elementary School Mathematics John A. Van de Walle p.229
A Collection of Math Lessons Marilyn Burns p. 224

Purpose: To use continuous length and area models to model fractions
To model fractions less than, equal to, and greater than one
To emphasizes that fractions are relative to a defined unit
To connect physical referents to fractional notation
To determine the fractional name (numerator and denominator) when the whole is defined and a fractional part is identified
To write proper, improper, and mixed fractions
To determine the whole when the fractional name is given and the fractional part is identified
To determine the fractional part when the fractional name and whole are identified
To determine that the denominator is the "denominations" of partitions of the whole and that the numerator "counts" the denominations.

Materials: Cuisennaire rods, questions on overhead (Appendix B),
Worksheet: Converting Fractions (Appendix B)

Explanation: Students were given a set of Cuisennaire rods and copied the question chart of the overhead into their math journals. After some initial instruction as a class, students worked in small groups and I circulated among the student groups to help with questions they had. Once finished the overhead material, students worked on a worksheet exercise with accompanying questions.

In the first tasks. the whole and fractional part were defined to be specific rods (e.g. whole = orange rod and fractional part = 1 yellow). The task was to compare the fractional rod (which initially was less than the whole) to the whole and determine the fractional name and write the symbol for the fractional rod (e.g. yellow = 1/2 of whole orange). In order to accomplish this task, students had to determine the denominator. This was accomplished by iterating the fractional part or smaller rods that evenly partitioned the defined whole and then counting the relevant denominations (e.g. 2 yellows = 1 orange therefore the denominator = 2 ; since the count is one yellow the numerator is 1 to give the fraction 1/2)

The second task was very similar to the first, except that the fractional part or rod was larger than the defined whole. This resulted in student writing improper fractions and writing mixed fractions. It emphasized the numerator as counting the denominations that could result in an improper fraction.

The third task involved defining the whole (e.g., brown) and giving the fraction (e.g., 7/8) and asking students to find the rod which represented this fractional part (e.g., black = 7/8 of brown). This task required, that students use the denominator to partition the whole (e.g., into 8 equal parts) and then iterate these until the numerator value is reached (e.g., 7) and then find the rod that equals this value (e.g., black). This task emphasizes the role of the denominator and numerator and their relationship to the whole.

The fourth task was a variation of the third except the fraction which was given was larger than the whole, therefore students had to look for a rod which was longer than the defined whole.

The fifth task gave the fraction (e.g., 7/9) and defined the fractional rod (e.g., blue) and students had to find the whole. This task was accomplished by iterating the defined rod (e.g., blue) into the number of parts given by the numerator (e.g., 7) and then iterating the remainder of equal parts to reach the whole (e.g., 7 and 2 more parts = 9/9 parts = black rod).

The sixth task was the same as the fifth, except the given fraction was greater than one whole, therefore the rod which represented the whole was shorter than the defined rod.

Work with the pattern blocks was very similar to the Cuisennaire rods. To emphasize that the whole is not always 1 object or area to be partitioned the whole, in the case of the pattern blocks, was sometimes given as 2 or more shapes (e.g., whole=2 yellow hexagons). This was done so that students would have to consider the fraction (numerator and denominator) in terms of the whole and to avoid the overgeneralization that a certain shape or color is half, as can happen with a model when the whole is always defined as the large circle, yellow hexagon etc.

Lesson # 4: Benchmark comparison to one whole 1 class

Reference: Elementary School Mathematics John A. Van de Walle p. 230

Purpose: To use manipulatives to look for patterns in the numerator and denominator to write fractions that are less than, equal to, and greater than one whole

To write improper fractions and mixed fractions in a variety of ways

Materials: Manipulatives (fraction circles, pattern blocks),
Worksheet: Fractions- Less Than, Equal to, Greater than 1 (Appendix B)

Explanation: Students worked with manipulatives and the worksheet to determine rules for writing fractions which were less than, equal to, and greater than one whole. This work was an extension of the previous work we had done in lessons 1-3. Students were encouraged to write a variety of equivalent mixed and improper fractions in order to prepare them for their work with subtracting which would require borrowing. (e.g., 2 4/6 = 1+ 6/6 + 4/6 = 1 10/6= 6/6 + 6/6 + 4/6 = 16/6 etc.)

Lesson #5: Equivalent Fractions 2 classes

Reference: Elementary School Mathematics John A. Van de Walle p.234

Purpose: To determine what equivalent means
To relate the known "equivalence" algorithm to a referent

Materials: Paper strips
Worksheet: Equivalent Fractions: Using Models (Appendix B)

Explanation: Students answered questions in their journal and then discussed as a class what fraction equivalence meant to them. They then explained the methods they used to create equivalent fractions. Students then folded paper strips to create fraction equivalents. The actions on the paper strip were related to the written algorithm. That is, multiplying numerator and denominator with the same number (e.g., 2) was an increase the number of sections by 2 fold. I also showed students that dividing by the same integer was the same as grouping or unpartitioning. I emphasized to students that since the fractions were equivalent that they were the same amount and therefore when they were multiplying they were actually multiplying by 1 to get the same quantity, except that the 1 was in the form 2/2 or 5/5 etc.

Lesson #6: Modeling fractions with discrete (set) models

Purpose:
To model fractions using sets
To re-emphasize that the denominator tells you how many equal groups
To understand that the denominator is the number of groups, not the number of objects within a group
To understand that the number of blocks within a group must be the same to be equal but is not necessarily the same number as the denominator
To re-emphasize that the numerator counts the equal groups
To model fractions greater than one whole with sets of objects

Materials:
colored blocks
Worksheet: Equivalent Fractions: Representations (Appendix B)

Explanation: Students were given a set of colored blocks and were shown how to model given fractions using the blocks. I emphasized to students that they needed to attend first to the denominator and partition the whole set of objects into that many equal groups. For example, if the whole set was defined as 15 blocks and the denominator was 3 (thirds) then the 15 blocks needed to separate into 3 equal groups of 5 blocks. Once the set was correctly partitioned, then the student needed to attend to the numerator and count that number of equal groups. If the numerator was 2 (e.g., 2/3) then the student needed to count 2 groups and then determine the total number of blocks within the 2 groups to give the answer that 2/3 of the whole of 15 blocks is 10. If the numerator was greater than the denominator (e.g., 4/3) then the student needed to get another set of blocks in order to count the correct number (e.g., 4). This would result in more blocks than the whole, which is consistent with the written notation (e.g.,4/3) which is greater than one whole.

Lesson #7: Benchmark comparison to half 1 class

Purpose:
To compare fractions to one half
To use half as a benchmark for comparison of fractions

Materials: Worksheet: Equivalent Benchmarks (Appendix B)
Overhead exercises (Appendix B)
Fraction circles

Explanation: Students were provided with worksheets and manipulatives. They were asked to determine how a fraction compared to 1/2. From the previous work with manipulatives, students had discovered that fractions are equivalent to half when the numerator is half the value of the denominator (e.g., 3/6 = 1/2 because 3 is half the value of 6). They also knew that a fraction is less than half when the numerator is less than half the value of the denominator (e.g., 2/6 is less than half because 2<3) and that a fraction is greater than half when the numerator is greater than half the value of the denominator (e.g., 4/6 is greater than half because 4 >3). Students then worked on comparing fractions with unlike denominators because comparing the benchmark half rather than converting to like denominators.

Lesson #8: Comparing Fractions using Benchmark Strategies 2 classes

Reference: Elementary School Mathematics John A. Van de Walle p. 233

Purpose: To use a variety of benchmark strategies to compare and order fractions with unlike denominators.

Materials: Worksheet: Compare and Test (Ordering) (Appendix B)
Manipulatives- fraction circles
Quiz: Comparing Fractions Using Concepts (Appendix B)

Explanation: Students were provided with the worksheet and were instructed on how to fill it out. We did the first few examples together and then students worked independently to fill in the worksheet. We discussed the answers when students were complete. Manipulatives were used to explain comparisons, especially the "less far away from one" strategy.

The worksheet encouraged the students to use the following strategies to compare the fractions:

1. More of the same size parts (e.g., 5/8 > 3/8 because 5 eighths > 3 eighths)
2. Same number of different size parts (4/7 > 4/9 because sevenths are larger pieces than ninths therefore 4 sevenths > 4 ninths)
3. Comparison to half (3/8 < 6/10 because 3/8 is < 1/2 and 6/10 > 1/2)
4. Comparison to one

a) more than/ less than one (3/5 < 3/2 because 3/5 < 1 but 3/2 >1)
b) less "far away" from one (4/5 < 7/8 because both are one piece away from the whole but 1/8 is smaller than 1/5 therefore 7/8 is closer to 1 than 4/5)

Lesson #9: Finding Approximate Benchmarks 1 class

Purpose: To use understanding of multiplication facts to estimate simpler fractions that could be used as benchmarks.

Materials: Worksheet: Close to But No Cigar (Appendix B)

Explanation: Students were given worksheet and I demonstrated how I to use their knowledge of multiplication facts and ratios to find "Friendly Fractions". For example, students would look at the fraction 4/11 and think that 4 and 12 are in a "family of facts" (multiplication facts) and therefore estimate the fraction 4/11 to be 4/12 which they can quickly change to 1/3 because they know that 3 is the other term in the 4 and 12 "fact family". Students who had weak recall of multiplication facts were encouraged to use a multiplication chart.

The remainder of the lessons in this unit involved operations with fractions. Since I did not focus on fraction operations in this study, I have not included these lesson plans.

Post Interview

Setting

Once the unit of fractions was complete, I re-interviewed the seven students who were involved in my study. These interviews occurred approximately 2.5 months after the initial interviews. Due to time constraints I interviewed the students in groups of two or three. The tasks for the final interviews were different from the first tasks but related to them in that they dealt with what understanding or referents students had for fractions and what strategies they used to compare and order fractions. Once again, manipulatives and paper and pencils were available for students to use. Below, I have outlined the tasks given to students during the final interviews and have listed them according to the framework categories that they correspond to.

Post-Interview Questions

Section 1.1 Sense of the relative and absolute magnitude of fractions

As in the first set of interview questions, these tasks examined the manner in which students modeled fractions using a variety of manipulatives, including a selection of continuous and discrete models. As in the first interviews, it is the part-whole meaning of fractions that I focused on. How students saw fractions in relation to the whole and what interpretation or meaning they gave to the numerator and denominator was examined.

Task # 1: Interpreting written fractions.

In this task students were first shown the fraction 3/6 and then 6/3 written on a card. They were asked what they thought or could say about each of the fractions and then were asked to model the fractions. I then gave the students blocks (24 with 3/6 and 12 with 6/3) and asked them to model the fractions using the blocks (set model). I then asked the students what the denominator and numerator in each of the fractions tells them. After these questions were complete, I then gave them photocopies of the diagrams in figure 10 and asked students to use them to model 3/6 and 6/3.

a) b)

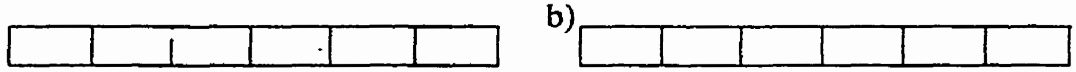

Figure 10: Diagrams provided to students to model 3/6 and 6/3.

The questions in this task were designed to see what meaning the students gave the numerator and denominator and how the students viewed the fractions in relation to the whole. I was particularly interested in how students would relate the fraction 6/3 to the whole. Initially students were free to model the fractions as they chose and then I asked them to use a set model and the diagrams in figure 10. I was interested in comparing how they handled the fractions with the different models. In particular, I was interested to see if students were equally comfortable using all models for the fraction 6/3 and how they would relate this fraction to the whole in order to model it.

<u>Task # 2: Finding fractional parts and wholes.</u>

In these tasks, students were given Cuisennaire rods (continuous length model) to model fractions. The wholes and fractional parts were identified as follows:

a) orange is 1, what part is 2 reds?

(fraction less than one whole- iterate the red)

b) yellow is 1, what part is 1 light green?

(fraction less than one whole- cannot iterate the green, need to get a smaller rod)

c) light green is 1, what part is yellow?

(fraction greater than one whole)

The students then needed to use the blocks and state the fraction. I asked them to explain how they determined the numerator and denominator. These tasks were designed to see how students used a different manipulative, in this case a continuous length model, to represent fractions. Again, I was interested in the process that each student used to determine the numerator and denominator and how they refereed to the whole in order to achieve this.

The next section of these tasks provided the fractional name of a given rod and asked the students to find the whole. The questions were as follows:

a) brown is 4/5, what is the whole? (whole is a larger rod)

b) black is 1 2/5, what is the whole? (whole is a smaller rod)

Before students were allowed to find the whole, I asked them if the rod they were looking for was smaller or larger than the brown or black one. I was interested to see if they could realize that if the black was 1 2/5 that this was larger than one and therefore the rod they were looking for was smaller than the black. I was interested if students could handle the concept that the whole was smaller than the fractional part.

Section 1.2 Sense of the orderliness of fractions

The tasks in this section of the interview were designed to determine which strategies students were using to compare and order fractions. They were also intended to shed more light on how students thought about the numerator, denominator, and the whole and how they coordinate this information.

Task # 3: Comparing and ordering fractions.

Students were shown a series of cards with pairs of fractions written on them. They were asked how they would go about determining which one is the largest of the pair. The questions were designed so that a variety of strategies could be used. The questions and the possible strategies that could be used to order them are provided below:

a)	3/4 and 3/5	Same number of pieces, but pieces are different in size. Needs to consider the inverse relationship of denominator to fraction size. The larger size pieces (fourths) are more.
b)	4/5 and 6/5	Same size pieces, the fraction with more pieces is more (6/5) OR 6/5 is more than one and 4/5 is less than one so 6/5 is more
c)	2/3 and 6/7	Both are one piece away from one whole, but sevenths are smaller therefore 6/7 is closer to 1 and therefore larger than 2/3
d)	3/4 and 5/12	Over half and under half- benchmark comparison
e)	3/5 and 5/7	Both two pieces away from one whole- sevenths are smaller therefore 5/7 is closer to one whole and therefore larger than 3/5
f)	3/5 and 6/9	Convert to percentages: 3/5 is 60 % whereas 6/9 is 66% therefore 6/9 is more.

g) 4/10 and 5/9 Benchmark to half- 4/10 is less than half and 5/9 is more than half therefore 5/9 is more.

The second part of this task was to order a set of fractions using a variety of strategies. The fractions were written on a card and shown to students. Students were provided with paper to write their answers on.

4/8	5/9	5/12	2/5	10/8
6/6	3/1	11/7	9/18	4/21

Figure 10: Comparing and ordering fractions problems

Both sets of fractions involved comparing to benchmarks of half and one whole in order to order them. The fractions, 5/12 and 2/5, in the first set were both under one half. The fraction 5/12 was one-twelfth below half and 2/5 was half a fifth or one-tenth below half. I wanted to see how students would resolve these fractions which were very near to each other in size.

Task # 4: Qualitative reasoning about fractions

These last questions were designed so that students would have to provide qualitative versus quantitative answers regarding fractions. The fraction "a/b" was written on a card and shown to students. I asked a number of questions that are listed below. Each question was intended to prompt students to think qualitatively about the size of fractions and the relationship of the numerator and denominator to size of the fraction

a) " If I told you "a" was (less than, equal to, greater than) "b", what can you tell me about the size of the fraction compared to one whole?

These questions were designed to have students think about the numerator relative to the denominator and to use this information to determine the fraction size relative to one whole.

b) "If I increased the number "a" what would happen to the size of the fraction?

This question focused on the numerator and its relation to the size of the fraction. I was interested in seeing if students understood that there is a direct relationship between the size of the fraction and the numerator. That is, an increase in the numerator results in a larger fraction.

c) "If I increased the number "b" what would happen to the size of the fraction?

d) "If I decreased the number "b" what would happen to the size of the fraction?

Questions "c" and "d" focused on the denominator and its relation to the size of the fraction. I was interested in determining if students understood that there is an indirect relationship between the size of the fraction and the denominator. That is, an increase in the denominator results in a decrease in the size of the fraction and vice versa.

Analysis Method

Once the interviews were completed I transcribed the videotapes. I watched the videotapes and recorded the audio verbatim onto index cards. I also wrote down notes regarding gestures, voice intonation, and work with the manipulatives that would serve as further evidence to student explanation. Each question was recorded on separate cards. Once this was complete, I coded each student's response for each task in order to generate similarities and differences in approach of the seven students I interviewed. The coding of the data is presented in tables in the next chapter (Chapter V: Data Analysis) along with a discussion of the analysis of the data.

CHAPTER V

DATA ANALYSIS

This chapter presents the analysis of the results of my study. First, the method used to analyze the data gathered from the videotaped pre- and post- interviews with the seven students is described. Then, the analysis of the results is summarized in eleven tables. A discussion of the analysis of the data follows the tables and completes the chapter. For the purpose of this thesis, only the results that related to the meaning that each student gave to fractions and the strategies that each student used to order and compare fractions was analyzed.

Analysis Method

Once the interviews were completed I transcribed the data from the videotapes. I watched the videotapes and recorded the audio verbatim onto index cards. I also wrote down notes regarding gestures, voice intonation, and work with the manipulatives that would serve as further evidence for student explanations. Each question was recorded on a set of index cards.

Once the transcription was complete, I looked at all the data collected for each student's response to each question. The responses from each student for each question were collated on a second set of index cards. From these collated responses I looked for similarities and differences in approach to answering each question. I then made notes about common trends (or lack thereof) that I noticed in the student responses for each question.

For the purpose of this thesis, only the data from the questions that addressed each student's *knowledge and facility with fractions* was analyzed (Section 1, p.12). In particular, sub-sections 1.1 and 1.2 of the adapted framework were examined (p.12). The first being an exploration of how each student viewed fractions in relation to the whole or unit and what meaning or interpretation s/he gave the numerator and denominator (section 1.1.1 and 1.1.2, p.12). Inherent in this exploration is a study of what models or referents each student utilized when thinking about fractions. If a student has a well-developed sense of the magnitude of fractions then s/he will, according to Hiebert (1988), be able to call to the *mind's eye* visual referents that were firmly connected during the initial *connecting* phase. If the notion of fraction is truly generalized, then the student is able to bring to mind many physical models that represent the various meanings of fractions. A sign, therefore, of an individual with a well-developed notion of fractions, is his or her ability to translate between these different visual modes with ease. As such, while I explored what meaning students had for the magnitude of fractions, I simultaneously explored what models they could use to represent them (section 1.1.1, p.12)

The exploration of sub-section 1.2 involved studying the processes that each student was using to order or compare fractions. In particular I was interested if each student understood the inverse relationship between the size of the denominator and the fraction size and whether s/he used this information to order fractions (section 1.2.1). I was also very interested in how students ordered a given set of fractions (section 1.2.3) and if they used benchmarks, such as half and one, to order fractions (section 1.2.4).

Although each question in the pre and post-unit interviews was designed to examine a particular aspect of fraction sense, in practice, each question elicited many aspects of each student's fraction sense. In order to manage the large amounts of data collected from the pre and post-interview questions, only data from the questions which best addressed the particular aspect of fraction sense that I wished to examine was analyzed. The questions and responses are grouped according to the following aspects of fraction sense given below.

<u>Pre-Unit Interview Data Analysis</u>

<u>Section 1.1 Student's ability to sense the relative and absolute magnitude of fractions</u>

Reference to Whole/Unit (task # 1, and # 4))

Meaning of Numerator and Denominator (task # 1, #2 and # 4- Fig. 9)

Referents for Fractions (task # 2, #3,)

<u>Section 1.2 Sense of the orderliness of fractions</u>

Inverse Relationship between Denominator and Fraction Size (task # 5)

Ordering and Comparing Strategies (task # 6)

Post-Unit Interview Data Analysis

Section 1.1 Student's ability to sense the relative and absolute magnitude of fractions

Reference to Whole/Unit (task #1 and # 2)

Meaning of Numerator and Denominator (task # 1, # 2, and # 5)

Referents for Fractions (task # 1, # 2)

Section 1.2 Sense of the orderliness of fractions

Inverse Relationship between Denominator and Fraction Size (task # 3)

Ordering and Comparing Strategies (task # 3)

The data is presented in a matrix chart format. Each aspect of fraction sense listed above is presented in a chart with behaviors that students displayed forming the categories (listed vertically). Each task number is listed horizontally. The data listed in the chart describes how many times the seven students interviewed displayed that particular behavior for each of the questions or tasks. The number of students displaying that behavior is presented in brackets in the chart as well. For example, the number 20(7) in the third line of Table # 3 (p.72) means that seven students were unable to correctly identify which model was partitioned in fourths and they demonstrated this behavior 20 times. Following these data tables I discuss the data in further detail and elaborate on student responses with references to specific students responses.

Pre-Unit Interview Data Results

Section 1.1 Student's ability to sense the relative and absolute magnitude of fractions

Table # 1: Reference to Whole/Unit (task # 1 and # 4)

Responses	Task (# 1 and # 4)			
Frequency of responses (Number of Students)	Interprets Written Fraction (4/6)	Given 1 ½ Find Whole (Fig. 8)	Find the Whole (4/4) (Fig. 9a)	Find the Whole (4/3) (Fig. 9b)
Interprets fractions as a whole to be partitioned	6 (6)	6 (6)	6 (6)	6 (6)
Does not interpret fractions as a whole to be partitioned	1 (1)	0 (0)	0 (0)	0 (0)
Mentions the word "whole" directly	0 (0)	2 (2)	3 (2)	5 (3)
Only mentions the "whole" indirectly	2 (2)	0 (0)	0 (0)	0 (0)
Models whole being partitioned but does not refer to it direct or indirectly	4 (4)	4 (4)	3 (3)	3 (3)
Does not model the whole	1 (1)	0 (0)	0 (0)	0 (0)
Interprets "whole" to be entire object/set to be partitioned	5 (5)	7 (4)	6 (6)	14 (4)
Does not indicate the "whole" when given an object/set to be partitioned	1 (1)	0 (0)	0 (0)	0 (0)
Able to interpret "whole" to be less than entire object/set	NA	2 (2)	NA	2 (2)
Able to interpret "whole" to be less than entire object/set after prompting	NA	3 (1)	NA	3 (3)
Unable to interpret "whole" to be less than entire object/set	NA	2 (3)	NA	1 (1)

Table # 2: Meaning of Numerator and Denominator (task #2 and # 4- Fig. 9)

Responses	Tasks # 1, #2, and #4 (Fig.9)						
Frequency of responses (Number of students)	Interprets written notation (4/6)	Geoboard Fourths (Fig. 3)	Examples and non-examples of 4ths (Fig. 4) Area (a-f)	Set (g-h)	Linear (i)	Fraction equal &greater than one whole Fig. 9 (4/4)	Fig. 9 (4/3)
Refers to numerator as what is there/removed	10 (6)	NA	NA	NA	NA	2 (2)	4 (4)
Refers to denominator as partitions of whole	2 (1)	8 (5)	0 (0)	0 (0)	4 (2)	4 (3)	4 (3)
Refers to denominator as partitions (no mention of whole)	7 (5)	11 (3)	11 (7)	13 (5)	4 (4)	5 (4)	8 (3)
Partitions whole correctly using denominator or fraction name (third etc.)	6 (6)	7 (7)	10 (7)	19 (4)	5 (5)	8 (6)	8 (4)
Does not partition whole correctly with denominator/ name	1 (1)	2 (1)	0 (0)	19 (5)	5 (4)	0 (0)	20 (3)
Models congruent/equal size partitions	5 (5)	17 (7)	NA	6 (4)	1 (1)	NA	NA
States parts need to be equal size	1 (1)	10 (5)	22 (6)	5 (1)	2 (1)	0 (0)	0 (0)
Uncertain if parts need to be equal in size	0 (0)	5 (3)	0 (0)	4 (4)	0 (0)	0 (0)	0 (0)
States partitions needs to be congruent	NA	2 (1)	0 (0)	NA	NA	NA	NA
States partitions do not need to be congruent	NA	3 (3)	3 (1)	NA	NA	NA	NA
Uncertain if partitions need to be congruent	NA	1 (1)	0 (0)	NA	NA	NA	NA
Has difficulty explaining or modeling fractions when numerator> denominator	2 (2)	NA	NA	NA	8 (7)	NA	22 (5)

Table # 3: Referents for Fractions (tasks # 1, # 2, # 3, # 4-9a)

Responses	Area		Set		Linear	
Frequency of responses (# of students)	Congruent Fig.4a,c,d,e	Noncongruent Fig.4b,f	Simple Fig.6	Cmpnd Fig.4g,h	<1 Fig.7,9a	>1 Fig.4i& 9b
Able to identify or explain the denominator correctly without prompting -able to partition/unpartition	51 (7)	15 (7)	8 (6)	7 (4)	18 (5)	4 (3)
Able to identify or explain denominator correctly with prompting	1 (1)	0 (0)	1 (1)	6 (3)	0 (0)	8 (6)
Unable to identify/ explain denominator correctly	0 (0)	0 (0)	3 (1)	20 (7)	3 (1)	5 (5)
Able to identify/ explain numerator correctly without prompting	NA	NA	7 (6)	3 (2)	19 (6)	2 (2)
Able to identify/ explain numerator correctly with prompting	NA	NA	1 (1)	2 (2)	0 (0)	3 (2)
Unable to identify/ explain numerator correctly	NA	NA	3 (1)	9 (3)	1 (1)	3 (2)
Attempts to invert fraction (4/3→3/4)	NA	NA	0 (0)	0 (0)	0 (0)	4 (3)
Abandons previously established meaning for numerator and/ or denominator in order to fit notation to the model	0 (0)	0 (0)	0 (0)	13 (6)	2 (1)	4 (2)
Incorrectly alters model in order to fit to fraction notation	0 (0)	0 (0)	2 (1)	2 (1)	2 (1)	1 (1)

Section 1.2 Sense of the orderliness of fractions

Table # 4: Inverse Relationship between Denominator and Fraction Size (task # 5)

Frequency of Responses (Number of students)	Order of 1/2, 1/3, 1/4, 1/6
Orders unit fractions correctly and is confident of order	3 (3)
Orders unit fractions correctly but is unsure or draws diagrams or uses known benchmarks to confirm order	2 (2)
Orders unit fractions incorrectly	4 (2)
Corrects order of unit fractions	2 (2)
Explains inverse relationship between denominator and fraction size correctly	3 (3)
Does not explain inverse relationship between denominator and fraction size correctly (uses whole number logic)	2 (2)
Corrects explanation of inverse relationship between denominator and fraction size after prompting/ work with manipulatives	3 (3)

Table # 5: Ordering and Comparing Strategies (task # 6)

Frequency of Response (Number of Students)	18/19 vs. 3/4
Correctly states the larger fraction without converting to like denominators	1 (1)
Uses a benchmark strategy to order fractions (reference to ½ or 1)	0 (0)
Correctly states larger fraction by using like denominator method or converting to decimals	1 (1)
Correctly states larger fraction by drawing a diagram	0 (0)
Correctly states larger fraction but states an incorrect reason (focuses on range between numerator and denominator)	0 (0)
Incorrectly or is unable to state larger fraction	8 (4)
States fractions are equal (both missing one piece)	6 (4)
Corrects answer after drawing diagram	2 (2)

Post-Unit Interview Data Results

Section 1.1: Students' ability to sense the relative and absolute magnitude of fractions

Table # 6: Reference to Whole (task # 1 and # 2)

Response	Tasks # 1 and # 2					
Frequency of responses (Number of Students)	Interprets written notation		Given the whole find the fraction		Given the partition and fraction name find the whole	
	(3/6)	(6/3)	<1	>1	<1	>1
Mentions the word "whole" directly	2 (2)	9 (5)	24 (3)	2 (2)	1 (1)	1 (1)
Mentions the whole indirectly	2 (2)	0 (0)	0 (0)	1 (1)	0 (0)	0 (0)
Models one whole but does not refer to it directly or indirectly	9 (3)	0 (0)	9 (7)	3 (3)	8 (5)	1 (1)
Interprets whole to be entire object/ set to be partitioned (whole > fractional part)	25 (7)	10 (5)	10 (7)	5 (5)	7 (4)	0 (0)
Able to interpret the whole to be less than entire object/ set (whole<fractional part)	NA	6 (2)	NA	6 (5)	NA	3 (3)
Able to interpret the whole to be less than entire object/ set with prompting (whole<fractional part)	NA	6 (2)	NA	0 (0)	NA	2 (2)
Unable to interpret the whole to be less than entire object/ set (whole<fractional part)	NA	3 (3)	NA	0 (0)	NA	0 (0)

Table # 7: Meaning of Numerator and Denominator (tasks # 1, and # 2)

Frequency of responses (Number of students)	Tasks # 1 and # 2					
	Written notation (3/6)	(6/3)	Given one find the partition		Given the partition find the whole	
	<1	>1	<1	>1	<1	>1
Refers to numerator as what is there or removed	5 (3)	1 (1)	1 (1)	0 (0)	0 (0)	0 (0)
Refers to numerator as "counts" of the partitions or the denominator	2 (2)	3 (2)	0 (0)	0 (0)	0 (0)	0 (0)
Refers to denominator as partitions of the whole	3 (2)	4 (3)	0 (0)	0 (0)	0 (0)	0 (0)
Refers to denominator as the "whole" number but no mention of partitions	6 (5)	4 (3)	1 (1)	1 (1)	1 (1)	2 (2)
Refers to denominator as parts but doesn't refer to the whole	1 (1)	1 (1)	0 (0)	0 (0)	0 (0)	0 (0)
Partitions whole correctly using denominator/fraction name (eg fifths)	18 (7)	4 (4)	16 (7)	3 (3)	4 (3)	4 (4)
Partitions whole correctly using denominator/fraction name (eg fifths) with prompting	2 (2)	9 (7)	0 (0)	2 (2)	2 (2)	0 (0)
Doesn't use denominator to partition whole (doesn't partition whole correctly)	0 (0)	3 (3)	0 (0)	0 (0)	0 (0)	0 (0)
Uses numerator to show appropriate numbers of partitions/ groups or correctly states the numerator	17 (5)	5 (3)	16 (7)	5 (5)	6 (4)	4 (4)
Uses numerator to show appropriate number of parts/groups or correctly states the numerator with prompting	1 (1)	4 (4)	0 (0)	0 (0)	0 (0)	0 (0)
Does not use numerator to show appropriate number of parts/groups or incorrectly states the numerator	0 (0)	3 (3)	0 (0)	0 (0)	0 (0)	0 (0)
Models equal partitions	20 (7)	16 (7)	10 (7)	7 (7)	6 (5)	4 (4)
Converts mixed to improper	NA	1 (1)	NA	2 (2)	NA	4 (3)

Table # 8: Referents for Fractions (tasks # 1 and # 2)

Frequency of Responses (Number of Students)	Area		Set		Linear	
	<1	>1	<1	>1	<1	>1
Able to identify/ explain denominator correctly without prompting -able to partition/ unpartition	26 (7)	7 (4)	6 (3)	6 (6)	22 (7)	4 (3)
Able to identify/ explain denominator correctly with prompting -partitions/ unpartitions with prompting	0 (0)	5 (3)	2 (2)	6 (5)	2 (1)	5 (5)
Unable to identify/ explain denominator correctly-unable to correctly partition/ unpartition	0 (0)	3 (1)	1 (1)	0 (0)	0 (0)	3 (3)
Able to identify/explain the Numerator correctly without Prompting	27 (7)	6 (2)	5 (3)	2 (2)	18 (7)	10 (5)
Able to identify/explain the Numerator correctly with Prompting	0 (0)	3 (3)	3 (3)	5 (5)	4 (1)	0 (0)
Unable to identify/explain the numerator correctly	0 (0)	2 (1)	1 (1)	0 (0)	0 (0)	11 (2)
Attempts to invert Fraction (i.e. 4/3 → 3/4)	0 (0)	2 (1)	0 (0)	2 (1)	0 (0)	1 (1)
Abandons previously Established meaning for Numerator and/or Denominator in order to "fit" Notation to model	0 (0)	0 (0)	4 (4)	3 (3)	0 (0)	3 (3)
Incorrectly alters model in order to "fit" fraction notation	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	1 (1)

Section 1.2 Sense of Orderliness of fractions.

Table # 9: Inverse relationship between denominator and fraction size (tasks # 3a; #4f; #5c, 5d)

Frequency of responses (Number of students)	¾ vs. 3/5	5/6 vs. 5/7	A/B Increase in B	A/B Decrease in B
Correctly identifies the larger fraction	4 (4)	4 (3)	7 (7)	5 (4)
Explains inverse relationship between denominator and fraction size correctly	4 (4)	4 (2)	4 (4)	2 (2)
Identifies the larger fractions correctly but is unsure and draws diagrams	0 (0)	0 (0)	0 (0)	0 (0)
Incorrectly identifies the larger fraction	1 (1)	2 (2)	0 (0)	1 (1)
Does not explain inverse relationship between denominator and fraction size correctly (uses whole number logic)	1 (1)	1 (1)	2 (2)	4 (3)
Corrects identification of larger fractions after prompting/ work with manipulatives	1 (1)	1 (1)	0 (0)	1 (1)
Corrects explanation of inverse relationship between denominator and fraction size after prompting/ work with manipulative	1 (1)	1 (1)	2 (2)	4 (2)

Table # 10: Ordering and Comparing Strategies (tasks # 3 a-g)

Frequency of responses (Number of students)	¾ vs. 3/5	4/5 vs. 6/5	2/3 vs. 6/7	¾ vs. 5/12	3/5 vs. 5/7	3/5 vs.6/9	4/10 vs. 5/9
Correctly states larger Fraction without converting to like denominators	4 (4)	3 (3)	2 (2)	5 (3)	3 (2)	1 (1)	3 (2)
Uses a benchmark Strategy to order fractions (reference to 1/2 or whole, "missing less therefore more, etc.)	8 (4)	3 (3)	1 (1)	1 (1)	1 (1)	1 (1)	2 (2)
Correctly states larger fraction by using like denominator method or by converting to decimals	0 (0)	0 (0)	0 (0)	0 (0)	1 (1)	4 (3)	1 (1)
Correctly states larger fraction by drawing a diagram	0 (0)	0 (0)	0 (0)	2 (2)	0 (0)	0 (0)	0 (0)
Correctly orders fractions but states an incorrect reason (focuses on range between numerator & denominator)	0 (0)	0 (0)	1 (1)	2 (2)	3 (1)	0 (0)	4 (4)
Incorrectly or is unable to state larger fraction (focus on "range")	1 (1)	2 (2)	2 (2)	0 (0)	1 (1)	5 (5)	0 (0)
States fractions are equal (both missing 1piece)	0 (0)	0 (0)	7 (4)	0 (0)	3 (2)	2 (2)	1 (1)
Corrects answer after drawing diagram	1 (1)	2 (2)	4 (2)	0 (0)	2 (2)	2 (2)	0 (0)

Table # 11: Ordering and Comparing Strategies (task # 3h and 3I)

Frequency of Responses (Number of Students)	Set 3h 4/8; 5/9; 5/12; 2/5; 10/8	Set 3i 6/6; 3/1; 11/7; 9/18; 4/21
Order set correctly	2 (2)	5 (5)
Evidence of references to one whole and half benchmarks	4 (4)	5 (5)
Orders set incorrectly	5 (5)	2 (2)
Orders set correctly when prompted to consider benchmarks	4 (4)	2 (2)
Orders fractions using range between numerator and denominator only	0 (0)	0 (0)
Orders fractions using benchmarks and "range" strategy	2 (2)	2 (1)
Recognizes that the "range" strategy is not correct but persists in using it	2 (2)	1 (1)
Evidence of whole number logic with denominators	0 (0)	1 (1)
Unsure how to order fractions in same range (both under/ over 1/2 etc)	5 (4)	NA

Analysis of Results

The analysis of the results is presented in two parts. The first part, addresses sub-section 1.1 (p.12) of the adapted framework for fractions. In this first section, data from the pre and post-unit interview that addressed the meaning that students gave to fractions was considered. In particular I attended to how each student related the fraction to the whole, what interpretation s/he had for the numerator and denominator, and what models or referents each student used or brought to his /her mind's eye when working with fractions (sections 1.1.1 and 1.1.2). In the second part of this analysis, the processes that each student used when comparing and/ or ordering fractions (section 1.2, p.12) was explored. In particular, I noted whether each student understood the inverse relationship between the denominator size and fraction size and whether or not each student used this information to compare fractions (section 1.2.1, p). I also attended to the processes that students used to compare and order fractions (section 1.2.3), including the use of benchmarks, such as half and one (section 1.2.4).

Section 1.1: Sense of the Relative and Absolute Magnitude of Fractions
Discussion of Pre and Post-Unit Results

According to the adapted framework for fraction sense (p.12), if a student is to develop fraction sense s/he must first have an understanding of how a fraction relates to the defined whole or unit. S/he must have meaning for both the denominator and numerator term and be able to consider them in relation to one another in order to develop a quantitative notion for fractions (Section 1.1). S/he must understand that the denominator represents the number of equal partitions of the defined whole and that the numerator represents the counts of those partitions.

Understanding the meaning of the denominator and numerator implies mental referents. If a student is to develop sound fraction sense, s/he must be able to bring to the mind's eye many different mental referents or models. Since it is my contention that these are the initial conditions required for the development of fraction sense, it follows that how the seven students understood the whole and the numerator and denominator is examined first. In particular, the awareness that the students had for the whole and how this guided or inhibited their work with fractions is considered. Next, the meaning that each student had for the denominator and numerator and, once again, how this guided or inhibited his/her work with fractions is examined. Finally, the models that each student used for understanding the meaning of the numerator and denominator is explored. In particular, the ability of each student to model fractions using a variety of referents and his/her ability to translate between the various modes of representation is considered.

Reference to Whole/ Unit

Focus on Partitioning.

In the pre-and post-unit interviews students were presented with a series of tasks which required them to consider the whole in order to solve the given fraction task. If we examine the data in Table # 1 (Reference to Whole/Unit tasks #1 and # 4, p.70), two very strong trends emerged with regard to the whole. The first being that all but one student interpreted a fraction to mean a "whole" (usually interpreted to mean a "pie" or square) which is equally partitioned with a certain portion colored in or taken away. Generally, students did not state or refer to the whole directly but rather focused on and discussed the parts as if they were discrete units. For example, when asked what the notation "4/6" meant Shari stated, "Altogether there's six and four out of sixes" and Jeremy offered, "Four-sixths. It means there's 4 like say there's 6 pieces and there's 4 there or 4 missing". When asked for a definition of fourths, responses included- "a group of four"; "four different sections"; " four equal pieces". These students focused on the partitions as discrete units with no reference to the whole being partitioned.

Vague Awareness of the Whole.

If students did refer to the whole, it was an indirect reference where the whole took the form of a geometric shape, such as a circle or square (Table #1, p.70). When asked to tell me about 4/6, Chris initially offered 4 divided by 6 (learned from algebra) and then explained "square divided into 6 parts. Four of the parts are taken away or a different color or whatever." Debby echoed this idea- " That, um, like say you had a pie and it was cut into 6 pieces and the fraction and 4 of it was eaten the fraction would be that, 4-sixths." Definitions of fourths included vague references to the whole as " cut *it* into 4" or " four boxes *in there.*" Once again the notion of the whole is not made explicit except by Tammy who states a fourth is " four pieces in a whole".

It is possible that each student assumed that the notion of the whole is understood by all and can be considered a given with no need to directly refer to it. From the models of squares and "pies" that students partitioned, one could be led to assume that these students understood the concept of a fraction as a partition of a defined whole. However, the lack of direct reference to the whole seemed to suggest that the concept of "the whole" was not fully developed or at least appreciated by the students. For example, even though all the students could name fractions for one whole (4/4, 3/3 etc.), only two of the six students who were presented with this problem could find one whole when given the fraction 4/3 (Figure 9, p. 48). When given the fraction 1 1/2 on Figure 8 (p.48), only two of the six students could successfully locate where one whole would be (Table # 1, p.70). Of these students, two were able to reason that one whole was two-halves and therefore they only need to partition the shaded portion in to three equal parts (three halves), locate two parts (two-halves), and label one appropriately. The other student who was successful with this task (after some prompting) reasoned that the shaded portion was 1 1/2 and the entire figure was two therefore one would be half way in the diagram. The other students were unable to apply the concept that one whole was two-halves, even though they did know this.

Perhaps, as I have suggested, the concept of one whole is not well developed with these students. I suspect that the concept of one whole is not really considered, but rather discrete partitions are focussed on. The difficulty in this becomes apparent when students are asked to work with fractions greater that one whole. If they are given models with more partitions than one whole, then they do not seem to know what to do. This idea is presented again when I discuss the meaning that each of the students has for the numerator and denominator.

Increased Attention Given to the Whole.

In the post-unit interview there was a noticeable increase in direct references to the whole by the students (Table # 6, p.75). This direct reference was most noticeable when students were working with fractions greater than one. When asked to model six-thirds (6/3), Chris showed two fraction circles covered in six blue pieces and explained, "Three makes a whole and so three added together, but there's six equal... two wholes." Jacquie defined 6/3 as "Top number is bigger than the lower one and it's one whole cause the six goes into the three once and then it can go in to it one more time, that's like two times so that's like two wholes." When asked to model 6/3 with the rectangle of six boxes, Liam said "two wholes" and then doubled the rectangular bar and shaded in both bars to create two wholes or 12/6 (Figure 11, p.85). After listening to me question his partner on thirds, he then changed this model and worked with the first bar by splitting the bar of six boxes in half and then shading in all six boxes (Figure 12, p.86). He explained that he changed it "because three means a whole" and then indicated the one whole was three boxes. This represented real growth in Liam's understanding of fractions from the pre-unit interview where his concept of a fraction was a multiplication array.

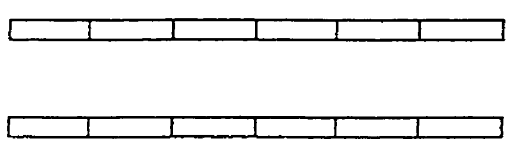

Figure 11: Liam's first model of 6/3 showing 2 wholes

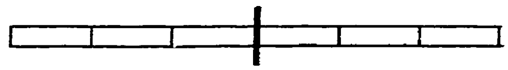

Figure 12: Liam's revised model of 6/3 showing two wholes

This increase in direct reference to the whole indicated to me that students were attending (more often) to the fact that a fraction was a partition of a defined whole, not just a collection of discrete partitions. This was evidenced in task # 2 (Table # 6, p.75) where students were asked to find the whole with Cuisennaire rods. Students were given the light green (3cm) rod as the whole and the yellow (5 cm) as the fractional part. Shari and Jacquie stated that the yellow part was 1 2/5 however Tammy suggested that it was 1 2/3. Shari and Jacquie then agreed that Tammy was correct and the fractional part was indeed 1 2/3. When I asked them why it was not 2/5, Shari said "Green, it's the bottom number" and Jacquie said "Three is the denominator, green is the bottom." Although they did not clearly articulate that the green represented the whole and the partitions to be considered were those of the whole, they most likely did understand this concept. This suggested to me that they were moving away from focussing on the partitions as discrete units and were considering the partitions as parts of a whole, which must have been taken in to account when they considered the denominator of the fraction.

Liam and Jeremy also seemed to be more aware of the concept of the whole as well. When presented with the black rod (7 cm) as being 1 2/5 or 7/5, they were able to quickly identify the yellow (5 cm) as the whole "cause you can fit five into the yellow one." Jeremy was able to explain that the yellow is one whole and the extra blocks represented "two out of five more."

Although there was more of an awareness of the whole, each of the students demonstrated that they would shift back to considering the partitions as discrete units with no consideration of the whole being partitioned (Table #10, p.80). This will be discussed more fully in the later sections on the meaning that students give to the numerator and denominator.

A Whole as An Entire Figure.

Another trend that is supported by the data in Table # 1 (p.70) is that students always interpreted the whole to be the "whole"- read "entire"- object or figure. When shown Figure 8 (p.48), students automatically assumed the whole was the entire bar and stated that 3/4 of the bar was shaded. When presented with Figures 9a and b (p.48), all six students presented with this problem were puzzled when asked to indicate the whole with 4/3. Chris recognized that 4/3 meant that the whole would need three-thirds (three boxes) with "...an extra box down here..." but his notion that the whole was represented by the entire bar was so strong that he felt that "...there's one box still not here." Upon further prompting to think how he could show thirds, he suddenly realized that the whole could be represented by a section less than the entire figure. He stated that the question confused him at first but then he just figured it out.

Jacquie, Jeremy, and Shari circled the entire bar of four boxes as the whole for both 4/4 and 4/3. All three were so insistent that the whole was the entire bar that they tried to invert the improper fraction from 4/3 to 3/4 in order to make the notation agree with the figure (Table #3, p.72). In fact, immediately after she had explained that it was the bottom number "4" in the fraction 4/4 that told her where the whole was, Jacquie circled all four boxes for the whole in 4/3 and even said the whole was separated into four. It was only upon further questioning by me that she recognized the discrepancy and proceeded to reconcile the difficulty by inverting the fraction to 3/4. When presented with the fraction 4/3, Jeremy's first response was "How can that be?" Then he began to partition the entire bar into three parts, ignoring the other lines. He recognized that this would not suffice- "I started putting the lines in to make it 3 pieces, but that wouldn't do it cause I need four." Like Jacquie, he reconciled the problem by inverting the fraction.

Of interest is the fact that both Tammy and Debby were able to identify that the whole was represented by less than the entire bar. They were also the only two to refer directly to the whole and to use the fractional name of thirds versus "three" when they talked about the fractions (Table #1, p.70). Tammy explained "Cause... I don't know...'cause it would be like three of them would be a *whole* cause it's 3-*thirds* or 3-*thirds* would be a *whole* and then another would be four-thirds." Debby stated "... it's not really a proper fraction in our mixed fraction it would be *one* and 1-*third* so the *whole* would be three 'cause there's one left over."

The other four students never mentioned the whole directly and did not talk of "thirds" but rather "three". Jacquie named 4/3 as "four three", Shari stated that she needed "four or three" to make the whole, Jeremy said "...I tried to make it three pieces...", and even Chris, who did come to a solution with minimal prompting stated that there should be "...three sections down here..." and explained his solution as "...so this part right here is three out of three right here is one out of three...which equals this (points to 4/3)." The lack of direct reference to the whole and the apparent aversion to the fractional names, may be semantics. On the other hand it could point to a deficit on the student's part to consider a fraction as a quantity relative to a whole. It could signal a fixation on the fractional parts as discrete units that students think can be considered independent of the whole.

Challenging the Notion of Whole with Improper Fractions.

When asked, most students reported that they were used to seeing fractions as a shape that was "cut up" and they had to shade in the parts. In the pre-unit interview, all seven students reported that they had not worked with questions that involved modeling an improper fraction. If this is indeed the case, then students would not have much need to consider the whole. The whole could always safely assumed to be the entire shape or group and all that need be considered is the number of partitions to be colored in. This notion was particularly strong with Shari. When she was asked to identify the whole given 4/3 (Figure 9b, p. 48) she repeatedly insisted that the whole was four. When I pointed to the notation, she then stated that she needed three to make the whole and circled three boxes on the diagram. I then asked, "and so if this (pointing to the three circled boxes) is the whole and yet you've got four-thirds (I point to the extra box), what have you got- more than the whole, less than the whole?" She then replied, "less than the whole....um, because altogether it's four and there's only three." Once again, Shari ignored her previous answer and the notation and focussed on the whole as being the entire rectangle of four boxes.

In order to guide her work with Figure 9b (p.48), I asked her to work with Cuisennaire rods and find the whole when provided with rods that represented the fractional part. While working with the rods, it became clear that Shari did not understand how the fractional parts got their names. She seemed to associate the fractional name with the color of the rod, therefore green rods (3 cm) were thirds or threes as she was more apt to call them. She did not associate the fractional name with the partitions of the whole and did not understand that the name of the rod would change when the whole as given changed. In order to progress with the question, I explained to her how the rods got their names in relation to defined whole. Once I was satisfied that she understood this concept, we moved on to finding the whole with a variety of rods. I gave her the green rod (3 cm) and told her that it was a third and asked her to show four-thirds. She showed me four green rods. I then asked her to find me the one whole at which point she claimed that there isn't one that fits. I removed one green and she said "three-thirds" and told me that it should equal one and then proceeded to get the blue rod (9 cm) to fit the three-thirds. I then replaced the fourth green rod and asked where the one whole was. She said "there's not a whole that is equals." Presumably, she still saw the "whole" as being the entire length to fit all the blocks and didn't understand that the whole could be less that the fractional part. Even though she knew the three-thirds (three green) were equivalent to one whole, she did not see the four green as four "one-thirds" (three-thirds and one-third) and therefore having the same whole (blue rod).

I decided to pursue this idea and asked her to compare three red rods (2 cm-partitions) to the black (7 cm-defined whole). She stated that the three reds are three-thirds. When asked for four-thirds, Shari showed me four red rods. However, even though she had just established the three-thirds and four-thirds, she was unable to show me the whole with the four-thirds- "it's... not that one... no it's he... um... I don't know what it is." I asked her if four-thirds was bigger that one whole. She answered that "yes, it was" but was not sure if it was allowed to be.

In the post-unit interviews, all the students once again assumed the whole to be represented by the entire object or set (Table 6, p.75). However, they were more comfortable with problems that presented the whole as less than the entire object or set, although this comfort did not necessarily result in total success with these types of problems. When presented with 6/3 all students except Jeremy saw two wholes and modeled this with fraction circles. In task # 2, students were given Cuisennaire rods and told that the light green rod (3 cm) represented one whole and the yellow (5 cm) represented the fractional part. All five students who worked with this problem were able to partition the whole correctly and, after some discussion, use this reference to the whole to state the fractional part as 1 2/3. When given the fractional part as 1 2/5 (black rod-7 cm) all of the five students were able to identify the whole as the yellow rod (5 cm), although two of the students required some discussion to come to the answer. In these questions, the students considered how many partitions were equivalent to the whole and used this information to solve these questions. This is an improvement from the pre-unit interview where several of the students had difficulty locating one whole given 1 1/2. The difference between the first and final interview seemed to be the students' awareness of the whole and consideration of the number of partitions that were equivalent to the whole when solving these problems.

Although the students seemed to have more success in the final interview with fractions that presented the whole as less than the entire object or set (fractions greater than one whole), this seemed to be limited to area and linear models of fractions. Questions that involved fractions greater than one whole with set models continued to pose a challenge to students (Table # 8, p. 77). I address this situation in a latter section entitled 'Referents for Fractions'.

Meaning of the Numerator and Denominator

I analyzed data from tasks in the pre and post-unit interviews that required students to explain and apply their understanding of the numerator and denominator. I wanted to determine if the students understood the intimate relationship between the denominator, numerator, and the whole. In particular, I wanted to know if students understood that the denominator represented the equal, but not necessarily congruent, partitions of the whole and that the number of these equal partitions of the whole is determined by the fraction name. I also wanted to know if the students understood that the numerator represented the count of the partitions and could be greater than the number of partitions (greater than one whole). Finally, I wanted to know if the students could apply their understanding of the relationship between the denominator, numerator, and the whole to solve tasks with fractions less than and greater than one whole.

Focus on the Denominator as Discrete from the Whole.

Students in the pre-unit interview demonstrated that they understood the denominator to be equal partitions (Table # 2, p.71). Student definitions for the denominator included: "number of groups"; "tells you what to cut"; "total"; and "number of pieces there are". The students did not, however, explicitly state that they were equal partitions *of the whole* but rather tended to talk of the denominator as if it were discrete units that could be contemplated without consideration of the whole.

Generally, students seemed to equate the fraction name or denominator with a whole number; therefore, sixths were "sixes" and fourths were "four". For example, in task # 2 (p.45), students were shown a variety of diagrams (Figures 4a-i, p.45) and asked to explain whether or not they were models of fourths. Explanations tended to focus on that fact that there were four equal parts: "Pretty even, four"; "Because they were equal and cut in fours"; "They're in four pieces"; "Four equal pieces…they're cut equally and they're each measured equally." This use of whole numbers for the fractional names could be a linguistic error or it could be a subtle reflection of the students' view of the partitions as discrete units that can be thought of using whole number logic.

Students' tendency to think of the denominations as discrete objects without regard to the whole was most evident when they compared fractions. When presented with the fractions 18/19 and 3/4, four of the six students challenged with this problem replied that they thought the fractions were the same (Table #5, p.74). Tammy said the fractions were the same because "there's only a difference of one between the top and bottom number". When asked to compare 1/2 and 1/6, Jacquie stated that both fractions would "…be the same…because, well, it's still like one piece out of two pieces and that's one piece out of six pieces." In these situations, these students seemed to focus on the denominations as discrete objects without regard to the whole. In other words, they did not appear to consider how the different denominations would partition the whole differently, thereby resulting in different sized pieces. They seemed merely to consider "pieces" as if they were objects disunited from the whole that could thought of using whole number logic.

This comparison of the "pieces", without regard to the relative size of the pieces, occurred even though students had just explained the inverse relationship of the number of fractional pieces to fraction size (Table #4, p.73). It seems that although they could explain that the size of the fraction decreases with an increase in the denominator or number of partitions, they tended to ignore or abandon this understanding when faced with abstract notation. When faced with the notation 18/19 and 3/4, four of the six students did not seem to consider the mental referents they had for the relative size of the denominators (Table #5, p.74). Or if they did, they did not seem able to coordinate all the variables together in order to correctly compare the two fractions. In Shari's case, she did make mention of the relative sizes of the denominations but then seemed to abandon it- "...they're the same size pizza, and one piece is gone but these will, these are into fourths *so these are bigger* so...so they're be same, equal amount... I know how to do it but I can't really explain it." It appeared that although she made an attempt to consider the relative size of the denominators, she was not able to coordinate this information with the quantity of pieces. It seemed it was easier for her to think of the denominations as discrete from the whole and therefore the same size. In this manner, a direct comparison of quantity (one piece) was possible which led her to confirm that the two fractions were the same.

Difficulty with Focus on Denominator as Discrete Units.

The focus on discrete pieces proved successful for the students as long as they worked with area models of fractions less than one whole. "Four equal pieces" worked for the Figure 4 a-f (p.45) which were area models with congruent and non-congruent fourths, thirds, and unequal partitions of four. All of the seven students were able to correctly identify the examples and non-examples of fourths with area models (Table #2, p.71). They recognized that there needed to be four pieces and that the pieces needed to be equal in size although not necessarily congruent in order to be fourths.

The difficulty with this limited focus on the "four parts" as discrete units surfaced when the students worked with the set models of fourths (Figure 4 g & h, p.45). When confronted with the set models all students expressed some uncertainty or confusion with the model (Table #2, p.71). Since they seemed to be working with a definition of fourths as "four", and not as the whole partitioned into four equal groups, they were uncertain what aspect of "four" they should focus on with the compound set model.

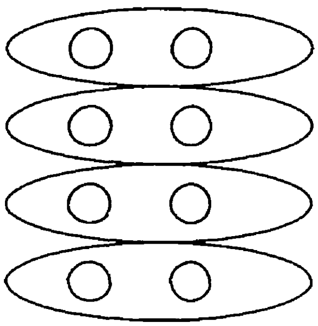

Figure 4 g: Compound Set Model of Fourths

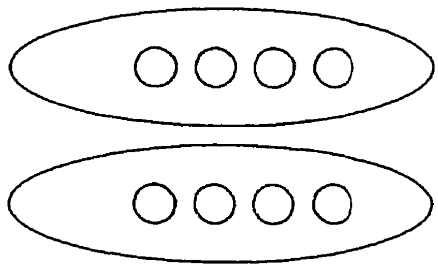

Figure 4 h: Compound Set Model of Halves

Shari decided that both models represented fourths because Figure 4g represented four (pointed to the groups) but she said that she didn't know what the two represented. She felt Figure 4h was also fourths because there was "...four in there...." She did, however, recognize that she "...didn't use the same rule." Debby also felt that both diagrams represented fourths because she could focus on four in both diagrams. For Figure 4g she stated that "...it's four groups of two...so I figured that was a group of four...." For Figure 4h she decided it was also fourths because "...there's four in these two groups so I figured that was a grou... um...like cut into four, split into fourths." A hesitation and a smile was noted in the last explanation. It appeared that she sensed that it was the "groups of" that she was to focus on since she began to say the word but then changed her mind and focused again on four. Jeremy, like Shari and Debby, focused on four as his criteria for fourths. Jeremy also felt that both figures could be fourths because Figure 4h had "four in it" and Figure 4g could be rearranged into groups with four in it. Jacquie, Tammy, and Liam did not feel that either figures 4g or 4h represented a fourth " because there's more than four" and "there's eight." These three students focused on the objects as discrete units and did not initially consider the groups as relevant.

Limited Awareness of the Part-Whole Relationship.

When presented with the comparison of 18/19 and 3/4 Shari was not able to maintain her thinking about the comparison of the relative sizes of the denominators to the whole. However, her attempts to do so (p.94) did demonstrate that she was aware of the need to do so. Other students also demonstrated that they were aware of the need to consider the denominator in relation to the whole. They demonstrated that they were aware that the whole needed to be identified first and then the partitions considered. They seemed cognizant of the fact that the partitions could not be considered independent of the whole, even though they did not consistently attend to this concept when working with fractions.

When confronted with Figure 4i, Tammy stated she did not feel the figure was an example of fourths "cause there's five things."

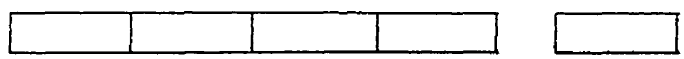

Figure 4i: Linear Model of Fourths

This initial response seemed to indicate that she was treating the denominations as discrete objects. In other words she interpreted fourths as "four" and rejected Figure 4i because she saw five objects. However, when I assured her that the "fifth" box was not attached to the others, she changed her answer and decided that Figure 4i did represent fourths and stated that the fifth box represented five-fourths or one and one-fourth.

Other students were also confused by this question because they were not certain if the "extra" box was "...attached so you really can't tell if it's part of the group or not." Chris reiterated this reference to the whole when he stated he was not sure if the box was "hanging around" or "if it's a part of *it*." Jeremy did not feel Figure 4i was fourths "because there's five pieces and they're not all stuck together." He seemed to be suggesting here awareness that the partitions need to be considered in relation to the whole. However, this understanding seemed to be based upon a physical joining of the partitions like one would find with an area model. When I explained that the extra piece was just there, he changed his answer and said that the diagram did represent fourths "...because it has four pieces" and then went on to say that if the fifth box was not "stuck together" then Figure 4i was fourths.

Figure 4i seemed to force students to consider the partitions in relation to the whole when determining the denominations rather than just considering the partitions as discrete units. Although only two students directly referred to the whole when they discussed the task, the others referred to it indirectly by questioning whether the fifth box was "part of it" or "part of the group". This suggested to me that the students did understand that denominations needed to be considered in relation to the whole and not merely as discrete objects. Their references to the whole in Figure 4i, their repeated modeling of a whole being partitioned, and their ability to explain the inverse relationship between the denominator and the size of the fraction would seem to suggest that the students understood the concept of the denominator as partitions of the whole. However, the students' difficulty with the compound set models and their comparison of 18/19 and 3/4 as the same fraction leads me to suspect that their working definition of the denominator is limited and sometimes ignored when faced with unfamiliar models or abstract notation. I suspect students have generalized the concept of the denominator to be the number of cuts or pieces of a geometric shape. This is probably a consequence of the overgeneralization from area models and is discussed in the section "Referents for Fractions".

Partitions Considered in Relation to the Whole Shows Some Improvement.

In the post-unit interviews, students seemed to have developed a broader working definition of the denominator to include equal groups of the whole. When presented with a set of blocks and asked to show 3/6 with 24 blocks, all of the students except Liam, were able to put the blocks in to six groups of four blocks (Table #7, p.76). When given this task, Shari immediately said "Divided them in to six... four in each group." When I asked her if it was correct to have four or six in the group she stated that four in the group was correct. Her partner, Jacquie stated that the four groups of six arrangement was not correct "...cause there's four groups." When given this same arrangement, Jeremy said it "...would represent 1/4 'cause there's four groups." These students seemed to have expanded their definition of the denominator to include the number of equal groups that the whole can be partitioned into versus their earlier focus on discrete objects in the pre-unit interview.

In the post-unit interview, all the students made reference to the whole when defining the denominator (Table # 7,p.76). This was a significant change from the pre-unit interview where students focused on the partitions without mention of the whole (Table #2, p.71). In the post-unit interview students defined the denominator as partitions of the whole or, interestingly, as the "whole" number without reference to the partitions. For example, Chris said "three makes the whole" and Shari said "three in the whole number" when discussing 6/3. Although their definition for the denominator now included the whole, they failed to mention the partitions (Table #7, p.76).

Focus on "Range" between the Numerator and Denominator.

Although all students in the post-unit interview seemed to demonstrate a broader understanding of the concept of the denominator and how it related to the whole, five of the seven students would continue at times to disregard this concept when comparing fractions (Table # 10, p.79). When presented with two written fractions, these students would use a "Range Strategy" to compare fractions. That is, they would focus on the range or the difference between the numerator and the denominator in order to determine the larger fraction. In other words, these students did not consider the relative size of the denominators when comparing them but rather treated the partitions as if they were discrete units. For example, when presented with the fraction 2/3 versus 6/7, four students claimed that the fractions were equal because they're both missing one piece. An excerpt from the interview with Jeremy and Liam illustrates this thinking.

Interviewer:	"Which is larger 2/3 or 6/7?"
Jeremy:	"Both are the same."
Interviewer:	"Why?"
Jeremy:	"Because I get 2/3 of a pizza which would be 3/4 and Liam would get 6/7 which would also be 3/4"
Interviewer:	"What do you think of that?"
Liam:	"I agree with him"
Interviewer:	"So 2/3 is the same as 3/4 and 6/7 is the same as 3/4?"
Jeremy:	"Mmhmn"
Interviewer:	"How come 3/4? Why did you choose 3/4?"
Jeremy:	"Because there's 1/4 missing out of both of these."
Interviewer:	""'k, if there's 1/4 missing , um, but what's the number here I'm working with?" [I point to the denominator of three]
Jeremy:	" Three. Oh hang on, or we'd each get the same amount of pizza, I mean pie, 'cause he'd be getting six out of seven and I would be getting two out of three pieces which would add up to the same thing."
Liam:	Nods in agreement
Interviewer:	"When we're dealing with thirds and sevenths, what can you tell me about the sizes, thirds compared to sevenths?"

Jeremy: "Thirds would be bigger than the sevenths, but we'd still get the exact same of the pies were the same size."

There are a number of interesting things to take from this conversation. Firstly, Jeremy compared the missing pieces as if they are both the same size; that is, he gave no consideration to the relative size of thirds compared to sevenths but treated them as discrete pieces. Even when he did articulate that the thirds were larger than the sevenths, he continued to state that the fractions were still the same size. The second thing to notice is that he compared the missing pieces to 1/4. Perhaps he realized that the missing piece is a fraction and therefore he cannot speak of one piece but must speak of fractional pieces. I suspect that he chose 3/4 and 1/4 because this is a fraction that he is familiar with. On the surface 2/3 and 6/7 are similar to 3/4 because all are "one piece" from a whole. The third thing to take note of is Jeremy's awareness that both wholes must be the same size in order for the comparison of the fractions to be valid. It is interesting to contrast this consideration of the size of the wholes with his apparent lack of consideration of the relative size of the two denominators. Although he seemed to understand the inverse or compensatory relationship between the size of the denominator and the size of the fraction, he appeared to disregard this information when faced with the abstract notation.

Shari and Jacquie also seem to disregard their understanding of the inverse relationship between the size of the denominator and the size of the fraction in favor of the range strategy. They focused on the range between the numerator and denominator and compared the differences of the two fractions in order to compare the fractions. The conversation below serves to illustrate this point.

Interviewer:	"Which is bigger 3/4 or 5/12?"
Jacquie:	" 3/4"
Shari:	" Bigger- 3/4"
Interviewer:	"Why?"
Jacquie:	" Because, um, 3/4 is like three out of four and that one's like five out of twelve and that's like, 3/4 is closer to the whole."
Shari:	"Yah, closer to the whole"
Interviewer:	" How do you know?"
Jacquie:	""Cause the top is three and bottom's four and the other one's five and the other one's twelve"
Shari:	" There's one difference and the other one's six difference"

When questioned about fourths and twelfths both Shari and Jacquie were able to describe the difference in their sizes. They explained that twelfths would be more pieces but that they would be smaller than fourths. Like Jeremy and Liam, they both seemed to understand the inverse relationship between the size of the denominator and the size of the fractions, but seemed to divorce this understanding when they worked with the abstract notation.

Physical Referents Dissuade Use of the Range Strategy.

All students were able to correct their thinking and recognize their errors when they worked with manipulatives to model the fractions (Table #10, p.79). However, the work with the concrete referents did not result in students abandoning this "range" strategy when confronted with symbolic notation. After establishing with manipulatives that the range strategy didn't work, all five students continued to use this strategy to compare and order fractions. When given the first written set of fractions to order (4/8;

5/9; 5/12; 2/5; 10/8), Jeremy ordered them: 2/5; 4/8; 5/9; 10/8; 5/12 and went on to explain his thinking:

> Jeremy: " By counting how many pieces went into after, two, then there's three pieces missing, four, and then there's four pieces missing, five, and then there's still four pieces missing but this is one is bigger than this one, then there's ten over eight which would be 1 2/8 and then there's 5/12 which would be seven missing."
>
> Interviewer: " Now, do you have any problems with what you've just talked about?"
>
> Jeremy: " I want to switch these two around (5/12 and 10/8) because there's already one whole in eight over ten and then you put ten over eight."

In this case, we can see that Jeremy was able to attend to the notation and used the benchmark of one whole to correct his thinking, however, he did not feel that there was anything else wrong with the order.

When given the same set, Jacquie also recognized that 10/8 was over one whole and therefore identified it as the largest fraction. However, she then reverted to the range strategy when she compared the remainder of the fractions. She explained that 5/12 was the smallest because it was farthest away from the whole and was unsure what to do with 4/8 and 5/9 since they were both four pieces away from the whole. After finding common denominators she determined that 5/9 was larger. I then questioned whether her range strategy was still working considering that both 4/8 and 5/9 were four away from the whole and yet she had just identified 5/9 as the larger fraction. Her response was non-committal and unsure. However, she then persisted with the strategy with the next set of fractions when she ordered 11/7 as greater than 3/1 because 11/7 was four greater than the whole and 3/1 was only two over the whole. Again, she used the range strategy when she compared the fractional pieces as if they were discrete units with no consideration of the relative sizes of the denominators.

Student Definitions of the Numerator: Influence of Area Models.

All students in the pre-unit interview understood the numerator to represent what is there or removed (Table # 2, p.71). Student definitions of the numerator included " the number of pieces colored in or taken away"; " the number of pieces eaten"; and "the number of pieces you want or filled in." All of these definitions suggest a mental referent of area models. Coloring in or eating suggests a "pie" or "cake" model of a circle or square being partitioned. These definitions also suggest referents to fractions less than one whole since "eating" or "coloring in" serves to guide one's work when a fraction is less than or equal to one whole.

Challenging Students' Definitions of the Numerator with Improper Fractions.

When the numerator becomes greater than the denominator, definitions of the numerator as "eating" or "coloring in" do not guide one toward a representation of the fraction. How does one "eat" or "take away" more than one has? Indeed, in the pre-unit interview, all students experienced difficulty with fractions that were over one whole (Table # 2, p.71). When shown the fraction "3/2" Jacquie's initial response was " I have no idea" and when shown "6/4" Tammy also responded in the same way. In both cases I encouraged the girls to draw a picture of the fraction. Both begun be drawing a circle and partitioning it correctly. Jacquie colored in the two parts and then stated " Oh, one and a half". She proceeded to draw another circle and color in half but then stated that her diagram was 3/4 not 3/2. We worked with the fraction circles and she was easily able to identify 1/2 and 2/2, however, she was very unsure if her model of 3/2 was correct. She even had difficulty saying "three-halves" but rather said "three-two".

When I reminded Tammy that her definition of the top number (numerator) was how many to color or take away and I encouraged her to use this idea to help her with 6/4, her response was "How?" Clearly her own definition was not leading her toward a solution. I encouraged her to see how far she could get at which point her response was "I have no idea, would it make it a negative number?" I then changed the fraction to 4/4, which she was immediately able to model. She used this model to suggest that 6/4 was four (fourths) and two-fourths and then proceeded to draw two circles partitioned into four pieces each and then six pieces in total colored in. When I asked her if what she had drawn made sense she said "... it just doesn't seem right." I then asked what she would draw for 1 2/4. She identified her diagram and was able to reason that 6/4 and 1 2/4 was the same thing.

In both these situations the two students were able to properly partition the whole and represent one whole, however, they were uncertain what to do when the numerator was greater than the denominator. Even when they did arrive at a solution, they were uncertain if it was correct. Tammy was sure, however, that her model was correct with the proper form of the fraction, where the numerator remained less than the denominator, but was skeptical with the improper form.

Other students also had difficulty with the fraction 4/3. Jeremy's first reaction was "How can that be?" Jacquie, Jeremy, and Shari all tried to invert the fraction to 3/4 and persisted in modeling 3/4 versus 4/3. Although they had all stated that it was the denominator that determined the partitions in the whole, all these students continued to state that there were four partitions in the whole. After repeated prompting to look at the denominator did the students eventually show the whole as being three partitions with four colored in.

Debby and Tammy were both able to correctly model the 4/3 fraction. Both students recognized that 4/3 represented 3/3 or one whole and an extra third and used this knowledge to model the fraction.

<u>Definition of the Numerator Expanded to Include "Counts".</u>

In the post-unit interview students continued to define the numerator as what is there or what is removed but they also expanded their definition of the numerator to include the count of the denominations (Table #7, p.76). I noticed an improvement in each student's ability to appropriately model the numerator, especially with fractions greater than one whole. When asked to model 6/3 all students except Jeremy modeled two whole circles. Chris, Debby, and Tammy all modeled two circles each partitioned into three pieces. Shari, Jacquie, and Liam all modeled two circles, however, they represented the partitions with sixths. When this was pointed out, they corrected their models to thirds. Liam explained that it was "...supposed to be three in a group... I showed you six, 12/6."

It is interesting to note that Shari, Jacquie, and Liam all recognized that 6/3 was two wholes, but then modeled the fraction with sixths. Perhaps they saw the larger number as the denominator, regardless of its position.

Jeremy was the only student to model this fraction incorrectly. He modeled 3/6 and stated that " I just thought it'd be the same." Even after he stated that the whole would be partitioned in to three and watched Liam's explanation, Jeremy was still not totally sure if his model of 3/6 or Liam's of 6/3 was correct. Jeremy was the student in the pre-unit interview that was most surprised by the improper fraction and insisted on dealing with it by inverting it. He continued this inverting in the post-unit interview as well. When asked for the proper form for 5/3 he offered 3/5 and when he discussed 10/8 he referred to it as eight over ten. It is possible that Jeremy considers "proper" form to mean the big number on the bottom of the fraction and simply inverts the fraction to satisfy this condition.

Cuisennaire Rod Activities Encourage Counting Aspect of the Numerator.

The tasks with the Cuisennaire rods forced students to consider the numerator, denominator and whole in relation to each other in order to solve the tasks. When the brown rod (8cm) was identified as 4/5 and students were required to find the whole, all students could iterate the brown rod with four smaller units (red blocks-2 cm) and then add one more red block to make a total of five blocks or 5/5 (orange rod-10 cm). They all recognized that the numerator represented the counts of the partitions and that the whole would require five of them. They realized that the brown rod represented four parts and iterated it accordingly.

Another question that forced the students to attend to the numerator was the question that identified the black rod (7cm) as 1 2/5. Shari, Jeremy, and Liam preferred to work with the improper form 7/5 to solve the problem. They did this by iterating the black rod with seven smaller white blocks (1 cm) and then counting off five of them and finding that the yellow rod (5 cm) was equivalent to these five iterations. These students recognized that the seven represented the counts of the denominations or partitions and that the five represented the denominations and not the other way around.

Generally, all students were able to articulate what the numerator was and were able to work with fractions with numerator less than, equal to, and greater than one whole in the post-unit interview (Table # 8, p.77). They were mostly successful with area models and moderately successful with set models. Since I felt that the problems that the students encountered with the different models were more a function of the model than the concept, I left discussion of these situations to the section below entitled 'Referents for Fractions'.

Visual and Concrete Referents for Fractions

In the pre-unit and post-unit interviews students were presented with a variety of models of fractions. These models included area models, simple and compound set models, and linear models[1] of fractions less than and greater than one whole. According to Hiebert (1988), if an individual has a well-developed sense of the magnitude of fractions, then s/he is able to bring to the mind's eye visual referents of fractions. If the concept of fractions is truly generalized, then the student is able to bring to mind many physical models that represent the various meanings of fractions.

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[1] Area models involve comparison of two dimensions (length and width) when modeling fractions, linear models involve comparison of one dimension (length) when representing fractions. Both area and linear models are considered continuous models because there is no visual break in the model. Set models are collections of objects that must be partitioned to represent the fraction. Set models can be simple, meaning the number of objects is exactly equal to the denominator, or compound, meaning that the objects need to be grouped in order to represent the denominator. Set models are considered discrete because the objects a physically separated or "discrete" from each other.

Most students begin their work with fractions by modeling fractions and by interpreting diagrams. This is what Hiebert (1988) refers to as the *connecting phase*. I believe the meaning that students give to the numerator and the denominator, and indeed the fraction itself, is a consequence of their work with these physical referents. I also suspect that the meaning students bring to their work with fractions is also limited by their work with manipulatives. That is, if they are only exposed to one type of model, then I suspect that they over-generalize from this one model and develop a narrow understanding of fractions. I examined the results from the students' work with the different models in order to see if this was the case.

Challenge of Non-Congruent Fourths with Area Models.

In the pre-unit interview all students, except Liam, demonstrated that they could correctly model fractions less than one whole with an area model (Table #1, p.70). The most common model that students used was a circle partitioned into a number of equal and congruent pieces dictated by the denominator, which the appropriate number of pieces shaded in accordance with the numerator.

When presented with a geoboard (area model) and asked to model fourths, most of the students partitioned the geoboard into four equal and congruent pieces. No one produced partitions that were equal but non-congruent (Table #2, p.71). When I presented non-congruent fourths on the geoboards (Figure 3, p.44), four students felt that they were still fourths whereas three students were unsure. The students who felt sure that the non-congruent fourths were indeed fourths did this by determining the interior area of each piece and comparing them to see if they matched or were equal. Two of these students were able to reason that fourths in one configuration (squares) were equal to fourths of another configuration (triangles) because they were all fourths of the same size area.

Surprisingly, when presented with the models in Figure 13 five of the students did not initially feel that the fourths of the square arrangement were equal in amount with the fourths of the triangle arrangement. They seemed to focus on the physical features of the shapes and not the logic of the situation. For example, Jeremy and Liam both saw the triangle fourth as larger than the square fourth.

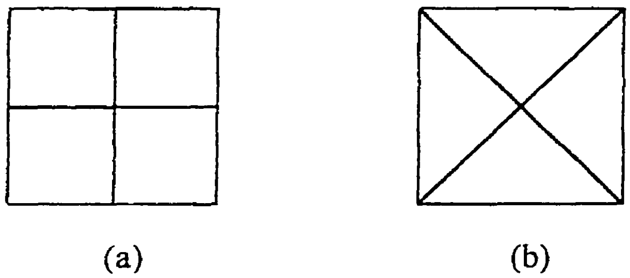

Figure 13: Two Geoboard models of fourths ("a" is non-congruent to "b")

Liam explained that he agreed that both arrangements were fourths but did not see the square and triangle partitions as equal in size because the base of the triangle was longer than the side of the square.

Two of the students who were not sure if the fourths of Figures 13a and 13b were examples of fourths were also unsure if being equal in measurement was important for fourths (Table # 2, p.71). Although they always modeled equal fourths, they were not certain if four unequal sections, such as Figure 14 (p.111), were acceptable as fourths. They were not sure if the conditions for fourths included equal partitions. However, they suspected that "equal" was a condition of fourths and did, at least tentatively, reject the unequal partitions when pressured for a definitive answer.

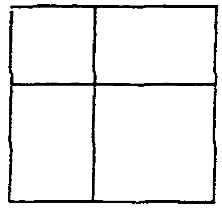

Figure 14: Non-example of fourths

When pursued for a definition of equal, these students indicated congruent meant equal. When I indicated that equal meant equal in amount or measurement, they were then able to determine that the non-congruent fourths (Figure 13) were equal by directly comparing their respective areas.

Challenge of Identifying Fourths with a Variety of Pictorial Models.

The next set of tasks asked students to examine a variety of diagrams to decide if they represented fourths. The diagrams (Figures 4 a-i, p.45) included area models, linear models, and compound set models. All students had no difficulty with the area models (Table #2, p.71). They were able to easily distinguish the examples from the non-examples. All of the students were able to recognize the non-congruent fourths as fourths. Presumably, the students who were uncertain with the non-congruent figures in the previous problem had learned that fourths could be non-congruent. The non-examples included a square partitioned into four, unequal pieces. All the students recognized this figure (Figure 4 b, p.45) as a non-example of fourths because of the unequal partitions.

Students were much less successful with the compound set models of fourths (Figure 4 g & h, p.45). I discussed earlier, in the section "Difficulty with Focus on Denominator as Discrete Units with Different Models" (p.95) that students were unsure what aspect of fourth was relevant when considering the set models. Most of the students were working on a concept of four equal "discrete" pieces, When they encountered Figures 4g and 4h, they were uncertain what aspect of "four" they should focus on. They were uncertain if it was the number of groups or the number within the group that they should focus on.

Challenge of Unpartitioning with Set and Linear Models.

It is interesting to note that representing fractions with a simple set model did not pose a problem for students. However, modeling fractions with a compound set model did cause students difficulty in the pre-unit interviews. For example, when students were asked to model 8/12 with a simple set model (Figure 15), all students were able to use the model correctly to represent the fraction (Table #3, p.72).

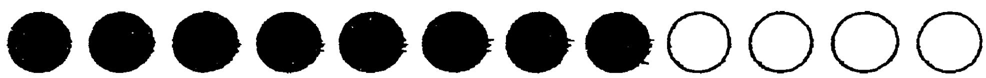

Figure 15: Simple set model of 8/12

However, when asked to model 4/6 with the same model, only three of the students were able to group the circles (Table # 3, p.72). Chris and Jacquie grouped the circles or "unpartitioned" the model by drawing divisions between the circles and considering the group of two circles as one partition (Figure 16, p.113).

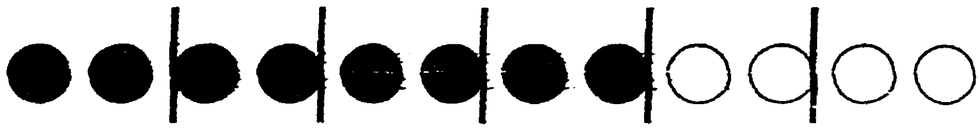

Figure 16: Four-sixths partitions of a set model

Although Debby was able to partition the circles, she could only do so if she converted the set model into a square area model (Figure 17).

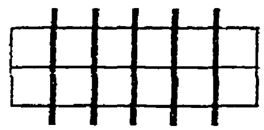

Figure 17: Debby's set model converted to an area model and partitioned

The other four students were unable to unpartition the simple set model to show 4/6. They were, however, able to unpartition the area model of 4/6 to show 2/3 (Figure 18).

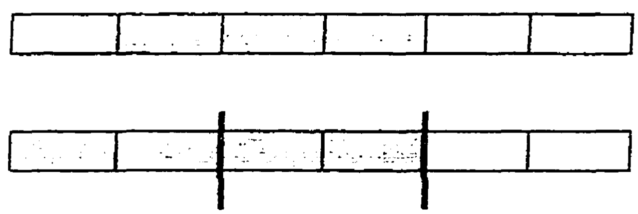

Figure 18: 4/6 area model unpartitioned to show 2/3

Why these students were able to unpartition the area model and not the simple set model may be a function of the fact that they were unfamiliar with the set model and therefore were not certain what was "allowed". Whereas, they seemed fairly sure that they were "allowed" to unpartition the area model by grouping the individual boxes and considering the group as one partition. Debbie's response would seem to support this hypothesis. She was able to convert the simple set model in to an area model and then unpartition it correctly (Figure 17,p.113). She seemed to understand the concept of unpartitioning, but did not seem aware how to do it with the set model. It is also possible that there are visual clues that differ between the two models. This could be an area for further research.

Challenge of Proper, Improper, and Mixed Fractions with Different Models.

Linear Models

The last model that students were presented with the in the pre-unit interview was a linear model. When they were asked to identify the fraction in Figure 8, all students gave either 3/4 or 5/7 as an estimate (Table #1, p.70). They did this by partitioning the length into four or seven partitions and then counting the number of partitions represented by the shaded portion.

Figure 8: Shaded bar used to identify the whole

When provided with Figure 9a, all students were able to represent 4/4.

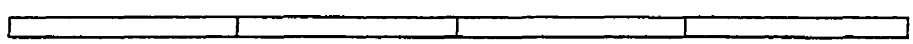

Figure 9a: Finding the whole with 4/4

It was when students had to work with the linear model with an improper fraction and a mixed fraction that they encountered difficulty. When told that Figure 8 represented

1 1/2, only two students were able, without assistance, to locate one whole, as I discussed in the section "A Whole as an Entire Figure" (p.87). When asked to model 4/3 with Figure 9a (p.114), again only two students could do so without prompting. When shown Figure 4i (p.97), only one student initially was able to identify the figure as being fourths, with 5/4 represented. As I discussed earlier, the other students rejected Figure 4i as fourths for one of two reasons. Either the students saw five objects and therefore rejected the figure as fourths, or, they were not certain if the "extra" box was attached to the other four and therefore could not identify the exact number of partitions in the whole.

I suspect the inability that these students initially had when working with Figure 9b (p.48) stemmed from their lack of experience modeling improper fractions. Since the models that most of the students reported working with were area models of fractions less than one whole, I suspect that they had over-generalized the whole to mean the "entire" figure. As a result, the students seemed to have difficulty reconciling Figure 9b to the fraction 4/3. Chris knew that was 4/3 was greater than one whole so he felt that he needed an extra box. Jeremy began by repartitioning the entire box into thirds but then became confused when he knew he needed four boxes. I suspect that these students did not have the experience to realize that the whole could be less than the entire figure. As a result, they found it very difficult to model improper fractions when the partitions on the model did not seem to correspond exactly with the denominator. For example, the students did not have difficulty with modeling 4/4 with Figure 9a because the number of partitions of the model (four) matched with the denominator (four). However, when asked to model 4/3, I suspect the students had difficulty because the partitions in Figure

9b (four boxes) did not match the denominator (three). I suspect that the students were not aware that they could identify the whole as being less than the entire figure. That is, that they were "allowed" to identify three of the partitions in Figure 9b as the whole and the fourth partition as the third of the next whole.

Area Models

This confusion with fractions greater than one whole (improper fractions) also arose with area models. In the first interview I asked Tammy and Jacquie to model 6/4 and 3/2 respectively. They were both fully able to model 4/4 and 2/2 with the fraction circles, but did not know how to proceed beyond that point. After encouraging them to consider what the numerator was telling them they did eventually "get more pieces" but were not certain if this was "allowed". When I asked Tammy if her model represented 1 2/4 she agreed that it did and indicated that she was comfortable with this notation. She also agreed that 6/4 was the same as 1 2/4, however, she still seemed skeptical about her model. Jacquie, after modeling 2/2, noted that 3/2 was 1 1/2, but she still had trouble understanding her model. When she used two fraction circles covered with three halves she then said her model was 3/4. She had re-defined the whole to be the two circles and therefore the 3/2 was now 3/4 (Figure 19).

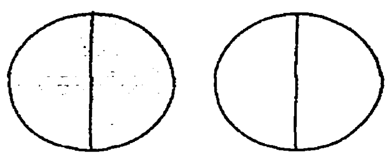

Figure 19: Jacquie's model of 3/2 that she interpreted to be 3/4

I suspect that this occurred because her experience had been that the whole was the entire figure or group. When I asked her to count the halves she hesitated after two-halves, calling three-halves "three-two". She did eventually come to say three halves and could identify that the whole was less than her fractional pieces (three "half" pieces).

Mixed Success Modeling Proper, Improper, and Mixed Fractions.

In the post-unit interview, students were also shown area models, set models, and linear models. All students were able to model fractions less than one whole with area models and linear models (Table #8, p.77). There was also an improvement in the students' abilities to model fractions less than one whole with compound set models. Only two of the students needed some guidance partitioning 24 blocks into sixths. All of the students except Liam, understood that partitioning the blocks into six groups of four represented sixths whereas four groups of six blocks represented fourths. They all also realized that the groups in the set model had to be equal in quantity in order to be considered sixths.

Three of the students, however, still had some difficulty with the numerator when working with the compound set model (Table #8, p.77). When I showed them four groups of six and asked for the name of what I had represented, Tammy offered "six-fourths" and then "four-sixths". Liam also referred to this arrangement as "four-sixths'. It seemed that these students had abandoned their definitions of the numerator and denominator when they offered these names. It appeared that the quantity of objects within the groups distracted their focus away from the focus on the groups, just as it had in the pre-unit interview.

Students also showed some improvement with their ability to model improper fractions in the post-unit interview (Table # 8, p.77). When given 6/3 to model, all students except Jeremy, stated that 6/3 was two wholes and modeled two circles that they identified as two wholes. However, as I discussed earlier, three students modeled 12/6 versus 6/3. This also happened when they were asked to model 6/3 on a linear model. Two of these students doubled the whole but recognized that their models demonstrated 12/6 so they changed their model to 6/3 (Figures 11, p.85; Figure 12, p.86). Chris and Liam both explained that they just doubled the whole to create two wholes but then recognized that their denominations were not correct and changed them to thirds. Although the fractions that the students modeled were equivalent to the one I suggested (6/3), they were not correct. I suspected although the students could convert the improper fraction of 6/3 to the proper form of two wholes , they still viewed the larger number as the denominator and therefore modeled it as such.

Three of the students continued to have difficulty with modeling 6/3 with a linear model, even though they were able to do so with the area model. When presented with this problem, Jacquie recognized that she needed thirds so she altered the model by drawing three extra boxes to create thirds (Figure 20b).

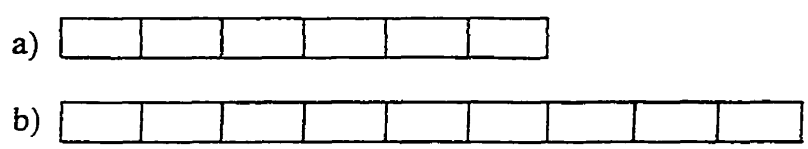

Figure 20: Jacquie's model of 6/3

When asked to show thirds she said she had forgotten that part and was just thinking about the two wholes. When she modified her diagram, she was able to show the thirds and as three groups of two and was able to show six parts colored in but she was unable to show the whole (Figure 21). She knew her diagram was incorrect because she needed two wholes but was unable to resolve it.

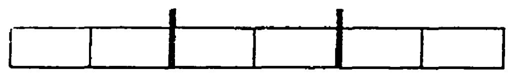

Figure 21: Jacquie's second modified model of 6-thirds

In Jacquie's second modified diagram (Figure 21) we can see that she has attended to the thirds by unpartitioning the figure provided and representing the thirds as three groups of two. To accommodate the six in the fraction she colored in six boxes (she repartitioned the figure to 6 boxes). In this situation it seemed that Jacquie had not understood or considered that the numerator counts or "colors in" the denominations. She had merely satisfied the need for six boxes to be colored. It seemed that Jacquie considered the denominator and the numerator as separate, independent terms. It is interesting to note that Jacquie was able to represent 6/3 with the fraction circles, but she was not able to transfer her understanding to the linear model. She knew that she needed two whole circles partitioned into thirds, but she was not able to recognize that she needed to modify Figure 20a (p.118) either by creating another whole (bar partitioned into thirds) or by redefining the whole as half the bar of six boxes.

Shari was also unable to satisfactorily model 6/3 with the bar model. She interpreted the whole to be the entire length of six boxes, however, she was unable to identify the thirds. Later, she doubled the entire length to represent two wholes, but was still unable to unpartition the figure to represent thirds and was not able to show where the six in the 6/3 would be.

Although only four students were able to model 6/3 with the length model diagram (Table # 8, p.77), all students were able to model improper fractions with the Cuisennaire rods (Table # 8, p.77). When presented with a black rod (7 cm) as 1 2/5, all students were able to change this to 7/5 and find the rod that represented the one whole (yellow- 5 cm). When given a smaller rod as the whole (green-3 cm) and the larger rod as the fractional part (yellow-5 cm), three of the five students presented with this problem were able to state the fraction was 5/3. Shari and Jacquie initially stated that the fraction was 1 2/5 but then changed their minds and explained that 1 2/3 was correct because the smaller (green- 3cm) rod was the bottom (whole). When Jeremy worked with the Cuisennaire rods to model 4/3, he was able to do so even though he repeatedly failed at properly modeling the fraction with the diagram (Figure 9, p.48). When he applied his understanding of the rods to the diagram he was able to represent the whole (3/3) as less than the entire diagram.

Working with the physical manipulatives seemed easier for students than working within the confines of a diagram. It is beyond the scope of this thesis to consider the reasons for this; however, it could serve as the basis for further research.

The final model that students encountered was the compound set model with the improper fraction 6/3. Debby and Chris were able to represent 6/3 with twelve blocks with no difficulty. Tammy and Shari were able to represent thirds with the twelve blocks however, they were not certain how to proceed. Shari stated that they needed to double it, but then did not continue with this idea. Tammy put the blocks into six groups of two and then said "That would be six out of the thirds." Shari replied "What thirds?" to which Tammy answered "I don't know." The group then put the blocks into four groups of three, and then three groups of four for thirds. They recognized this satisfied the condition for thirds, but were not able to reconcile the six in 6/3. When I asked them to remember what the first thing they noticed about 6/3, Tammy said " two wholes" and separated the blocks into two halves and then quickly rejected this. I encouraged the students to count the groups. They agreed that they could count three but needed to count six. After reconfiguring the set of twelve blocks into fourths and half, Tammy finally said "we need like more." After they realized that they needed more blocks, they were quickly able to model 6/3 and relate the compound set model to the area (circle) model they had made earlier.

It seemed the difficulty Tammy, Shari, and Jacquie had with the compound set model and 6/3 was understanding that they were able to "get more blocks". They knew that 6/3 was two wholes, and had no trouble doubling the whole with the fraction circles, however, they did not seem to transfer this idea to the set model. They were able to partition the set correctly, but then changed these partitions in an effort to accommodate the six in 6/3. I noted two behaviors when this was occurring in this post-unit interview. Firstly, the group was much more certain that thirds were "three groups" and not a "group of three" when they worked with the set. Secondly, they had abandoned the meaning of "counting" they had previously established for the numerator. This leads me to suspect that when students encounter “new territory’ they do not readily transfer their understanding of the meaning of the numerator and the denominator, but rather tend to ignore it in an effort to make the notation “fit” the model or vice versa. This appears to be the case when student work with models with which they are unfamiliar and with symbolic notation, as discussed in the next section ‘ Sense of Orderliness of Fractions’.

Section 1.2 Sense of Orderliness of Fractions

Analysis of Pre-unit and Post-unit Interview Results

The exploration of section 1.2 of the adapted framework for fraction sense (p. 12) involved studying the processes that each student used to order or compare fractions. In particular I was interested to see if each student understood the inverse relationship between the size of the denominator and the fraction size and whether s/he used this information to order fractions (section 1.2.1, p.12). I was also very interested how students ordered a given set of fractions (section 1.2.3, p.12). I was interested to know if students used benchmarks, such as half and one, to order fractions (section 1.2.4, p.12).

Understanding the Inverse Relationship between Denominator Size and Fraction Size

In the pre-unit interview, only three of the seven students was confident in their order of a set of unit fractions (1/2; 1/3; 1/4; 1/5; 1/6) and were able to explain the inverse relationship between the denominator size and the fraction size (Table #4, p.73). Another two of the students were able to order the set correctly but were unsure of the order and needed to confirm the order with diagrams or known benchmarks. The remaining two students ordered the set incorrectly and did not explain the inverse relationship between the denominator and fraction size correctly.

Whole Number Logic Used to Order Unit Fractions.

Jacquie and Jeremy both ordered the set of unit fractions incorrectly. Jeremy had the order totally reversed, but was able to quickly recognize this and correct the order of his fractions. He was also able to explain the inverse or "compensatory" relationship between the size of the denominator and the size of the fraction. Jacquie ordered the set, from smallest to greatest: 1/2; 1/3; 1/4; 1/5; 1/6. She then reversed this order and then reversed it again saying that one-sixth was "bigger". She continued in this vein even after I described a pizza scenario where a pizza was cut into two pieces and six pieces. When I then reiterated the pizza example by stating that "you get one of six pieces and I get one of two pieces, who would get more pizza?" Jacquie replied that we'd both get the same "because well it's still like one piece out of two pieces and that's one piece out of six pieces." When I asked her if both our pieces would be the same size she then answered that they would not be and her sixth would be bigger. She even drew an area diagram to explain how her sixths would be bigger. It was not until I had her model one-half and one-sixth with the fraction circles that she was able to correctly identify the larger fraction and explain the inverse relationship between the denominator and fraction size.

Three things stand out when analyzing Jacquie's response to this question. First, she seemed to be using a whole number strategy when comparing the unit fractions and stating that the sixth (six) was more than two. She seemed to persist in this thinking, even with the pizza scenario and her own model of sixths. The second thing that was of interest was her response that both pieces were the same because they were both one piece. It seemed that she was considering the pieces as discrete units and giving no consideration to the relationship between the denominator, the whole, and the size of the fraction. The final thing that was noticed was that it was after she worked with the manipulatives that she was able to understand the concept. Her diagram did not help her understanding of the concept. As was mentioned previously (p.120), there appears to be a difference in some student's minds between a diagram and physical referent of the same model.

Using Benchmarks to Order Unit Fractions.

Initially, Shari was also uncertain how to order the set of unit fractions but then she applied her understanding of "half" in order to do so. She knew that half was larger than the other fractions and used this to order them, but was puzzled by this because "...this [points to the six] is a bigger number than half." After she worked with the fraction circles, she was able to explain why one-sixth was less than one-half, however, she still seemed a little puzzled when she said "...so the more... is the smallest?"

Like Jacquie, it was noted in Shari's response evidence of whole number thinking when she commented that the six was more than half. It was interesting that she said "half" and not "two". She seemed to identify "half" as a number and seemed to have a strong sense of its magnitude. This sense was so strong that she used it to order the other fractions even though she did not seem to totally understand why she was doing it that way.

Although Shari had a strong sense of the magnitude of 1/2 as a number, the understanding of this fraction did not transfer to other fractions. For example, when presented with the set of unit fractions, she did not see the fraction 1/6 as one-sixth but rather as "six". It appeared that when she was faced with fractions other than 1/2, she did not coordinate the meaning of the numerator and denominator into a number with a single value, but rather tended to disassociate the denominator from the numerator. That is, in the case of 1/6, she treated the denominator "six" as separate from the numerator "one". I suspect that she focussed on the six partitions as discrete units, disunited from the whole. As such, the six partitions became the whole number "six". However, I suspect that although she indicated 1/6 as being "six" she was aware that she was dealing with fractions, and that 1/6 was not the same as six, even though she could not articulate the reason.

Jeremy, like Shari, used benchmark fractions to compare other fractions. In Jeremy's case, his benchmark fractions were 1/2 and 3/4. Similar to Shari, Jeremy's understanding of the magnitude of 1/2 and 3/4 did not transfer to other fractions. For example, in the pre-unit interview, he stated that 18/19 was the same as 3/4 and in the post-unit interview he claimed that 2/3 and 6/7 were both 3/4 and missing 1/4. I suspect that 3/4 was a strong mental referent for Jeremy. He was able to call to his mind's eye an image of 3/4 and could compare this correctly to half. However, he did not seem to have a solid understanding of other fractions and therefore appeared to equate any fraction that is "one piece" away from the whole as 3/4. That is, 3/4 with its 1/4 piece missing, was a strong mental referent and took the place for all fractions that were missing "one piece" from the whole.

Use of Manipulatives and Diagrams to Order Unit Fractions.

Jacquie, Shari, and Debby all benefited from their work with the fraction circles to model unit fractions. Shari was able to explain the relationship between the denominator size and fraction size after modeling the various unit fractions. She was able to come to an understanding of why 1/6, with its six partitions, was less than 1/2 with its two partitions.

Debby, like Shari, was initially unsure how to order the fractions because "...they're all cut differently, like the denominators are different so like....the denominators would have to be equal." When I encouraged her to work with the fractions she knew for sure she stated that she knew a half and then a fourth and ordered these two. She then explained with a diagram how one-half compared to one-fourth. She then drew a diagram of thirds and made it into sixths and then ordered these fractions accordingly. She then placed one-fifth and one-seventh in the set correctly because "there's a pattern."

Debby was used to comparing fractions by finding like denominators and was not totally sure how to compare them when the denominators were different. Like Shari, she had a strong sense of "half" which she used, along with the aid of diagrams, to order the unit fractions. She was able to explain the inverse relationship between the denominator and the fraction size, however, this did not seem to be a concept that she had really considered before. I suspected that this was the case because of the length of time it took her to complete this question (2 minutes) and the fact that she needed diagrams to work with fractions as basic as one-third and one-sixth.

Understanding of Fraction Magnitude is Weak.

Although all seven students were eventually able to order the set of unit fractions correctly, only three of these were able to do so based on their understanding of the inverse relationship between the denominator size and the fraction size (Table # 4, p.73). Two students relied on diagrams and benchmarks to establish the order of these basic unit fractions and two students relied on my guidance and manipulatives in order to establish the correct order of these fractions. I would have expected that students at grade 8 would have a fairly solid understanding of the inverse relationship between the denominator size and fraction size. At the very least, I would have expected that the students would have ordered these common unit fractions based on their everyday experience with them. The fact that four of the seven students had difficulty with this task led me to three considerations. The first being that these students did not have an adequate understanding of the inverse relationship between the denominator and the fraction size and therefore were unable to order the fractions based on this understanding. The second being that they did not have a sense of the magnitude of these numbers and therefore ordered them incorrectly. The third possible conclusion was that they did understand the relationship between the denominator and fraction size, but that they tended to ignore it. Instead, when presented with the fraction notation, they focused on the surface details of the notation and applied their whole number experience to order the set without regard to the meaning of the fraction notation.

Although each of these three conclusions is different, they all point to a lack of "feel" for fractions. Either these students had not developed a sense of the magnitude of these numbers, or if they had, this sense had not been adequately connected to their abstract symbolic representations. Either way, I felt that these students required more work with a variety of manipulatives in order to establish mental referents for the fractions so that they could develop a solid sense of the size of these fractions relative to the whole and to each other. I also felt that they needed more work connecting their understanding of the magnitude of these fractions to the symbolic representations. The goal, of course being students who are able to work with the notation independent of their mental referents. In other words, the notation embodies the meaning of the magnitude of the fraction just as the whole number notation does for the students.

Attention to "Inverse Relationship" Inconsistent.

In the post-unit interview, there was an improvement in each student's ability to explain the inverse relationship between the denominator size and the fraction size (Table # 9,p.78). Four of five students presented with the problem of comparing 3/4 and 3/5 were able to correctly identify the larger fraction and explain the inverse relationship between the denominator and fraction size. Jeremy explained that 3/4 would be more because "...he gets 3/4 and I only get like a bit more than half.... I think 3/4 would be more...cause if you break, if you break this whole into five pieces and that whole into four pieces, his pieces would be bigger than my five pieces." In this explanation, Jeremy very clearly explained the inverse relationship between the denominator and the size of the fraction, but also he made reference to the benchmarks "half" and "three-quarters".

He mentioned that Liam got 3/4 whereas he himself only got a bit more than half. It appeared that he assumed it was self-evident that 3/4 is more than 1/2.

Liam was the only one to compare 3/4 to 3/5 inconsistently. He stated "except he gets five" when Jeremy stated that 3/4 is the larger of the two fractions. In this question, Liam seemed to focus on the five pieces alone. He did not consider the relative size the five pieces to the four pieces, nor did he seem to even acknowledge the numerator (3). This is in contrast to his ability in the pre-unit interview to correctly order the unit fractions and explain the inverse relationship between the size of the denominator and fraction size. It seemed that this understanding seems to "tune" in and out.

Shari also seemed to have an understanding of the inverse relationship between the denominator and fraction size that seemed to "tune" in and out. She was able to correctly identify 3/4 as larger than 3/5 and explained the difference in the relative sizes of the pieces. However, when asked to compare 5/6, 5/7, and 5/8, Shari initially claimed that sixths had smaller pieces, but then changed her mind to eighths and stated "more pieces is smaller." However, she then contradicted this statement when she offered 5/9 as a fraction larger than 5/6 and 5/7.

All of the students were able to correctly explain what would happen to the size of the fraction "a/b" when "b" increases, and when "b" would decrease. However, the explanations did not always relate the increase or decrease in the fraction size to the corresponding decrease or increase in the number of partitions (b). Jacquie explained that when "b" decreased the fraction would become "bigger… 'cause you're taking like less pieces say that's away to begin with and you're making it less than the top number so you're getting closer to the denominator" to which Tammy added " the actual whole." This explanation highlights Jacquie's use of the "range" strategy to compare fractions. That is, her preference to determine the relative size of fractions by finding the difference between the numerator and denominator and comparing these differences without regard for the relative sizes of the pieces she's comparing.

When presented with these two questions, Chris and Debby both went into an elaborate discussion, complete with models of how the whole itself became greater or smaller. Chris stated that "when you increase or decrease the denominator you're either losing or gaining some more wholes." When I questioned this thinking with an example of 3/4 and 3/3, both students were able to correct their explanations and recognize that it was not the whole that was changing but the partitions of the whole that were changing. This explanation was unusual from Chris and Debby and was a complete departure from everything else they had explained and modeled for me previously. I wondered whether they really believed the whole itself changed or whether this unique explanation was a consequence of the use of variables in the fraction notation that might have confused them. I also wondered if this explanation was connected to their earlier definition of the denominator as being "the whole" without reference to partitions. I wondered if indeed they saw the denominator as "the whole". That is, when the denominator changed the whole as defined also changed. That was what they were modeling with the "a/b" question. However, since this did not occur for any other question in either the pre-unit or post-unit interview, I considered it an aberration in their thinking.

From the analysis of the data in the pre-unit and post-unit interviews, I believed that all the students had shown improvement in their understanding of the inverse relationship between the denominator size and fraction size. However, I felt that although this understanding had improved, I did not feel that it was a sufficiently well established or connected to the written notation. Students were still using whole number logic and the "range" strategy to compare the fractions. I believe that if the relationship between the denominator and the fraction size was well enough understood and connected strongly enough to the written notation then this would not happen. I cannot imagine these students ordering whole numbers incorrectly. The sense of the magnitude of these numbers is too well understood and connected to the written notation. I feel that this sense of the magnitude of fractions is still lacking in these students.

<u>Ordering and Comparing Strategies</u>

The ordering task of the pre-unit interview involved ordering a set of unit fractions and comparing the fraction 18/19 and 3/4. From the responses to these problems, I identified a variety of appropriate and inappropriate strategies that students used in order to compare and order fractions. The appropriate strategies included: using like denominators to compare fractions, using area model diagrams and manipulatives to determine the relative size of fractions, using a pattern to order the fractions, using benchmarks such as half and fourth to order fractions, and comparing the relative sizes of the denominators. Although all of these strategies were appropriate, the last two strategies indicated that the individuals who used these strategies had a good sense of fractions. Using benchmarks to compare and order fractions points to an individual who has a strong sense of the magnitude of fractions. Ordering and comparing fractions using an understanding of the inverse relationship between the size of the denominator and fraction size also points to a strong understanding of the concept of fractions.

In contrast, the first three strategies did not require a sense of fractions. Strategies such as the like-denominator method can be applied with little or no sense of the magnitude of fractions. For example, Chris was only able to correctly compare 18/19 and 3/4 by finding like denominators for both fractions. He said he was not sure of another way of to compare the fractions without finding like denominators first. He did acknowledge that this method was tedious with denominators such as 19 and 4.

If Chris truly understood the magnitude of both fractions, then he would have been able to recognize that they were 1/19 and 1/4 away from the whole. If he had strong fraction sense then he would have compared these "missing parts" and then reasoned that 18/19 was the larger of the two fractions.

Similarly, students who required the physical manipulatives did not have a strong sense of the magnitude of fractions, even those as basic as 1/3 and 1/6. Shari, Jacquie, and Debby all needed the manipulatives to order the unit fractions. Clearly, these students did not have a strong sense of the magnitude of fractions or else they would not have needed the manipulatives to order the set of basic unit fractions.

The use of a pattern to order the unit fractions also did not require a sense of the relative size of the fractions. Debby, after establishing the order of 1/2, 1/3, 1/4, and 1,6 using diagrams, placed 1/5 and 1/7 based on a whole number pattern of the denominators (e.g. 2,3,4,5,6,7). Even though she had used diagrams to determine the relative size of the unit fractions 1/2, 1/3, 1/4, and 1/6, this concept did not appear to transfer to 1/5 and 1/7. The use of the pattern in the denominators did not require or demonstrate a sense of fractions.

Some students often used inappropriate strategies to order and compare fractions. These strategies included: comparing the denominators incorrectly without regard to the inverse relationship between the denominator and the whole- "a whole number strategy"; and focussing on the "range" between the numerator and denominator to order fractions- considering the partitions of the fractions as discrete units.

The "whole number logic" strategy was explained in the section "Whole Number Logic" (p.123). Students who used this strategy ordered the set of unit fraction inappropriately. The order of the set was in accordance with whole number logic so 1/6 was identified as the largest fraction and 1/2 was identified as the smallest.

The "range" strategy proved to be one that students applied frequently to compare and order fractions. When presented with 18/19 and 3/4, three students felt that both fractions were equal because they were both missing "one piece."

Shari was not sure she could answer the question so I encouraged her to visualize a pizza cut into fourths and nineteenths with one piece missing from both. She then claimed that both would be equal "because they're the same size pizza and one piece is gone...." She went on to mention that the fourths are bigger but she didn't seem able to sustain her explanation and told me " so they're, be same, equal amount... I know how to do it but I can't really explain it."

Tammy immediately stated that 18/19 and 3/4 were the same because "there's only a difference of one between the top and bottom number." When I asked her to consider that both are missing one piece, she told me that the 3/4 was more because the pieces are bigger. I then asked her to consider which fraction was missing less. She told me that 18/19 was missing less and then indicated, albeit with some uncertainty, that the 18/19 was the larger fractions.

Jeremy stated that 18/19 and 3/4 were actually the same because "there's one less than the bottom number." He went on to explain to me that 2/3 and 3/4 were also the same, even after imagining a pizza scenario. He even persisted in saying that they were the same even after modeling the fractions with fraction circles. It was not until he lay the fraction pieces overtop each other that he changed his answer to identify 3/4 as the larger fraction. He then stated that he would not be able to just tell by looking at the two fractions, which one was larger.

Improvement Noted in Ordering Strategies.

In the post-unit interview, students were asked to compare a number of pairs of fractions and to order two sets of fractions (Table # 10, p. 79 and Table # 11, p.80). Three trends were noticed in the post-unit interview. The first was that students used the benchmarks of one-half and one-whole more frequently to order fractions. The second trend was that students seemed to have difficulty comparing fractions by comparing what was missing from each fraction. The third trend noted was that students persisted in using the "range" strategy of ordering fractions even when it was repeatedly proven to be an inappropriate strategy.

Use of "Inverse Relationship".

All of the students correctly ordered 3/4 and 3/5 by focusing on the relative size of the denominator, except Liam who used a whole number strategy and claimed that five pieces was more than four pieces (Table # 10, p.79). In this question, most of the students correctly applied their understanding of the inverse relationship between the denominator and fraction size. Nobody suggested using a like-denominator method, as had been suggested in the pre-unit interview. Also, no one needed manipulatives to explain the inverse relationship between the size of the denominator and the size of the fractions.

Use of Benchmarks.

When presented with 4/5 versus 6/5, three of the five students presented with this problem, named 6/5 as the larger fraction because it was greater than one whole (Table # 10, p.79). Liam and Jeremy both gave 4/5 as the larger fraction, however when I repeated the question Jeremy changed his answer and explained that 6/5 was larger "...cause there's one whole and still some left". Liam continued to insist on 4/5 "...'cause um, four, cut in four is bigger." In this case, Liam seemed to interpret the numerators (which are different) as the denominator. He seemed to reverse the meaning of both terms that he had previously established.

Except for Liam, an immediate recognition that 6/5 was greater than one whole was noticed. Four of the five students used this benchmark in order to order the two fractions. This was a significant improvement from the pre-unit interview where most students were confused with the improper form of fractions.

Use of the "Range" Strategy Persists.

Tammy, Shari, and Jacquie all identified 6/7 as larger than 2/3. However, Shari and Jacquie went on to explain that both fractions are missing "one piece" and are really the same but not the same size. When I asked them how we decide which is the larger fraction, Shari replied "because six is bigger than two" to which her partners both laughed. After working with the manipulatives all three students could identify that 6/7 was missing less and was therefore a larger fraction.

Like Shari and Jacquie, Liam and Jeremy focused on the difference or "range" between the numerator and the denominator and stated that 2/3 and 6/7 were the same. Jeremy even stated that both were equivalent to 3/4, perhaps because this is a mental referent for him of a fraction that is missing "one piece". Even after discussing a pizza scenario, both students continued to state that both fractions would be the same. Jeremy was able to state that thirds would be bigger than sevenths, but continued to insist that both fractions would still be the same amount.

In this question, Shari employed whole number logic when she claimed that " ...six is bigger than two." Four of the five students used the range strategy to order the fractions (incorrectly). These students compared the missing "piece" of both fractions as if they were discrete units. They seemed to have totally disregarded their understanding of the inverse relationship between the denominator and fraction size that they had clearly described in the first question.

To aid the students, I encouraged them to draw diagrams and use the manipulatives to model the fractions. I wanted them to notice which fraction was missing less and was therefore a larger fraction ("missing less therefore bigger" strategy). After considering the models, all three girls could identify the larger fraction, however, the two boys continued to have difficulty. Jeremy and Liam recognized that thirds would have larger pieces and then identified the 2/3 as being more. They eventually were able to explain that the fraction missing the smaller piece (6/7) was the larger fraction, however, this required a great deal of support from me and I did not feel it was a strategy they were comfortable with. It seemed that they were comfortable with a direct comparison of the size and quantity of the pieces. However, the indirect comparison that required students to compare the missing pieces, decide which fraction was missing less, and then use this information to decide which fraction was more seemed difficult for all students, especially Jeremy and Liam.

The next fraction pair was 3/4 and 5/12. All five students correctly identified 3/4 as the larger fraction but only Tammy used the benchmark of half to order the two fractions (Table # 10, p.79). Shari and Jacquie once again focussed on the range between the denominator and numerator, even though they had just demonstrated to themselves that that strategy did not work. Jeremy explained the difference in the relative sizes of the denominators and seemed to be able to visualize making the twelfths equivalent to the fourths and then comparing the equivalent with the 3/4. When I asked them what they noticed right away about 5/12, Liam noted and Jeremy agreed it was almost half. When asked about 3/4 Jeremy replied "it's 3/4" and Liam said "over half" to which

Jeremy replied " It's also 3/4." Here again, Jeremy seemed to be demonstrating his strong identification with 3/4.

When shown 3/5 and 5/7, Shari immediately identified them as equal but then seemed to change her mind and explained that "well, they would be equal, the, like they're equal in... 5/7 is bigger but 3/5 and 5/7 are both two apart." Jacquie agreed with this statement but claimed that 5/7 was bigger because "...that denominator (7) is bigger than that one (5)." Tammy was able to identify 3/5 as being larger after drawing a diagram and explaining the "missing less therefore bigger" strategy. Neither Shari nor Jacquie seemed to be able to follow her argument. Shari stated that she could just look at the fraction numbers but she claimed she couldn't really explain it like Tammy had done.

Again, Shari and Jacquie relied on the range strategy and whole number logic to order these fractions, even though they have admitted that these strategies were not valid.

When confronted with same question, Liam and Jeremy recognized that both fractions were greater than one-half. Both then went on to describe the size of the pieces relative to each other and the quantity of pieces, but they did not seem able to coordinate these two pieces of information simultaneously. After drawing diagrams, Jeremy once again considered the pieces as discrete units and stated that both fractions were the same because they were both missing two pieces. I encouraged his to consider the relative size of the pieces. To which he answered that the fifths were bigger pieces and therefore 3/5 was the larger fraction. Again, the indirect comparison proved difficult for both students.

The "range" strategy proved itself to be a stubborn one. Even though they had repeatedly proven to themselves that the range strategy would not lead to the correct solution, several of the students continued to employ it. Jeremy's first comment when he compared 3/5 to 6/9 was "Three-fifths... because there's three out of five pieces there and the 6/9 there's only six pieces out of nine. The three out of five is two pieces...." Jeremy did, however, begin to recognize his error "...oh, hang on, it's just like the last one...." He then tried to consider not only the size but also the quantity of the missing pieces simultaneously. He came to the conclusion that both fractions were the same because the smaller size of the nineths was compensated for by the fact that there were three of them which equaled the two larger fifths. After some discussion and modeling, Liam suggests that we use a calculator but they are not sure how to proceed beyond this point.

Tammy, Jacquie, and Shari all used the range strategy to compare the fractions and determined that 3/5 was larger because it was " two apart". I suggested the "missing less therefore bigger" strategy but they found this very difficult and suggested the "like denominator" algorithm.

Chris used the like denominator algorithm for all the questions whereas Debby used benchmark to half in order to solve them. Both students felt that the benchmark strategy worked but that the like denominator algorithm was more accurate and suggested this was a better strategy to use, especially if the numbers were quite small and easy to multiply.

Range and Benchmark Strategies used Simultaneously.

The last two tasks involved ordering two different sets of five fractions with unlike denominators (Table # 11, p.80). All the students found the first set ((4/8; 5/9; 5/12; 2/5; 10/8) the most difficult. They felt the second set (6/6; 3/1; 11/7; 9/18; 4/21) was easier since the fractions were more spread out. Like in the comparing questions (Table # 10, p.79), the range strategy continued to be employed by Jeremy and Jacquie. I did notice, however, the application of benchmarks to the tasks. I felt this was an improvement over the pre-unit interview where none of the students used this strategy or reported ever having seen it.

Debby and Chris were able to order the second set correctly and only disagreed on the order of 4/8 and 5/12 in the first set. I noted that they use benchmarks to half in order to settle the disagreement. Chris explained that 4/8 was equal to half and 5/12 was less than half, therefore 4/8 was more. Debby quickly agreed and changed her answer. Both students said the second set was easier to order because the "...estimates were closer to half or whole" as compared to the first set.

Tammy also used the benchmarks of half and whole to order the two sets of fractions correctly and used these benchmarks to point out the errors in her partner's order. Jacquie used a combination of benchmarks and the "range" strategy to order the set as did Jeremy. Both recognized that 10/8 was over the whole and used this to order it as the largest fraction. However, both of them continued to use the difference between the numerator and denominator in order to decide the order of the fractions. When I pointed out that 4/8 and 5/9 were both " Four away from the whole", Jacquie expressed confusion with these and said " Yah, I know, I didn't know what to do with that one." She clearly did not consider the relationship of these two fractions to half. Jeremy resolved this problem by saying that 5/9 was "bigger" than 4/8 even though both were "missing four". He did not, however, explain how he came to decide this.

Liam had more success with the ordering and explained his use of half and whole as benchmarks. He was unsure what to do, however, with 2/5 and 5/12 since both were under half. Jeremy felt that 2/5 was larger since it was "... half away from half." In this response, I noticed Jeremy attempting to apply the "missing less is more strategy" to 2/5 and 5/12. However, when comparing 0.5/5 and 1/12 he considered these "missing" pieces are discrete units. As a result, he compared the "0.5" piece to the "1" piece, determined that the "0.5" piece was smaller, and therefore decided that 2/5 was larger than 5/12.

Jeremy continued to change the order of his fractions. He would vacillate between the benchmark strategy and the "range" strategy. At one point he considered 5/12 to be larger than 10/8 since 5/12 was equal to 2 2/12. I noted that this inverting of 5/12 to 12/5 was the opposite of his earlier tendency to invert the improper fraction.

The last set of fractions once again saw the "range" strategy appear. Jacquie ordered 3/1 before 11/7 because 11/7 was four over the whole whereas 3/1 was two over the whole. She immediately recognized her error after Tammy explained that 3/1 was three wholes whereas 11/7 was only 1 4/7. Although Jacquie seemed able to interpret the notation, she nonetheless seemed to disregard the meaning of the fraction notation and persisted in using the inappropriate "range" strategy to compare the fractions.

Jeremy and Liam were both successful in ordering the last set and both used benchmarks when comparing and ordering the fractions. Interestingly, Jeremy once again inverted 11/7 to 7/11 but only for a moment and quickly caught the error. Both students went on to clearly explain that 3/1 was the largest fraction since it was three wholes and 4/21 was the smallest since it was under half.

The post-unit interview demonstrated to me that students could interpret the notation correctly, could explain the inverse relationship between the denominator and fraction size, and could relate fractions to known benchmarks (half and one whole). However, it also demonstrated that students could also ignore or abandon all of this understanding and employ whole number logic to the fractions. The persistence of the "range" strategy, in spite of the student's ability to correctly interpret fractions, needs further exploration and further consideration.

CHAPTER VI

DISCUSSION

In this chapter the results of the pre and post-unit interviews are discussed. The findings of my research into students' fraction sense are summarized and the pedagogical implications of these findings are discussed. As well some areas for further research are suggested.

Discussion of Results and Pedagogical Implications

In this section the results of the pre and post-unit interviews are presented in accordance with the adapted framework for fraction sense (Figure 2, p.12). First, the major findings of my research into students' sense of the relative and absolute magnitude of fractions are discussed, which is subsection 1.1 of the adapted framework. Then, the results of the exploration into students' sense of the orderliness of fractions are presented (subsection 1.2). As a part of this discussion, the impact that classroom activities and instruction had on these two components of students' fraction sense is considered. Recommendations for activities and instruction to help students improve their number sense are also included.

Subsection 1.1: Students' Sense of the Relative and Absolute Magnitude of Fractions

Students' sense of the relative and absolute magnitude of fractions involves understanding the meaning of the numerator and denominator and how these two entities combine and relate to the whole in order to produce a number with a single value. In my research into this understanding, I examined the awareness and significance that students had for the whole. I also looked at the meaning students had for the denominator and its relationship to the whole. In addition, I investigated the meaning students had for the numerator and its relationship to the denominator and the unit. Inherent in these explorations, was an investigation into the understanding that students had for the written notation of fractions and how this understanding affected their work with fractions. As well as an investigation in to the different models of fractions that students understood and those that caused them difficulty.

Students' Understandings of the Whole: Pre-unit Interview Results.

Students disregard whole after it is partitioned.

Students in the pre-unit interview demonstrated a part-whole understanding of fractions. That is, they viewed a fraction as a whole that is partitioned (into congruent segments) with a section of the partitions shaded in or removed. Students' behaviors and dialogue suggested that their model of the whole was always a geometric figure, such as a circle or square. No other model of the whole was produced voluntarily by students.

In the pre-unit interview, once the whole was partitioned, students usually did not consider it again. Rather, they focused on the partitions and treated them as distinct units that could be thought about without consideration of the whole from whence they came. This manifested itself in both student speech and action. Students in the pre-unit interview rarely mentioned the whole when defining the denominator. Instead, they spoke of the partitions as "parts" or "pieces" without reference to the whole. When asked to identify the whole on diagrams that modeled the fractions 1 1/2 and 4/3 (Figures 8 & 9, p.48), fewer than half of the seven students could do so confidently.

Whole is viewed as the entire figure.

Another trend that was revealed in the pre-unit interviews was that students thought of the whole as being the entire figure or model. They had a great deal of difficulty seeing the whole as less than the entire set or object. As such, most of the students had difficulty interpreting and modeling fractions greater than one whole.

Students' Understandings of the Whole: Post-unit interview results.

Whole given more consideration.

In the post-unit interviews there was a noticeable increase in direct references to the whole, especially when students worked with fractions greater than one whole. Students demonstrated an increased awareness that the denominator represented partitions of the whole rather than the total number of partitions present. Students defended their selection of a denominator by making direct reference to the number of partitions of the whole versus the total number of partitions. This was an improvement from the pre-unit interviews where students focused on the total number of objects as the denominator. For example, when shown the model of 5/4, many students rejected it as fourths because there were five objects.

Whole viewed flexibly- students model improper fractions.

The students also demonstrated that they were more willing to view the whole as represented by less than the entire set or object. As such, students were able to model fractions greater than one whole whereas students were generally confused with these fractions in the pre-unit interview. This improved ability to model and interpret fractions greater than one whole did appear to be model specific, however. Students were able to model fractions greater than one whole with concrete referents and diagrams of area models and with concrete referents of linear models (Cuisennaire rods). However, some of the students could not model 6/3 on a diagram of a linear model (with partitions already provided). Modeling fractions greater than one whole with set models also proved difficult for all but two students. However, once students were aware that they were "allowed" to get more blocks (in the set model), they were able to model the fraction greater than one whole. I concluded that the differences in success in modeling the fractions greater than one whole was a factor of the model and not the concept. As such, this discussion is continued in the section "Students' Referents for Fractions".

Consideration of whole disregarded when comparing fractions.

Although students demonstrated more awareness of the relationship between the whole and the denominator, they also demonstrated they would abandon this awareness when comparing or ordering fractions. They would compare the "pieces" of each fraction as if they were independent of the whole and therefore the same size. This situation is discussed further in the section "Students' Understandings of the Denominator".

Pedagogical considerations.

In order for students to understand fully the part-whole concept of fractions, they must develop a conscious awareness of the whole and its relationship to the denominator and numerator. To keep the thought of the whole foremost in students' minds, I instructed them to utilize a three-step process when solving problems with fractions. The first step had students question "What represents the whole?" and then ask "How many equal groups or parts of the whole?" The final step had students ask themselves "How many partitions do I count?" I encouraged students to always attempt to identify the whole before they tried to model the fraction because I wanted them to be very conscious that the partitions were not discrete but parts of the whole. To develop this awareness, I would constantly re-define the whole so students could not always assume it to be a certain object or shape. Constantly changing the whole, encouraged students to ask me to define the whole first before we began any other work. My goal with this approach was that students would keep the concept of the whole and its relationship to the partitions foremost in their minds.

Although I encouraged students to verbalize the questions listed above, I saw no evidence in the post-unit interview that they would do so voluntarily. I suspect this was the case because this was an unfamiliar activity that I had introduced late in grade 8. As such, the students had not developed the habit of using this approach. Also, I suspect that they did not really believe they needed to use it since it no other teacher had presented it except me. I would be interested to try this approach with students just beginning their study of fractions in order to see if they would apply it to their work with the referents and if they were successful with it.

I was aware that Armstrong and Larson (1995) had concluded that the area models that are typically utilized in fraction instruction do not focus attention on the whole but rather tend to focus attention on the parts as discrete units. As such, I tried to design activities that would raise awareness of the whole. These activities asked the student to identify the whole given a model of a fraction less than or greater than one whole (Lesson # 2, p.53) and identify the fraction when the whole as defined was repeatedly changed and the fractional piece as defined remained constant (Lesson # 3, p.53). Since these problems could not be solved without consideration of the whole and its relationship to the denominator and numerator, it was my goal that these activities would increase students' awareness of the role of the whole when working with fractions.

I felt that the activities that students worked on during the instructional unit were effective in improving students' awareness and consideration of the whole. In the pre-unit interview, students were unsure how to find the whole when provided with a model of an improper fraction. After the instructional activities with the Cuisennaire rods in lesson # 2 (p.53), students in the post-unit interview were able to identify the whole with fraction less than and greater than one whole. I also noticed that there was an improvement in students' abilities to identify the name of the fractional piece (denominator) in relation to the whole. When the whole was constantly redefined (task #2, p.53), all students were able to reference the partitions of the whole as the reason for their choice of denominator. This was in contrast to the pre-unit activities where students focused on the total number of objects or partitions as the reason for their choice of denominator.

The best results were seen with tasks in the post-unit interview that matched activities that students worked on in class. For example, identifying the whole with Cuisennaire rods given a fraction greater than one whole or modeling fractions greater than one whole with area and linear models were practiced in class and showed good results in the final interview. Since we did not model a lot of fractions with set models, I was not totally surprised that students were unsure what to do with this model. This finding agrees with that of James Hiebert (1988) who found that "...the success of instructional efforts seems to be restricted to the referents and actions which children have experienced and about which they are knowledgeable" (p. 348).

Although the activities described above did improve students' awareness of the whole, I did not feel that they alone were enough to overcome students' tendencies to ignore the relationship between the whole and the denominator when they compared fractions, especially symbolic fractions. Suggestions for activities that force students to keep in mind the relationship between the whole and the denominator are provided in the next section entitled "Students' Understandings of the Denominator".

Students' Understandings of the Denominator: Pre-unit interview results.

Partitions seen as discrete objects.

Students demonstrated that they understood the denominator to represent the number of congruent partitions of a geometric figure, such as a circle or square. It appeared, however, that students interpreted these partitions to be discrete units independent of the whole, not as partitions or groups *of the whole*. This finding was similar to those found by other researchers (Mack, 1990; Markovits & Sowder, 1994; Armstrong & Larson, 1995).

This interpretation of the denominator as discrete units manifested itself in many different situations. Several of the students had difficulty with models that had a number of partitions or objects that were different from the denominator. For example, only three students could represent 4/6 with Figure 6 (p.47) although all could explain its equivalent 8/12 with the model. I suspect that the four students who could not reconcile 4/6 to Figure 6, were working with the "discrete" approach to the denominator. That is, they saw twelve objects and therefore concluded that the correct denominator in their minds was twelve, not six. The three that could reconcile 4/6 with Figure 6 did so by grouping or "unpartitioning" the twelve circles into six groups of two. I suspect the working definition of the denominator for these three students was the "groups of the whole" approach. Indeed, I find support for this hypothesis in the students' definitions of the denominator. Those in the "discrete" camp defined the denominator as parts without any mention of the whole, whereas those in the "groups of the whole" camp, would sometimes mention the whole, albeit indirectly as "parts of *it*" or "the square is cut into 6 parts".

Another manifestation of the "discrete" approach to the denominator surfaced when four students claimed that fractions 18/19 and 3/4 were the same. Explanations mentioned that both fractions were missing one piece and therefore represented the same amount. These explanations from students seemed to indicate they were not considering the partitions in relation to the whole. That is, they were not considering the difference in size of the partitions of the two fractions. Rather, they treated the partitions of the two fractions as independent from the whole; that is, as discrete units of the same size. Stating both fractions were the same also pointed to a weak understanding of the concept of equivalence of fractions.

Confusion between equal and congruent partitions.

Another conclusion drawn from the first interview was that students interpreted "equal" as congruent. This conclusion was based on the fact that all students modeled congruent partitions only and that most students were not certain, at least initially, if non-congruent equal sized parts were acceptable.

I also discovered that some students' understanding of congruent or equal parts did not transfer between models. Although four students were certain that equal parts were required for partitions of area models, only one was certain that this was also the case with set models. Four students reported that they were uncertain if the groups needed to be equal when working with sets. These results would seem to agree with those of Peck and Jencks (1981) who found that fewer than half of the grade 6 students interviewed, knew that the partitions needed to be equal in size.

Weak understanding of conservation of area .

Another surprising discovery was that most students were not aware that fourths of one arrangement were equal in size to those of another arrangement of the same whole (Figure 13, p.110). Only one student was able to reason that since both were fourths of the same size whole then the fourths were equal, even if they didn't look to be so. The other students either determined the equality in the figures by a strategy similar to the Area of Parts/ Decompose-Recompose strategy of the students in Armstrong and Larson's study (1995) or they rejected the figures as being equal based on visual information.

I believe that this behavior agrees with my earlier conclusion that students viewed the partitions as discrete units, not as parts of the whole. When students viewed the two arrangements of fourths (Figure 13, p.110) I suspect they agreed that they were fourths because they saw four congruent parts in each model. However, I suspect they did not see four equal parts *of the whole* and therefore did not consider the concept of conservation of area of the whole.

Students' understanding of "inverse" relationship weak.

The pre-unit results those four students had difficulty ordering a set of basic unit fractions (1/2, 1/3, 1/4, 1/5, 1/6) was most surprising. Although two of these students (one was an "A" student in math) were able to order the set correctly, they needed diagrams or manipulatives in order to do so. The other two students used whole number logic to order the set incorrectly. By grade 8, I would have expected that most students would have developed a sense of the magnitude of these unit fractions either through everyday experience or classroom instruction and would have been able to order the fractions on this understanding alone.

Equally surprising was that only three students could correctly explain the compensatory or inverse relationship between the number of partitions (denominator) and the size of the fraction. Perhaps, not surprisingly, these were the same three students who could confidently order the set. Once again, I would have thought that this concept would have been firmly established in all students' minds by grade 8. However, according to Behr and his colleagues (1984), the understanding of this concept is highly variable and can remain elusive for some students even when they've "...had ample opportunities to learn and practice" (p.338).

There might exist a link between students' focus on the partitions as discrete units and their lack of consideration of the inverse relationship between the number of partitions of the whole and the size of those same partitions. Perhaps if a student sees the partitions as discrete from the whole then the inverse relationship is of no real consequence in the student's mind.

Students' Understandings of the Denominator: Post-unit interview results.

Partitions as discrete units continues to appear.

All students showed improvement in their understanding that the denominator represented equal partitions or groups *of the whole*. Definitions and explanations of the denominator made more frequent references to the whole. Students had shown real growth in this area compared to the pre-unit interview when either they focussed on the total number of objects ("discrete" approach) and ignored the groups altogether, or else they focussed on both the groups and the number of objects within the groups when working with set models.

Further evidence was provided when students correctly named the denominator of an improper fraction modeled with Cuisennaire rods. Students referenced the partitions of the whole when they defended their choice of denominator. When I asked them why the total number of partitions was not the denominator, four students explained that it was the partitions of the whole that determined the denominator, not the total number of partitions. I did notice in these explanations however, that only two students directly mentioned the word "whole". The other two students referred to the whole indirectly by citing the color of the block that represented the whole. Although I accepted that these two students were discussing the parts of the whole, their lack of direct reference to it was noticed. These two students were also the most likely to shift into thinking about the denominator as discrete units, whereas the two students who made the most direct references to the whole did not employ this type of thinking.

Even though all the students demonstrated growth in their understanding of the denominator as partitions or groups of the whole, four students would abandon this understanding when confronted with comparing and ordering problems. These students used the "range strategy" to compare fractions. This strategy is explained and exemplified in Chapter 5 (p.136).

Interestingly, students were able to use the "missing less therefore bigger" strategy with this "range" strategy. That is, students recognized that the fraction that was missing the smallest amount was the smallest away from the whole and therefore was the largest fraction. I have assumed here that students were comparing the fractions to the benchmark one whole. However, they could also be ignoring the whole and merely comparing the differences based on the fraction that has the smallest piece missing before the denominator matches the numerator, which all students had demonstrated to understand as one. This type of thinking would not be based on a part-whole concept of fractions, but rather on a surface matching of the denominator to the numerator. Further research is needed to distinguish between these two possibilities.

Range strategy proved powerful.

The "range" strategy proved to be a powerful one. The four students who used this strategy would continue to do so even though they would demonstrate to themselves that the strategy did not work consistently. I suspect that this was the case for a few reasons. One was the fact that the strategy was easy and fast and proved itself successful more often than it did not. The four students who used this method were not the strongest math students and did not profess an interest in math. I believe that these students were not troubled with the fact that the range strategy was not consistent and therefore not mathematically valid. Conversely, the students who did not employ the strategy were the ones I considered better math students. I believe that these students rejected the range strategy because they understood that it was not valid or correct, and also because they could mentally handle the complexities of the other strategies. That is, they could maintain the inverse relationship of the denominator to the whole and coordinate this thinking with the numerator and the whole in order to compare fractions. This is perhaps the second reason why the other students opted for the range strategy. That is, these students had difficulty maintaining and coordinating all the components of fractions, as a result they clung to the range strategy because it was the only strategy they could be moderately successful with. I detailed earlier how two students tried to explain the comparison of two fractions but could not maintain and coordinate the different aspects of the denominator and numerator. As a result, they abandoned the explanation for the range strategy.

Although four students continually returned to the range strategy in the post-unit interview, I was encouraged to note that they would abandon this strategy for the benchmark strategy when the fractions were fairly easy to compare to a benchmark (i.e., one fraction under half and the other above). It was when both fractions were both under or over half (or one) that students were most likely to use the range strategy.

Confusion between equal and congruent eliminated.

In the final interview, all students demonstrated that they understood that the partitions or groups of the whole needed to be equal, but not necessarily congruent. Students explained that the groups of a whole set needed to be equal in quantity and that the number of objects within the group need not equal the denominator. This was a significant improvement from the pre-unit interview where many students did not see the groups as relevant or else they did not believe the groups needed to be equal (although they believed this necessary for area models). Students also discussed with confidence that the partitions of an area model could be non-congruent and, as long as they were equal in area, they were valid as a representation of a given fraction.

"Inverse" relationship more clearly explained.

Along with growth in referencing the partitions to the whole, was students' abilities to explain the inverse relationship between the number of partitions and the size of the partitions. Only one student did not explain this relationship correctly. The others were able to very clearly relate the number of partitions to the size of each partition and use this relationship to compare simple fractions such as 3/4 and 3/5.

Similar to referencing the partitions to the whole, some students would abandon their understanding of the inverse relationship between partition number and partition size when ordering fractions that could not be easily compared using benchmarks. That is, they would treat the partitions of two different fractions as if they were the same size.

For example, four students stated that 2/3 and 6/7 were the same because they were both missing one piece. The inverse relationship that they had previously explained for thirds and sevenths was ignored. My thoughts on the reasons for this behavior are presented in the section "Comparing fractions using the range strategy".

Pedagogical Implications.

A number of instructional activities proved to be very effective in aiding students in their understanding of the denominator. The first activity, "Tige's Treats" (Lesson # 1, p.52) was very effective in teaching students about the conditions imposed by fractions. Students learned that a fourth implied partitioning the area into four equal parts. They learned the difference between equal and congruent and also that non-congruent pieces could be equal in size.

Work with a variety of manipulatives modeling examples and non-examples fractions helped students generate a more generalized understanding of the meaning of the denominator. Students learned that the denominator indicates the number of equal in size parts or groups to separate the whole into. They were able to apply this understanding to area, linear, and set models which was a significant improvement from the first interviews.

Activities such as lesson # 3 (p.53) where students had to find the whole given the model of the fraction or the fraction given the whole, proved very effective in focusing students' attention to the relationship between the numerator, denominator, and the whole. These problems could not be solved without doing so. I relate the improvement in students' abilities to find the whole, given a model of an improper fraction to these activities and as such, I would use them again.

Activities that I would change are some of the benchmark comparing problems that I presented to students. Upon examination of the questions presented to students, I noticed that the majority of the comparisons could be solved using the "range strategy". I discovered that the problems provided were inadvertently promoting this type of approach. Namely, the problems encouraged students to consider the partitions as discrete from the whole because it was easy, fast, and usually effective to think of them that way. To correct this situation, I would create problems where using the range strategy would result in the wrong answer more frequently than the right one. Hopefully, the lack of success would extinguish this strategy from students' minds. It would be an interesting study, in fact, to see if students would drop this strategy if they repeatedly proved to themselves that it did not work. If they did not, then more research would be required to explain why it was so attractive to them.

Other recommendations are that students need to spend a great deal of time in the intermediate grades working with a variety of referents to model fractions. In particular, the inverse relationship between the number of partitions of the whole and the size of those partitions should be stressed. However, students should not be told "the bigger the denominator the smaller the fraction". They should come to this conclusion themselves through work with manipulatives. I recommend that students should be encouraged to explain their thinking to other students and in their math journal. I found that communicating their ideas helped students to clarify their thinking. The NCTM also supports communication in math classrooms as a means to develop deeper understandings of math concepts.

Other activities to emphasize the part-whole concept that I recommend are ones like lesson # 3 (p.53) where students need to find the whole given a model of the fraction, or name the fraction based on the model of a fraction, or model a fraction that is specified. In all cases, fractions should be proper, improper, and mixed and a variety of concrete and visual models should used.

It is important that students have many opportunities to partition models and diagrams themselves rather than providing the partitions "ready made" on diagrams. Students need to learn the subtle differences between partitioning an area model versus a set model. Students also need to group or "unpartition" models and diagrams They need to experience models and diagrams where the partitions or object in the set are not the same as the denominator. Students need to be given experiences where they learn to partition and "unpartition" models correctly to match the fraction notation. The ability to partition and unpartition is an important precursor to understanding equivalent fractions.

In order to keep students' minds focussed on the relationship between the whole and the partitions, the whole should be re-defined regularly so students don't come to associate the whole or the fractional parts as any one color of manipulative or shape. Changing the size and shape of the whole would lead students to appreciate that the name of a "piece" is relative to the whole. Activities such as those in the addenda series "Understanding Rational Numbers" that compare the fractional sections of two different sized pizza force students to consider that the wholes need to be the same size if a comparison of the fractions is to valid. Students in my class remembered this trick I played on them- two students in the final interview referenced the need for the wholes to be the same size when two fractions are compared.

Students' Understanding of the Numerator : Pre-unit interview results

Influence of area models.

Student definitions of the numerator were consistent with a mental referent of a diagram of an area model. Their definitions referred to coloring in or taking away or "eating" parts or sections. These definitions do not suggest much experience with linear models such as number lines since one doesn't "color in" or "eat" the partitions with a number line. Nor do these definitions suggest much experience with set models since all the definitions referred to the number of pieces not groups.

Students' understanding of improper fractions weak.

Five of the seven students were confused when presented with notation of an improper fraction. Efforts to model the fraction 4/3 with a prepared diagram, resulted in some students trying to invert the fraction to 3/4. Students were able to model fractions where the numerator was equal to the denominator (equivalent to one whole). They were also able to model mixed fractions, such as 1 2/4. However, they appeared very confused when the numerator became larger than the denominator.

I suspect that most students' experience with fractions had been modeling proper or mixed fractions with diagrams of area models, like those found in most textbooks. If this was indeed the case, then most students had probably developed a limited understanding of the numerator as being the parts on the diagram that you color in. That is, the number of parts you color in (the numerator) was never more than the number of parts that were there (the denominator). Since I felt that "coloring in" or "eating" did not inspire a mental image of getting more pieces, I decided to encourage students in instructional activities to consider that the numerator "numerates" or counts the parts. It was reasoned that the students' experiences with counting-on would encourage them to count-on more partitions once the numerator became larger than the denominator.

Students' Understanding of the Numerator: Post-unit interview results.

Numerator definition included "counts" occasionally.

Most students continued to define the numerator what you color in, take away, or eat. However, two students did mention the "counts" aspect of the numerator. Given that only two students mentioned "counting" as a definition of the numerator, and no one applied this understanding when they were unsure how to model improper fractions with sets, I concluded that students had not adopted this definition for the numerator as I had intended they would.

Modeling improper fractions shows some improvement.

All students except one were able to model 6/3 with fraction circles. However, this success with the area models did not transfer to the diagram of the linear model. Three of the seven students were able to model the fraction (6/3) on the diagram, however success was not immediate. Two students initially modeled 12/6 but then changed their model to reflect 6/3. I wondered whether students were once again seeing the larger number (6) as the partitions, regardless of its position. Three students could not resolve the problem of modeling 6/3 with the diagram with the linear model provided.

This inability to model the improper fraction with the partitioned linear model was contrasted with students' abilities to model such fractions with the physical referents of Cuisennaire rods. I wondered whether the freedom to partition the rods as they saw fit, combined with the active manipulation of the materials was responsible for the students' success with this model. Armstrong and Larson (1995) cite Kieren as suggesting that partitioning experiences "... may be as important to the development of rational number concepts as counting experiences are to the development of whole number concepts" (p.17). Perhaps it was the partitions on diagrams of the linear models that caused students difficulty. A study that examined student success with modeling improper fractions on diagrams of linear and area models with and without partitions might shed some light into this problem.

Five students could not model 6/3 with a set model without my intervention. They were able to partition the set correctly but became confused with representing the numerator greater than the number of groups. It required me to suggest that the numerator counts the groups before two students offered that they needed to get more blocks. Once "more blocks" was acknowledged as being "allowed" students were able to solve the problem. I had to acknowledge here that students did not consider the numerator as counting the partitions. They had not developed this concept as I had hoped they would.

Pedagogical implications.

The fact that students did not adopt the "counting" definition of the numerator that I advocated during instruction, and continued to speak of "coloring in" or "eating" pieces leads me to conclude that it is very difficult to replace prior constructs with new ones. Indeed several researchers had commented on this same phenomenon (Hiebert, 1988; Wearne & Hiebert, 1988; Mack, 1990; Markovits & Sowder, 1994). I recommend, therefore that when students are first introduced to fractions, the numerator be defined as the term that counts the partitions. Since children already have experience "counting on", I think they would readily understand "counting on" more partitions or groups when the numerator is larger than the denominator if this was stressed in their early work with fractions. This would serve as an interesting basis for a new study.

The mixed success in the post-unit interviews with modeling improper fractions with different models points to the fact that children should, in their earliest work with fractions, model proper and improper fractions so that they do not distinguish between them. I propose that teachers have students model fractions less than one whole and then count up with the number of partitions until the whole is reached and then keep on going. Students would come to learn that the numerator can be larger than the denominator and would understand how to model this situation. In this way, proper, whole, and improper, and mixed fractions would all be viewed as a continuum of the same concept, not as different entities.

I felt that the instructional activities I provided were effective in helping students understand the numerator and improper fractions. However, I felt that more time and experience modeling proper and improper fractions with a variety of different models and diagrams was needed. Unfortunately, this extra time was not available at the grade 8 level. If students are to thoroughly understand these concepts, they need to begin this work in the early intermediate grades and continue practicing them every year until the late intermediate grades. Teachers also need to realize that a variety of models would need to be utilized in order for the concept of fractions- proper, improper, and mixed- to be generalized.

Students' Abilities to Model Fractions with Different Referents: Pre-unit interview results.

Area models dominate thinking.

Students' descriptions of fractions suggested that diagrams of area models were the overwhelming mental referent for students. All students opted to model a fraction with an area model. Linear and set models were never used voluntarily by students. Using an area model as mental referent limited students' understanding of fractions, however. Students modeled only congruent partitions and were not sure it non-congruence was "allowed". Four of the seven students were certain that equal partitions were necessary with models of fractions. However, when asked to work with set models, four students were not certain that the groups needed to be equal. In fact, five students were not certain that the groups were relevant at all with the set models

Improper fractions were difficult for students to model.

Students were able to model fractions that were less than one whole with area, linear, and set models as long as the number of partitions or objects in the set matched the denominator of the fraction. However, when the denominator was different from the number of partitions or objects, all of the students experienced some difficulty modeling the fractions- four of the students were unable to resolve the difficulty without my prompting. Compound set models proved the most difficult for students.

Modeling fractions greater than one whole also proved confusing to all of the students, especially when provided with a diagram with the partitions already indicated. Only two students were able to independently model fractions greater than one whole. The rest all required prompting on my part, even with a fraction as simple as 3/2. All students could model fractions equivalent to one whole with diagrams and manipulatives of area and linear models, however, they did not know how to move beyond that point. Students were unable to model fractions when presented with a set of blocks. No one confidently partitioned the set correctly.

Students' Abilities to Model Fractions with Different Referents: Post-unit interview results.

Area models continued to dominate thinking.

In the final interviews, it was obvious that area models continued to be an overwhelming mental referent for all students. All students continued to define the denominator and numerator with descriptions consistent with area models and all students continued to model fractions only with area models, even though all models were available to the students.

Ability to work with one model didn't readily transfer to another.

All students except one were able to model proper and improper fractions with area models. However, this ability did not transfer automatically to other concrete and visual referents. For example, one student who could model and explain 6/3 with the fraction circles could not transfer this understanding to a diagram of a linear model with the partition lines already provided. Similarly, another student could not transfer her understanding of modeling 6/3 with the fraction circles to the set models, even though she had partitioned the set model correctly. In fact, this student completely abandoned the meaning she had established for the numerator and denominator when confronted with the conundrum of modeling the improper fraction with the set model. One student who was not able to model an improper fraction with the diagram of a linear model, was able to model the fraction (with some guidance) with a concrete linear model. He was then able to transfer this understanding back to the diagram of the same model. Analysis of the results with the different concrete models and diagrams suggested that most students could model proper and improper fractions with concrete area models and linear models which allowed the students to determine their own partitions. The presence of established partitions seemed to have negative effect on the students' abilities to model fractions. More research needs to be done to determine what role partitioning plays in students' understanding of the part-whole concept of fractions.

Pedagogical implications.

The difference in ability of the individual to model fractions with the different physical and visual referents and the lack of transfer between the different models raises some interesting points. First, it emphasizes that all models must be presented to students during the connecting phase. Students must be presented with a variety of area models, linear models, and set models so that they can develop a generalized understanding of the part-whole concept of fractions. Hiebert (1988) warns us that using any one model may limit a student's understanding of fractions to that model. The results of my study and others' (Peck and Jencks, 1981; Armstrong & Larson, 1995) suggest that most students' primary mental referent for fractions is the area model. I felt that this model limited students' understanding of the part-whole concept of fractions. In particular, I felt it limited their understanding of partitioning to congruent pieces and I suspected that it allowed students to focus on the pieces when comparing fractions versus the relationship between the pieces and the whole.

Reviewing my lessons, I realized that we had spent most of our time with fraction circles and pattern blocks (area) and Cuisennaire rods (linear) as models of fractions. Far less time was given to other diagrams and set models. This was due in part to time constraints and partly because I failed to recognize that a concrete model was not the same in the students' minds as a diagram. I did not recognize that the partitions on the diagrams would restrict students and cause them difficulty whereas the concrete models allowed students freedom to partition the area or length as they liked.

This is the second point that needs to be considered. Students need experience with a variety of concrete models, but they also need experience with a variety of diagrams. "...Physical aids are just one component in the development of representational systems and ...other modes of representation-verbal, pictorial, and symbolic- also play a role in the acquisition and use of concepts (Behr et al., 1984, p.325). As I have discovered for myself, students do not readily transfer their part-whole understanding between different models, diagrams, and notation. The teacher must carefully plan lesson sequences and experiences for students to purposefully encourage the transfer of understanding from one mode to another. In the future, I will ensure that not only will I use manipulatives in my lessons, but also I will consider the variety of manipulatives, diagrams, and verbal descriptions that I present to students. And I will strive to plan lessons that are structured so students can make links between the different models. In this manner, I hope to create "...experts [who] view all modes of presentation of information as equivalent..." (Bright, Behr, Post, Wachsmuth, 1988, p. 230). Further research needs to be conducted to determine what, if any, aspects of models and diagrams encourage or limit transfer of understanding of fractions between the different modes of presentation. Also, a study that determines the optimum sequence in which models and diagrams should be presented in order to encourage transfer of the part-whole concept between the different modes of presentation needs to be conducted.

Although I recognize that more time was needed to be spent on the other models and diagrams and comparing the similarities and differences in each of the models, I did not feel that the grade 8 curriculum allowed time for this exploration. As such, I do not feel that I could have given it any more time than I already did. I believe that if students are to develop a generalized understanding of the part-whole concept with fractions, then the extensive work connecting the physical referents with the symbolic notation needs to be done in the earlier grades (grades 3-6).

I suspect that students are not getting sufficient experience during the "connecting" phase (and therefore have weak fraction sense) for several reasons. First, I suspect that the referents that students are provided with are limited primarily to area models. Several researchers have suggested this is indeed the case (Peck and Jencks, 1981; Armstrong & Larson, 1995). If this is the case then students' understanding of the part-whole concept will not be generalized, but rather will be limited to this model. A study that examines the models that teachers use in the classroom needs to conducted to see if indeed this is the case.

Second, I suspect that time spent of connecting referents to written notation is limited. I believe, that either students' experience with fractions is limited before they are moved on to other mathematical topics or that their experience connecting referents to fraction notation is limited before they are moved into "routinizing" activities. I suspect that early intermediate teachers feel they are doing a service to students by introducing the algorithms of fractions early to students to give them a "head start" for their work in later grades. Unfortunately, research seems to be telling us that rushing students into routines results in students who mechanistically apply, often incorrectly, the algorithms they have been shown. In other words, rushing the early stages of "connecting" to get to the "routinizing" stage results in students with poor fraction sense. A study of teacher beliefs and attitudes about fraction instruction is required to determine whether students are actually receiving ample time in the "connecting" phase or if indeed they are being hurried into the "routinizing" stage.

The third suspicion I have is that there is a mistaken belief by some intermediate teachers that fractions are no longer stressed in the intermediate grades since operations with fractions have been moved to the grade 8 year in the BC Mathematics Instructional Resource Package (IRP). As such, I suspect that fractions are not being given the same attention they used to in order that teachers can teach the other topics more thoroughly. A study of teacher beliefs would reveal whether or not this was the case.

I believe that all three of my suspicions are correct and all three are valid explanations that partly explain why students lack fraction sense. I suggest that a research project that explores the "implemented curriculum" and teacher beliefs and attitudes about fraction instruction in grades three through seven would provide valuable insight into the actual amount of time students have to connect referents to abstract symbols in the "connecting phase" and also reveal the scope of the referents presented to students.

Subsection 1.2 Sense of Orderliness of Fractions

In my research I was interested in the understanding students had for the size of the fractions and in their ability to order them. I was most interested in the strategies that students used to order and compare fractions. It was my suspicion that the "like-denominator" method would be the most common strategy cited by students for comparing fractions.

Students' Ordering and Comparing Strategies: Pre-unit interview results.

"Inverse relationship" not applied to ordering unit fractions.

As has already been discussed in the section "Students' understanding of 'inverse' relationship weak" (p.152), most students did not order the set of basic unit fractions with confidence. These students did not demonstrate a strong grasp of the inverse relationship between the number of partitions of the whole (denominator) and the size of the partitions. Only three students, in fact, used their understanding of this inverse relationship to order the fractions. The others used inappropriate whole number logic or manipulatives and diagrams to order the fractions. There was also some evidence of reference to 1/2 as a benchmark for ordering the other fractions.

Other ordering strategies employed by students.

The students demonstrated that they had essentially no effective strategies for ordering fractions. Only one student suggested using the like-denominator method to compare the fractions 18/19 and 3/4. Four of the students used the "range" strategy and incorrectly stated that both the fractions were the same. Surprisingly, these students could suggest no other methods to compare the fractions. Even though they demonstrated in other problems an understanding, albeit a faulty one, of the like-denominator method, they did not consider it as a strategy to compare fractions. This was contrary to my expectations. I would have expected students to say that the denominators had to be the same (like-denominators) before they could be compared since this is the algorithm most commonly taught to students. Either these students did not have a lot of exposure to this method or they did not retain it or see it as applicable to the problem at hand.

Students' Ordering and Comparing Strategies: Post-unit interview results.

Inconsistent attention to "inverse" relationship.

In the post-unit interview there was an improvement in each students' ability to explain the inverse relationship between the number of partitions and the size of those partitions. However, this understanding was only applied by four of the five students to the comparison of 3/4 and 3/5 and by two students to the comparison of 5/6 and 5/7. The remainder of the comparisons was based on either the use of benchmarks or more often, the range strategy.

Although I felt that all the students had a better understanding of the compensatory relationship between the denominator and fraction size, I did not feel that this understanding was deep. Students seemed to be able to apply this thinking to a direct comparison of fractions with the same numerator and a difference of one between the denominators. Even then, they did not do so consistently. When an indirect comparison of fractions was involved (comparing missing pieces), students abandoned their understanding of the inverse relationship of denominators and compared the denominators as discrete pieces of the same size. This focus on pieces did not appear to happen when an easy, direct comparison of the fractions was possible. Further research is needed to determine what aspects of the notation encourage students to focus on discrete pieces. For example, would students focus on discrete pieces when directly comparing fractions whose denominators were greatly different but whose numerators were the same (e.g. 4/5 and 4/20)?

Benchmark and "range" strategies favored.

There was significant growth in the use of benchmark strategies amongst most of the students. A few different benchmark strategies appeared in the final interview that were not present in the first one. Unfortunately, the use of the "range" strategy also appeared and was used extensively by two of the students and occasionally by another three. The good news was that either the benchmark strategy seemed to "win out" in students' minds when the fractions were close and easy to compare to known benchmarks or students would quickly abandon the range strategy if they judged their answers incorrect by referencing them to benchmarks.

The most commonly applied benchmark strategy was comparing two or more fractions to 1/2 or 1. This strategy was most consistently and correctly applied when the fractions were "more spread out" according to the students.

The students who experienced the most success ordering the sets of fractions (task # 3, p. 62) generally avoided using the range strategy. These students would order the fractions they could by using 1/2 and 1 as benchmarks. The fractions that they could not order were usually left out. Only two students recommended using the like-denominator strategy to resolve the order of the fractions that could not be ordered easily with the benchmark strategy. The others could suggest no solution. As I mentioned previously, I was surprised by how few students suggested the like-denominator method of comparing fractions given how prevalent it is in math textbooks and most classrooms.

The two students who had the most difficulty ordering the sets of fractions used the benchmark strategy and range strategy, even though they had proven the range strategy to be less than 100% accurate and valid. There did not seem to be any pattern in the fractions that would predict which strategy these students would choose. It seemed that if certain fractions that were easy to compare to benchmarks "jumped out" at them, then they would use a benchmark approach to order them. Otherwise, the "range" strategy was employed.

A strategy new to the final interview was "missing less therefore bigger strategy". This strategy was introduced to students during instruction in order to compare fractions within the same domain (either over or under half or one). Most of the students tried to apply this strategy however, they were often unsuccessful with it. They were able to follow all the steps except one. When they had to compare the missing addends of the fractions (to reach one whole) students did not compare fractional parts that were relative to one whole but spoke of pieces as if they were discrete units. For example, when students compared 2/3 and 6/7 they said both were missing one piece and were therefore the same. For students, determining whether 1/3 or 1/7 was smaller based on the inverse relationship between the size of the denominator and the fraction size, and then relating this information to the original fractions to determine which fraction was larger based on their smaller fractional addends was too much to handle intellectually. The students who did try to consider the relative size of the fractional addends, could not relate this information back properly to the original fraction pair being compared. When asked to compare 3/4 and 18/19, Jeremy claimed that 3/4 was more because 1/4 was a bigger piece missing and so fourths were bigger so 3/4 was more. He could not mentally coordinate the inverse relationship between the relative sizes of the "missing parts" (themselves an inverse relationship) with the relative sizes of the pair of fractions. This relationship, however, was easy for students once they drew a picture of it or made a model of it (or a similar pair) with fraction circles.

Pedagogical Implications.

Comparing and ordering activities in class emphasized the use of benchmarks and the inverse relationship between the denominator size and fraction size as a means to determine the relative size of fractions. When fractions were both under or over half or one, I encouraged students to use a "missing less therefore bigger" approach.

The first lessons I presented had students use manipulatives to find fractions equivalent to half and one and then to compare fractions less than and greater than half and one to these benchmarks. I believe that students' first experiences determining the relative size of fractions must include manipulatives so that they can develop mental referents in their minds. Unfortunately, I don't believe that some students in my class had developed strong mental referents since they could not handle the "missing less therefore bigger" type problems. During instruction, most students were unreceptive to working with the manipulatives to compare fractions. They said they could just look at the notation and state which fraction was more, especially compared to half and one. Classroom management became an issue so I moved them into comparing fractions in notational form. Unfortunately, I believe this had the negative effects of encouraging students to look at the numerator independently from the denominator and use the range strategy.

When I questioned students about their understanding of fractions equivalent to half, all explanations focused on the numerator relative to the denominator. That is, students determined that a fraction was equal to half if the numerator was half the denominator (3/6 is equivalent to 1/2 because 3 is half of 6). This thinking concerned me because it seemed to be based on a whole number schema of multiplication and division rules versus a rational number schema of three-sixths is half of six-sixths. Although I recognize that the students' approach was an efficient way to make rapid comparisons to benchmark fractions, I was concerned that this thinking encouraged students to view the numerator and denominator as separate terms.

I suspect that having some students try to mentally manage the "missing less therefore bigger" strategy without the benefit of extensive modeling with manipulatives led to the range strategy being engrained in their thinking. Without the benefit of manipulatives, I suspect that these students couldn't keep the relative size of the missing pieces in mind, so they simply thought of the missing pieces as discrete units and determined that the fraction that was missing fewer pieces was larger.

The other unfortunate result of the instruction with the notation only, was that the range strategy worked more often than it didn't given the examples that were on the worksheets. Students were getting good results on the tests and assignments so I did not realize they were using this faulty approach. I thought they were using the strategies that I had taught them. Unfortunately, that didn't appear to be the case with some students.

The students in my research who did not use the range strategy did seem to manage the mental complexity of coordinating the numerator and denominator into a number with a specific value. They also seemed to manage the mental demands of the "missing less therefore bigger" strategy. Perhaps, they did not require extensive work with the manipulatives whereas the other students did.

Although I suspect my instruction may have had some negative side effects, I do feel there were some positive outcomes. In particular, most students used the benchmarks of 1/2 and 1 properly to order fractions. This behavior was not demonstrated nor suggested in the first interviews so I attribute the change to the work we did in class. Also, even though some students used the range strategy they could also use the benchmark strategies to order fractions and recognize errors in their thinking with the range strategy.

In the future I will not introduce the "missing less therefore bigger" strategy without manipulatives. Only when students demonstrate a thorough understanding of the concept will I allow them to work with symbols alone. Student communication will be a large part of this instruction. Students will have to repeatedly explain their thinking to ensure they are not comparing "pieces" but fractional pieces. To this end, I will insist that students compare the missing addends by using the fractional name, not the word "pieces". For example, to compare 3/4 and 8/9, students would be required to say "one-fourth and one-ninth is missing" saying both are missing "one piece" would be taboo!

In the future I would also try to discourage the use of the range strategy by carefully structuring the problems on worksheets such that use of the range strategy would lead to incorrect solutions. I would also provide examples of faulty thinking with the range strategy and have students explain to me why they know the solution is incorrect. For example, I would provide the solution of 2/4 and 1/3 are the same because they are both missing two pieces. Students would explain to me why this solution and thinking is incorrect.

Although I recognize that I did not provide enough opportunity for some students to work with the manipulatives in my class, I also recognize that most of my students had developed strategies to compare fraction notation to 1/2 and 1 and did not want to work with the manipulatives. I concluded from my experience, therefore, that it is difficult to go backwards in time. That is, it is difficult to take students back to working with manipulatives when they have already been working with the notation. Indeed, Hiebert (1988) stated that "...if these initial processes are not engaged when children first encounter a particular symbol system, it may be more difficult to go back and engage them later" (p.349). As such, I recommend that teachers in the intermediate grades begin students on manipulative activities that encourage them to develop a generalized understanding of the absolute magnitude of fractions and then the relative magnitude of fractions. Finally, students should compare fraction notation and explain their thinking in discussion and math journals. If there is evidence of the range strategy or lack of attendance to the absolute magnitude of the fractions, then students should be moved back to manipulatives to strengthen their mental referents. I believe these concepts must be established in sequence in the intermediate years. By grade 7 they must be firmly understood by all students or else I fear they will remain undeveloped since there is not time in the middle school math curriculum to address these topics adequately.

I have assumed here that developing a generalized understanding of the absolute and relative magnitude of fractions would prevent the use of the range strategy to compare fractions. Research needs to be done to determine if this assumption is correct.

Self Reflection

Participating in the process of research has given me new appreciation for the position that research has in educational practice. As a teacher, evaluation of student learning is usually based on fleeting conversations and pencil and paper work. The teacher usually does not get an in-depth view in to her student's thoughts. By taking the role of teacher-researcher I have been given that opportunity to look closer at my student's thoughts which have informed my instructional practice. In particular, I have a new appreciation for the impact that manipulatives have on student learning. I recognize the need for the connecting phase to be thoroughly engaged before students are asked to manipulate symbols. I recognize that rushing this stage is futile. I also have come to understand that not all manipulatives are created equal. The features of any one referent limit "Meaning". Therefore, students must be provided with many different referents, physical and pictorial, in order to develop a generalized understanding of the symbol system they are studying. I also have come to realize that students do not readily transfer between different physical and pictorial forms and I must keep this in mind and help them to make those transfers. Communication of ideas in the math classroom has taken on more importance in my mind. In my classroom, I probably would not have known about the mis-learning that occurred when students compared fractions using the "range" strategy. On paper, their scores would have been fine and I would have thought that students had learned what I had intended. However, communication of thinking highlighted this was not the case. As such, it is important that communication be an integral part of mathematics learning to avoid inappropriate learning.

I also have new appreciation for the complexity of the part-whole concept of fractions. I recognize that much effort has to go into helping students generalize this concept. I feel that the amount of time and effort given to this phase is completely inadequate. Students must be given as much time and experience (or more) developing an understanding of the magnitude of fractions as they are given with whole numbers if students are to develop any sense of the magnitude of fractions at all. I understand that activities that encourage fraction sense, such as ordering fractions by comparison to known benchmarks, must be practiced and mastered before procedures and algorithms are introduced. If students are to be successful with fractions, they must develop as solid an understanding for these numbers as they have for whole numbers. If not, I fear that the vast majority of students will be confined to memorizing procedures with fractions that hold little or no meaning for them.

The most important concept I have learned from my role as teacher-researcher is that students do strive to make sense of fractions and numbers in general, even though it may not appear that they are successful in doing so. I have come to appreciate that this development of understanding, however, does not progress in a tidy, linear fashion, but rather is erratic in its path. I realize that as students encounter new mathematical situations, their understanding of fractions may regress or be abandoned altogether. As the teacher, I must be ever vigilant to this situation and not assume that understanding of fractions will be transferred from one setting to another. I am convinced more than ever that strong student-teacher and student-student communication is crucial if students are to properly develop fraction sense.

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APPENDIX A
PRE AND POST-UNIT INTERVIEW MATERIALS

Rational Number Sense Interview Questions

Tige Woodward

The purpose of my study is to examine the number sense strategies that selected grade eight students use when they are trying to solve various problems that relate to the size, order, and equivalence of proper fractions.

Meaning of Fractions

Show: 4/6 (have manipulatives, paper, etc.)

Ask: " What does this mean? Can you tell me or show me?"

" What does the '6' tell you?"

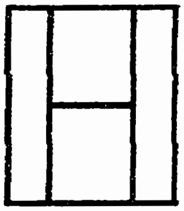
"What does the '4' tell you?"

Ask: Using the whole geoboard as the whole please separate it into fourths in 4 different ways.

Ask: "Please explain how you know that these are fourths."

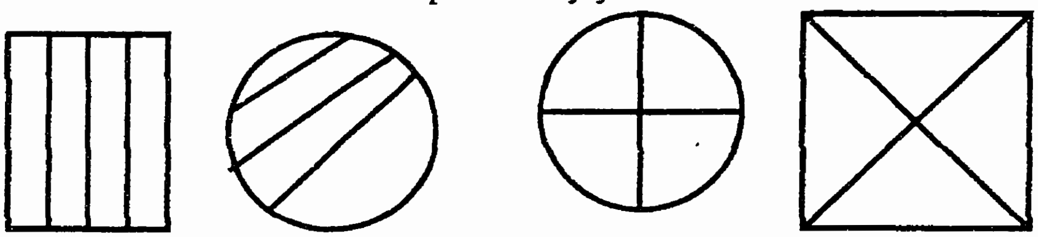
"Do the fourths all have to be the same shape?"

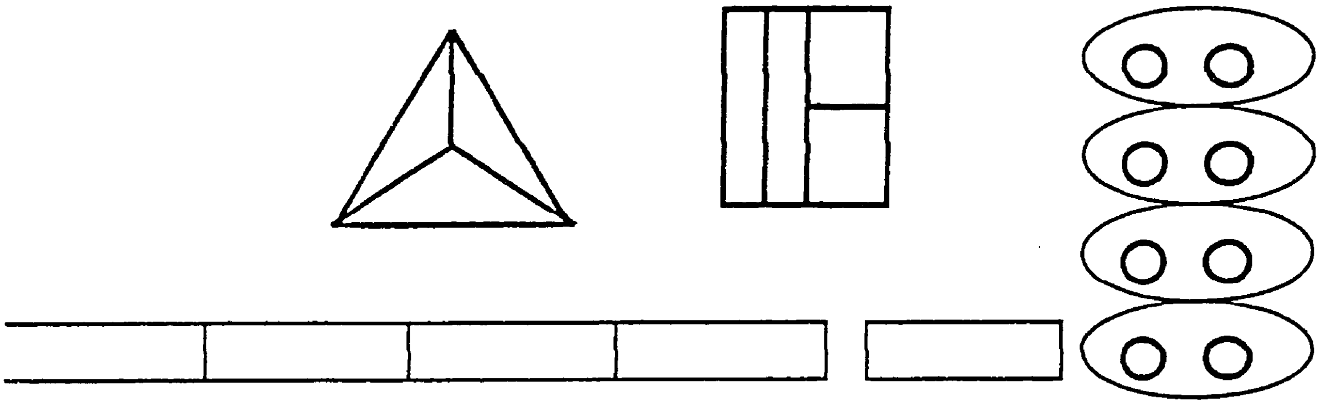
Show



Ask: "Are these fourths?" Why do you think so?"

Ask: " Look at the examples below, which ones are fourths and which ones aren't?" Please explain why you think so."





Show: 

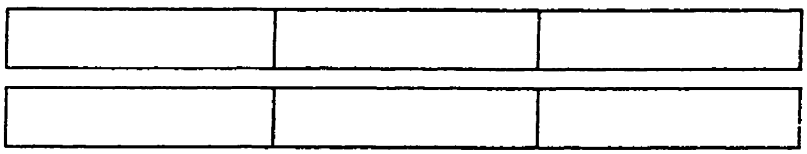
"Can you tell me what part of the set is shaded?"

"How did you figure that out?"

" If I told you that 4/6 is shaded, what would you say?"

" Please explain how you know?"

Ask: "Please shade in 2/3 on each of the diagrams"



Show: 

"Can you tell me what part of the bar is shaded?"

"Please tell/show me how you figured that out?

"Could there be another fraction for the part that is shaded?"

"What would you say if I told you that 1 1/2 is shaded?"

Show: 

Ask: "The first example is 4/4 and the second is 4/3. Please show me what the 'whole' or 'unit' is."

Ordering Unit Fractions

Show: fraction circles (1/2, 1/3, 1/4, 1/6), paper, pens

Ask: "What part of a circle are these?"

"Please explain how you figured that out?"

"Please write the names of each of these fraction pieces in symbolic notation."

"Please explain why you wrote 1/x for this piece?"

"Why did you think to write x as the denominator for this piece? Why did you write 1?"

"Please order these written symbols from smallest to biggest? Please explain your thinking out loud."

"What made you decide that 1/x is the smallest? Biggest?"

"Please order each of these fraction pieces from smallest to biggest."

"Look at the order of the fraction pieces and what you wrote for the fraction symbols. Do they agree? Which one is correct? Can you explain your thinking?" What do you notice about the denominators?"

Ordering Non-Unit Fractions

" Which fraction is larger 18/19 or 3/4?" Please explain your thinking."

Equivalence of Fractions to Common Benchmark

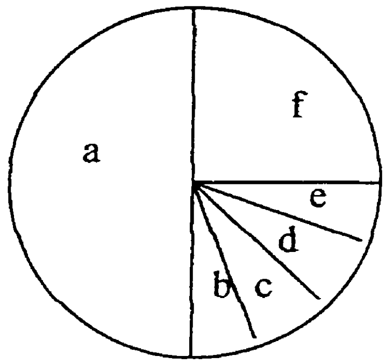
Ask: "What does it mean when two fractions are equivalent?" Please show me with the fraction pieces."

"Is 5/10 equivalent to 1/2? Please explain your thinking?"

"How would you use the blocks to check your thinking/ figure out the answer?" Please show me."

"Is 7/15 equivalent to 1/2? How do you know?"

Show:



Ask: "f" is what fraction of the whole?

"b,c,d,and e together is what fraction of the whole?" Can it have a different fraction name?

Using Benchmarks to Determine Size for Order

Ask: "Is 7/15 smaller than, equivalent to, or greater than 1/2? Please explain your thinking."

" Is 6/10 smaller than, equivalent to, or greater than 1/2? Please explain your thinking."

"Thinking about your previous answers, tell me which fraction is larger 6/10 or 7/15? Please explain your thinking."

Size of Fraction: Changes to Numerator/ Denominator

Ask: "When I have a fraction (2/5), and I change the '2' to a '3', what happens to the size of my fraction: decreases, stays the same, increases? Please explain your thinking." Can you show me with blocks/ draw?

"When I have the fraction (2/5) and I change the '5' to a '6', what happens to the size of my fraction: Decr, same, incr? Explain.

"When I have the fraction (2/5) and I change the '5' to a '4', what happens to the size of the fraction?... Explain"

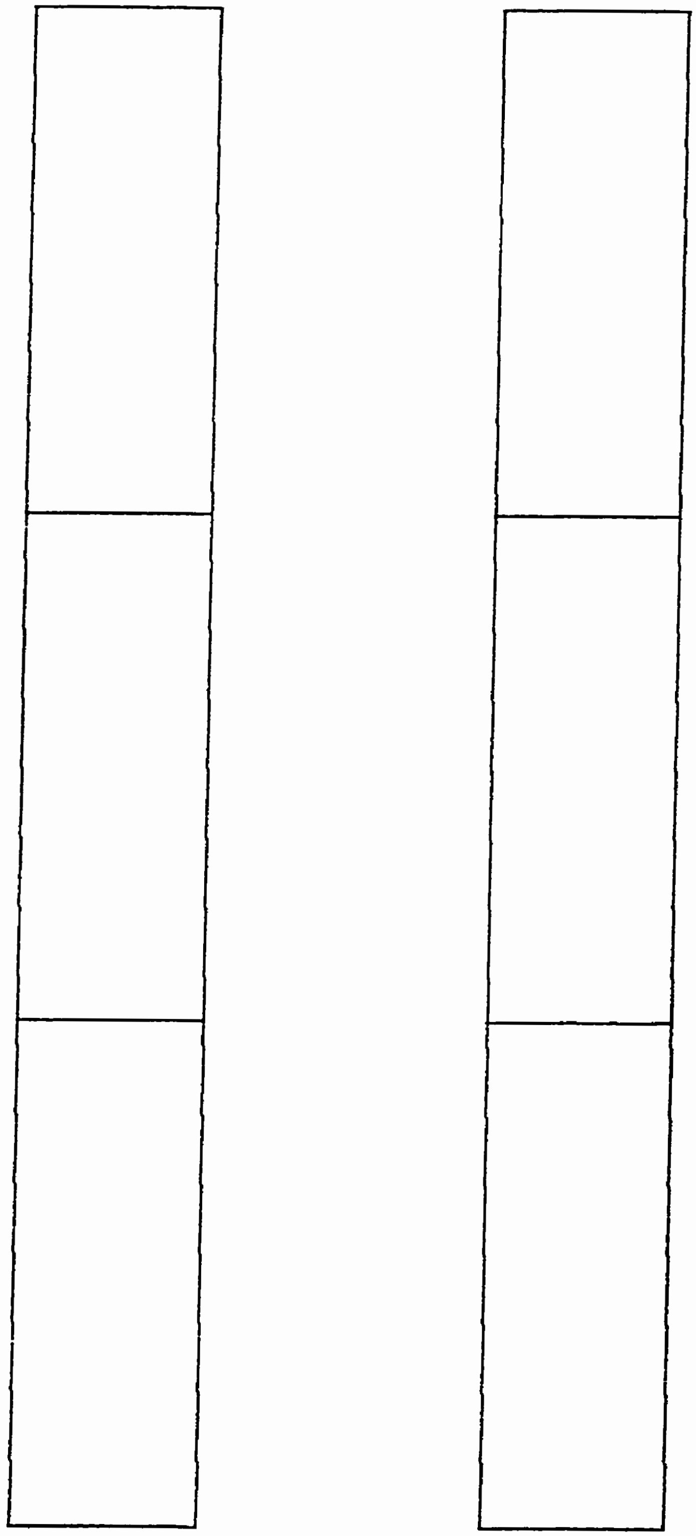
When I have the fraction (2/5) and I change the '2' to a '3' and the '5' to a '6', what happens to the size of my fraction?" Ditto

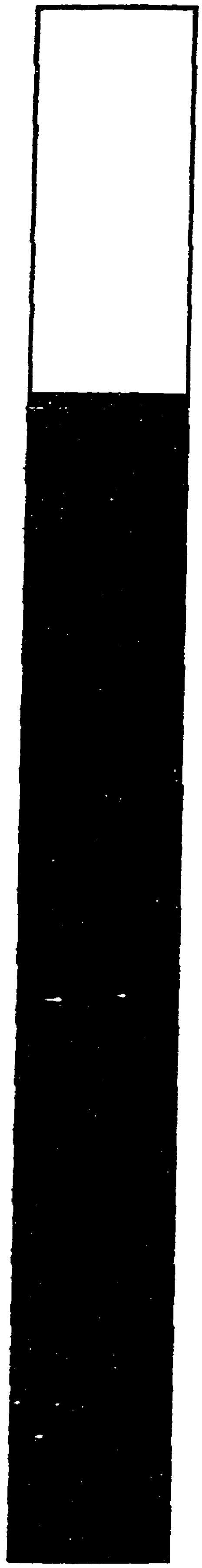
Rational Number Sense and Addition of Fractions

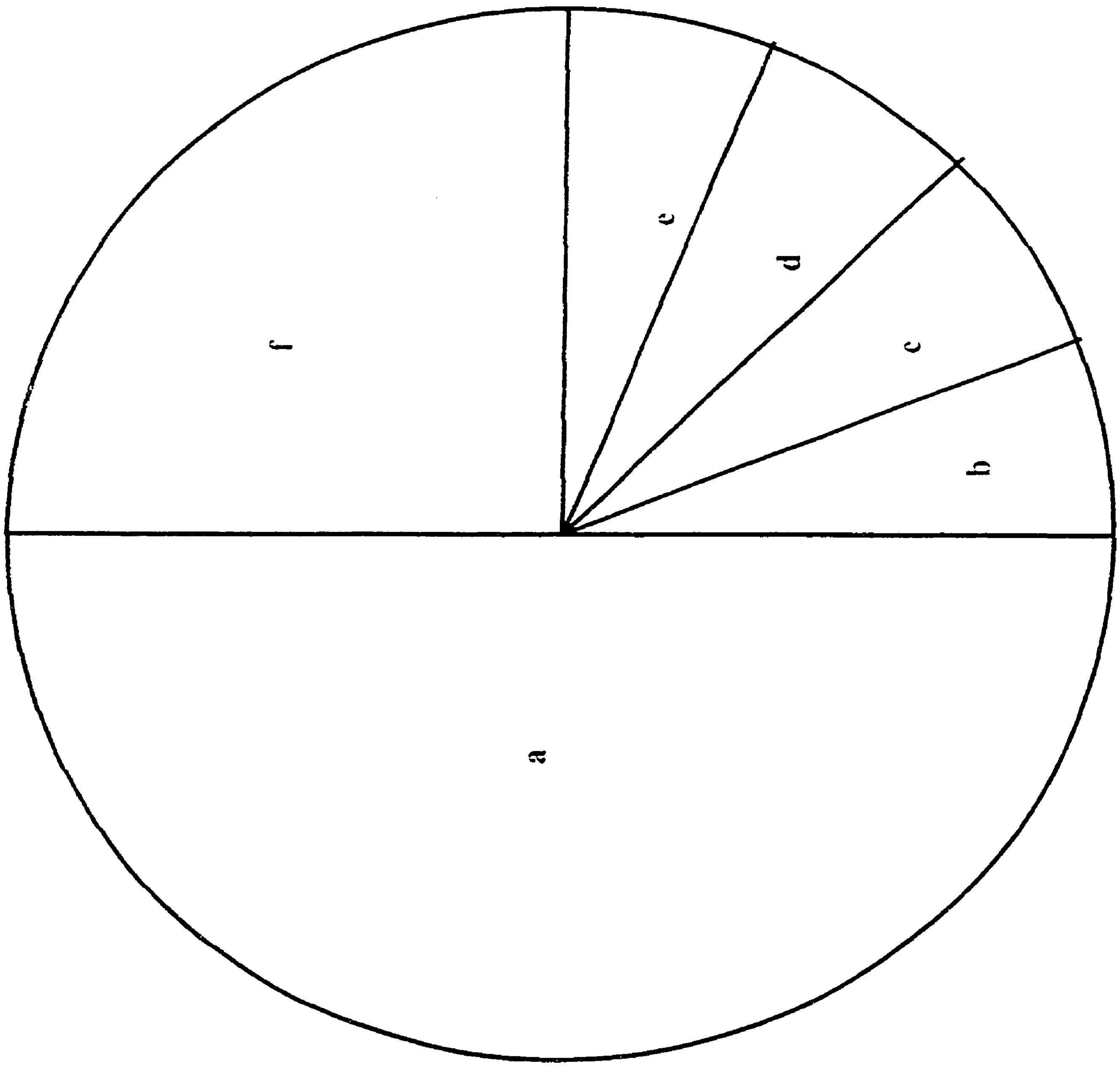
Ask: "Please tell me if 6/10 + 7/12 is smaller or larger than one? 1/2?

" What if I told you that 6/10 is larger that 1/2 and 7/12 is also larger than 1/2. Would the answer to 6/10 + 7/12 be smaller or larger than one? 1/2? Please explain your thinking."

"Please tell me if 5/11 + 7/19 is smaller than or larger than one? 1/2?



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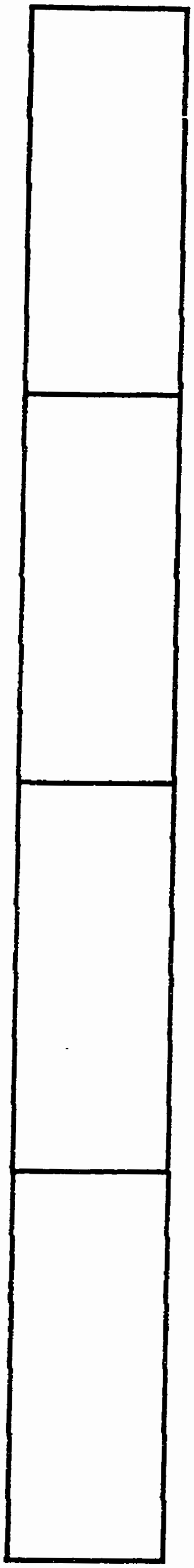


Fractions Unit- Final Interview (Woodward) **INTERVIEW WITH:** ______________

DATE: ______________

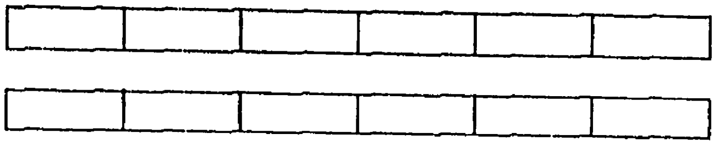
Section One- Meaning of Fraction Notation

RESPONSE:

Given 3/6: What do you think/ say when you see this fraction?
What model can you make for me?
What if I gave you 24 blocks?
What does this number tell you? (6)
What does this number tell you? (3)

Given 6/3: What do you think/ say when you see this fraction?
What model can you make for me?
What if I give you 12 blocks?
What does this number tell you?(3)
What does this number tell you?(6)

Given:



How can one of these models be 3/6 and the other be 6/3?

How can one of these models be 1/2? How can it be 9/12?

Given: Orange is one whole, what part is 2 reds?
Yellow is one whole, what part is 1 light green?
Light green is one whole, what part is one yellow?

"Tell me about the written fraction- how big will the numerator be compared to the denominator?

Given: Brown is 4/5, which rod is the whole?
Black is 1 2/5, which rod is the whole?

"Tell me about the mystery rod, will it be bigger or smaller than the fractional part? How do you know?

RESPONSE:

Section Two- Fraction Number Sense

RESPONSE

Given: 3/4 and 3/5

"If I give you these two fractions, how do you go about deciding which one is the larger number?

Given: 4/5 and 6/5

"If I give you these two fractions, how do go about deciding which one is the larger number?

Given: 2/3 and 6/7

"If I give you these two fractions, how do go about deciding which one is the larger number?

Given: 3/4 and 5/12

"If I give you these two fractions, how do go about deciding which one is the larger number?

Given: 3/5 and 5/7

"If I give you these two fractions, how do go about deciding which one is the larger number?

Given: 3/5 and 6/9

"If I give you these two fractions, how do go about deciding which one is the larger number?

Given: 4/10 and 5/9

"If I give you these two fractions, how do go about deciding which one is the larger number?

Please order the following sets of numbers and explain how you are deciding (numbers on index cards)

Given: 4/8 5/9 5/12 2/5 10/8

6/6 3/1 11/7 9/18 4/21

Given: 5/7 and 7/7 (on cards) (new)

"Can you name a fraction that is between these two fractions?"

Given: 5/7 and 6/7

"Can you name a fraction that is between these two fractions?"

Given: 5/6 and 5/7

"Can you name a fraction that is between these two fractions?"

RESPONSE:

Given: 1/2 and 1/3

"Can you name a fraction that is between these two fractions?"

Given: 5/6 and 5/7

"Can you think of a fraction that is smaller than these two?"

Given: 2/3 and 3/5

"Can you name a fraction that is between these two fractions?"

Given: 2/3 and 3/4

"Can you name a fraction that is between these two fractions?"

Given: a/b

" If I told you "a" was_______ "b" what can you tell me about the size of the fraction compared to 1 whole?"

1) less than
2) equal to
3) greater than

"If I increased the number "a" what would happen to the size of my fraction?"
"If I increased the number "b" what would happen to the size of my fraction?"

RESPONSE:

" If I decreased the number "b" what would happen to the size of my fraction?"

Part Three- Operation Sense

" I am thinking of two fractions that add together and their sum is between 1/2 and 1. What can you tell me about these fractions?"

"I am thinking of two fractions that add together and their sum is between 0 and 1/2. What can you tell me about these fractions?

"I am thinking about two fractions and the difference of these two fractions is between 1/2 and 1, what can you tell me about these two fractions?"

Given: 2/3 x 5/4

"Will the product be greater than 5/4? Less than 5/4? Equal to 5/4? How do you know?

Given: 4/6 x 2
"Will the product be greater than 2? Less than 2? Equal to 2? How do you know?"
" Will the product be greater than 1? Less than 1? Equal to 1? How do you know?"

Given 5/3 x 3

"Will the product be greater than 3? Less than 3? Equal? How?

Response:

APPENDIX B
INSTRUCTIONAL UNIT MATERIALS

Compare and Test (Ordering)

Name: ________________
Date: ________________
Math Journal

Predict which fraction below is the larger of the two (assume the whole is the same size). Do NOT change to like denominators or cross multiply. Justify your prediction. Test your prediction with the fraction circles.

Fractions	Larger	Justification	Test
3/8 5/8			
2/3 3/3			
7/6 8/6			
4/6 4/8			
2/3 2/6 2/4			
5/4 5/3			
1/3 1/2 1/4			

Fraction	Larger	Justification	Test
2/3 2/2 2/6			
3/7 5/8			
5/4 7/8			
7/8 4/3			
6/6 5/4			
5/6 3/4			
2/3 3/4			
5/4 4/3			
2/6 3/8			

Converting Fractions

Name: ________________
Date: ________________
Math Work

Complete the questions below. Use your Cuisennaire rods and pattern blocks to help you.

Whole	Fractional Part	Symbol(s)
brown		1/4
dark green		2/3
dark green		3/2
	purple	1/3
	dark green	2/3
	yellow	5/4
dark green	yellow	
dark green	blue	
dark green	2 orange	
brown	2 blue	
pink	2 black	
light green		1 1/3 or 4/3
brown		2 1/2 or 5/2
	2 dark greens	1 5/7 or 12/7
	3 yellow	5/1 or 5
black	2 blue	
pink	2 blue	

Answer the questions in below in detail please!

1. Use the light green block as the whole and the brown as the fractional part.

a) Will this fraction be greater than one whole? Greater than two?

b) What are 3 different ways that you can express this fraction (2 mixed, 1 improper)?

c) What other fractional values could you give this fraction?

2. State whether the following fractions will be less than one, equal to one, or greater than one.

Fraction	Less than or Equal to or Greater than One
4/3	
6/1	
7/10	
5/2	
4/6	
2/3	

3. If I was to write an improper fraction for an amount greater than one, what can you tell me about the number on top (numerator) as compared to the number on the bottom (denominator)?

4. "a" and "b" represent whole numbers.
I) If a< b, then is the fraction less than, equal to, or greater than one whole? Explain.

ii) If a = b, then is the fraction less than, equal to, or greater than one whole? Explain.

iii) If a > b, then is the fraction less than, equal to, or greater than one whole? Explain.

5. Can you explain your method of converting the following mixed and improper fractions?

4/3 =

5/2=

7/3=

1 1/2=

2 2/4=

10/2=

6. Use a diagram to explain why 7/3 = 1 4/3 = 2 1/3

7. How can I know just by looking at a fraction if it is less than or greater than one?

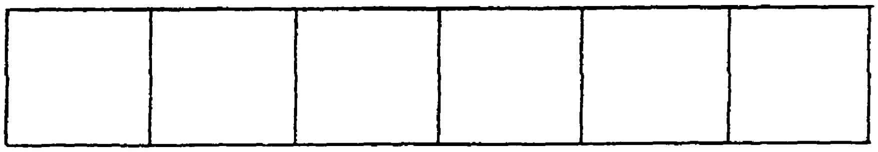
8. How can I know just by looking at a fraction if it is greater than two?

Equivalent Fractions: Using Models

Name: ________________
Date: ________________
Math Journal

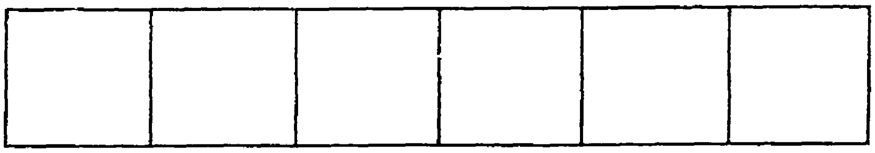
Use the models below to demonstrate equivalent fractions.

Show 4/6



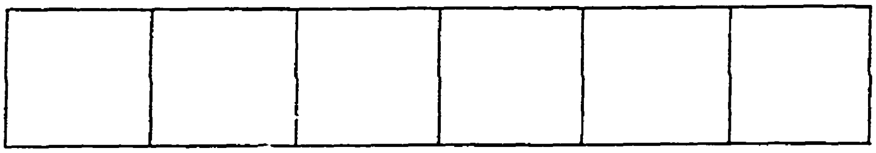
Show

4/6 x 2/2



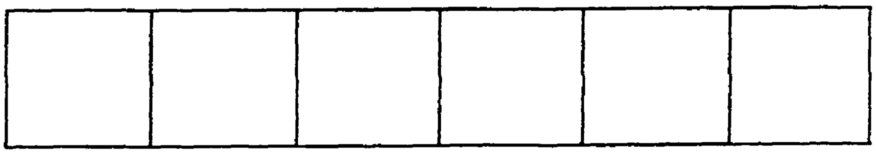
Show

4/6 x 4/4



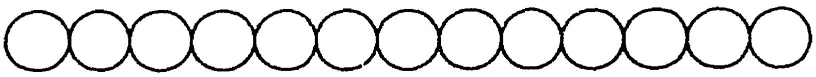
Show

4/6 -- 2/2



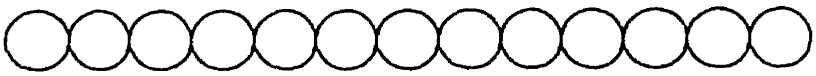
Show 8/12

Show
8/12 x 2/2

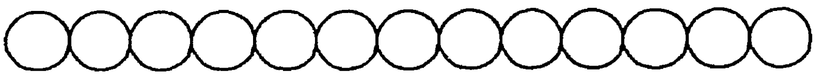


Show

8/12 ÷ 2/2



Show
8/12 ÷ 4/4



Answer the following questions in detail. Use labeled diagrams to support your answer.

1. When you find equivalent fractions, you multiply the numerator and denominator by the same number.

a) If I were to multiply the fraction a/b by 2/2, explain how many pieces I would cut each of the fractional parts into?

b) If I multiplied by 3/3, 4/4, 5/5 or n/n, how many pieces would I cut each of the fractional parts into?

c) Explain the relationship between the number you multiply the numerator and denominator with and the number of parts you cut the fractional pieces into?

2. When you multiply numerator and denominator with the same number (e.g. 2/3 x 2/2), you are actually multiplying by 1.

a) What happens to a number when you multiply it by 1?

b) If when I multiply a fraction by 1 (e.g. 2/3 x 2/2 = 4/6) how come it looks like a different number (e.g. 4/6)?

Equivalent Fractions: Representations

Name: ____________
Date: _____________
Math Journal

Materials: Grid paper
Fraction circles
Cuisennaire rods
Blocks

Use the equipment listed above to demonstrate the following fraction equivalencies. Record all your data.

1. On the grid paper, the square outline represents one whole. A fractional part has been shaded in. State what fractional names you could give this part. For each fractional name, indicate (by drawing on the grid paper) what equal sized piece you are considering the denominator. The first one is done for you.

2. Below areas of circles have been drawn for you. If one white circle represents the whole, what fractional names can you give for the areas. Explain your answers by using labeled diagrams.

3. Complete the chart below:

Whole	Fractional Part	Fractional Names
blue	dark green	
brown	pink	

brown orange

4. Use 24 blocks as the one whole. Represent the different fractions with different colors. Record your data using X's and O's. The first one has been done for you.

Fractional Part	Record	
1/2 and 1/2	x x x x x x x x x x x x o o o o o o o o o o o o	1/2 of 24 = 12
2/3 and 1/3		2/3 of 24 = 1/3 of 24 =
1/6 and 5/6		1/6 of 24 = 5/6 of 24 =
3/4 and 1/4		3/4 of 24 = 1/4 of 24 =
3/8 and 5/8		3/8 of 24 = 1/8 of 24 =
9/6		9/6 of 24 =
5/4		5/4 of 24 =

Journal: Ordering Fractions

1. Which fraction below has more fractional pieces? Explain how you know.

a/3 or a/8

3. Which fraction below has larger fractional pieces. Explain how you know.

a/3 or a/8

3. What is the relationship between the number of fractional pieces and the size of the pieces?

4. Order the unit fractions below. Justify your order.

1/3 1/5 1/2 1/4 1/6 1/1

Fraction Unit Test- Estimation

Name: ____________
Date: ______________
Math Journal

This is a time-limited test. You must use estimation to answer the questions below or you will not get through the questions.

A. Order the fractions below from

Smallest-- Biggest
(Least) (Greatest)

1) $\frac{2}{7}$ $\frac{2}{5}$ $\frac{2}{3}$ $\frac{2}{9}$ $\frac{2}{6}$ $\frac{2}{1}$

2) $\frac{3}{4}$ $\frac{4}{5}$ $\frac{5}{6}$ $\frac{2}{3}$ $\frac{6}{7}$

3) $\frac{18}{19}$ $\frac{4}{5}$ $\frac{7}{2}$ $\frac{1}{4}$ $\frac{4}{20}$

4) $\frac{2}{1}$ $\frac{5}{7}$ $\frac{2}{4}$ $\frac{6}{8}$ $\frac{12}{27}$

5) $\frac{3}{6}$ $\frac{6}{7}$ $\frac{8}{10}$ $\frac{10}{8}$ $\frac{3}{1}$ $\frac{6}{6}$

B. Perform the operations indicated below. Remember to give your answers as estimates only. Think about friendly fractions.

1) $\frac{5}{6} + \frac{4}{9} =$

2) $\frac{7}{8} - \frac{4}{7} =$

3) $1\frac{3}{5} + 4\frac{5}{6} =$

4) $8 - \frac{6}{8} =$

5) $\frac{3}{5} \times \frac{7}{8} =$

6) $\frac{6}{12} \times \frac{3}{5} =$

7) $\frac{6}{12} \div \frac{3}{5} =$

8) $2\frac{1}{2} - 1\frac{4}{5} =$

9) $2\frac{1}{2} \times 1\frac{4}{5} =$

10) $\frac{7}{8} \div \frac{4}{7} =$

APPENDIX C
ETHICS REQUIREMENTS/PERMISSION DOCUMENTS

SIMON FRASER UNIVERSITY

PARENT INFORMATION SHEET

Title of Project: **An Investigation of Children's Understanding of Fractions and the Role that Number Sense Plays in that Understanding.**

Student performance on rational number concepts (fractions, ratio, decimals) and their applications have been weak on every provincial mathematics assessment. Research indicates that most students are merely memorizing procedures for operations with fractions without a conceptual understanding of such rules. As a result, students often confuse the various rules which govern the operations on fractions. Since students are not being given the opportunity to develop a conceptual understanding of the meaning and size of fractions, they have little ability to judge the reasonableness of their answers arrived at when applying memorized procedures. Even if they try, students are not able to recognize their own faulty application of rules because they have little or no number sense of fractions which would help them to detect the ridiculousness of their answers.

I would like to examine how children understand the meaning of fractions and how they understand fraction size. I would then like to examine the impact that "number sense" activities have on children's understanding of fraction meaning and fraction size. I will conclude my research by exploring the effect that number sense strategies have on the ability of students to judge the reasonableness of answers obtained by adding fractions.

My research will be conducted by me with students in my mathematics 8 class. All students will participate in the fraction number sense activities. I would like to conduct videotaped interviews with selected students in order to collect detailed information about their understanding of fractions. I will also use student writing in their math journals and assignments, field notes (written by myself), and selected videotaped/audiotaped lessons, as other sources for my data.

I hope that you will be willing to have your child participate in this study. Please be sure that all activities will support the current mathematics 8 curriculum and the children should gain a better understanding of fractions at the grade 8 level. By signing the permission slip, you will allow me to videotape and audiotape your child as they participate in the classroom and/or in personal interviews and to use their journal for my research.

You and your child can be assured of the following:
1) Students' participation in my study is completely voluntary. There will be no prejudice whatsoever toward those students that initially elect not to participate.
2) Parental consent is required for participation and may be withdrawn at any time.
3) Students can withdraw from the study at any time, with no negative consequences.

4) There will be complete anonymity and confidentiality of research findings. Students will be protected because of the use of a pseudonym in my thesis, in any publication or conference presentation or in any discussion about the study.
5) Only myself, my research advisors (Dr. Sandy Dawson and Dr. Rina Zazkis), and my technological support advisor, Linda Hoff, will ever view the videotapes or listen to the audiotapes.
6) The data collected in my research assessment (e.g. personal interviews) will not be used for evaluation or reporting purposes. Assignments, math journals, and tests, which all children will take part in, will be used as part of my normal reporting process.
7) Any concerns about the study may be addressed to me at any time or to Dean Robin Barrow at Simon Fraser University, Burnaby BC
8) A copy of the results of this study , once completed, may be obtained by contacting Dr. S. Dawson. Faculty of Education, Simon Fraser University.

If you would like more detailed information on my thesis research, please feel free to contact me at 942-1835. Thank-you for helping me with my master's research in mathematics education.

INFORMED CONSENT OF MINORS BY PARENT OR GUARDIAN TO PARTICIPATE IN THE PROJECT "AN INVESTIGATION OF CHILDREN'S UNDERSTANDING OF FRACTIONS AND THE ROLE THAT NUMBER SENSE PLAYS IN THAT UNDERSTANDING"

Further clarification on any issue regarding the project "An Investigation of Children's Understanding of Fractions and the Role that Number Sense Plays in that Understanding" may be obtained from Ms. Tige Woodward, mathematics teacher at Kwayhquitlum Middle School (942-1835) or Dr. Rina Zaskis, Assistant Professor of Education (291-3662), Simon Fraser University, or Dr. Sandy Dawson, Professor of Education, Simon Fraser University (291-5969).

As a parent or guardian of (name of child) ______________________________,
I consent to his/her engagement in the procedures specified in the Participant Information Sheet for the project "An Investigation of Children's Understanding of Fractions and the Role that Number Sense Plays in that Understanding." I have read the Participant Information Sheet. I understand that any documentation resulting from this study will guarantee the anonymity of the above-named child and that his/her name will not appear in any publication.

I give my consent for the above-named child to participate in this project, and acknowledge receipt of the Participant Information Sheet.

______________________________ ______________________

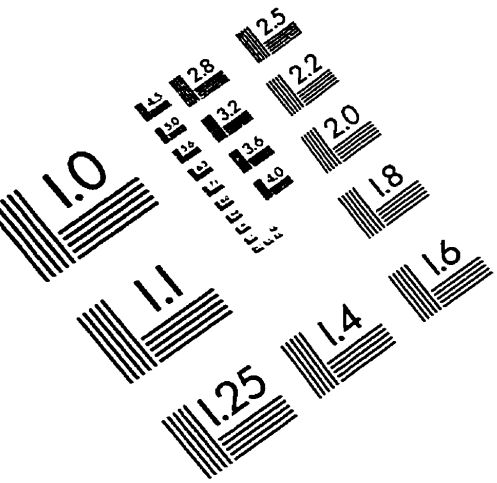
signature date

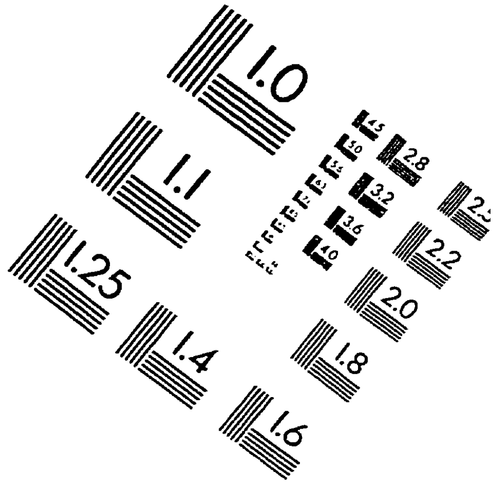
I do not give my consent for the above named child to participate in this project.

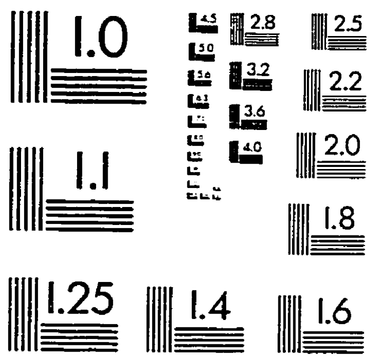
______________________________ ______________________

signature date

IMAGE EVALUATION TEST TARGET (QA-3)







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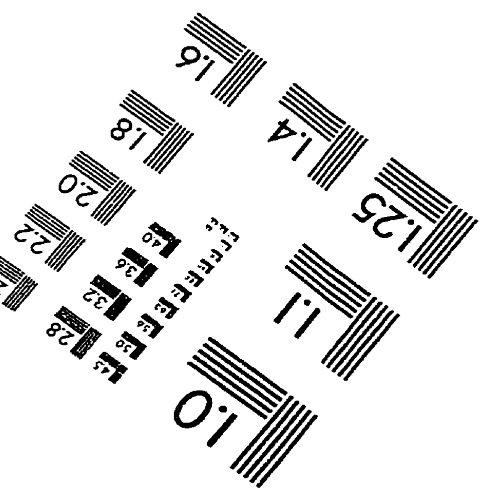

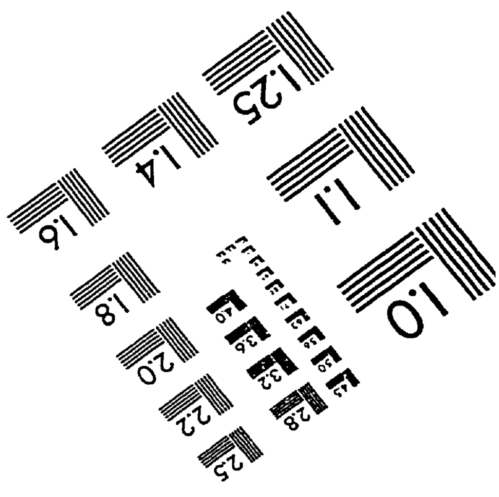

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