THE FIFTEEN-CENT GUITAR:
RE-TEMPERING THE STANDARD SIX-STRING GUITAR

JOHN PAUL SWOGER-RUSTON

A thesis submitted to the Faculty of Graduate Studies
in partial fulfillment of the requirements
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Abstract

The Fifteen-Cent Guitar: *Retempering the Standard Six-String Guitar* is an attempt to address several issues pertinent to the expansion of harmonic/melodic resources available to composers and performers interested in just-intonation. Included is the presentation of a system of tuning for the standard six-string guitar that allows for the exploration of nearly-just intervals based on ratios occurring in the harmonic series up to and beyond the 32nd partial through the detuning of strings by a fifteen-cent increment or decrement.

It is an educational tool, allowing the user to explore otherwise foreign relationships on an instrument that is very nearly a household item in North America, which requires no special adaptation of the instrument except for the precise retuning of strings. It is a compositional tool that allows the composer to use complex harmonic relationships even if the performers have no previous experience with expanded just-intonation. It can be used as a source of reference tones for instruments capable of adjustable intonation but otherwise not predisposed to microtonality.

Included in the body of the thesis is an introduction to basic tuning theory, a historical survey of melodic/harmonic lattices, and an addressing of issues concerning the defining of temperaments and a comparison of the most common twentieth century equal-temperaments. The central chapters of the thesis describe the fifteen-cent temperament for guitar tuning, which was invented by the author, and a set of compositions that utilize this

system in a variety of ensemble settings. It is believed that this system is original and that this is a highly effective temperament when applied in appropriate contexts, which the compositions serve to display. A variety of compositional approaches are utilized addressing larger and more general compositional concerns such as form, rhythm, dynamics, and timbre, but the application of just-intonation to the guitar is the central unifying theme of the works involved.

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Introduction

We have been handed down a musical system that only allows for the exploration of a small set of harmonic relationships, most of which are unsatisfactorily out of tune. Twelve-tone equal temperament is a system that robs the purity of intervals based on whole number ratios for the convenience of modulation and an economy of pitches with which to deal.

The attempt to quantify almost everything perceptible has led to the establishment of prescriptive technologies that limit our compositional resources. Pitch is not the only parameter that has suffered; rhythmic notation rarely attempts to quantify beyond the resolution of factors of two and three, dynamic levels are reduced to no more than ten increments, and standard notation does not directly address the various parameters associated with timbre.

While the present thesis attempts to at least address a few of the issues listed above, its primary concern is the expansion of harmonic resources available to a composer or performer, primarily facilitated through the use of the six-string guitar.

The Historical Background

An understanding of the harmonic series leads to an understanding of pure harmonic relationships based on whole number ratios and insight into how European music has progressively corrupted these relationships in service of the West's obsession for quantification, standardization, and prescriptive technologies. While the harmonic series was not properly explained until 1673 by William Noble and Thomas Pigot, independently, the awareness for harmonic proportions has been present since at least the ancient Greeks. This discovery is generally attributed to Pythagoras of Samos, although it is unlikely that he was the first to describe or understand the properties of pure harmonic relationship.

In Harry Partch's openly anti equal-temperament treatise Genesis of a Music, the author provides a list of the "institutions" for which Pythagoras is responsible; *institutions* because they are at the root of many of the biases of our current popular musical system. These institutions established a bias for perfect fifths (3/2's), the generation of scales based on fifths, the use of low-limit ratios, the acknowledgement of the Pythagorean comma, the division of a string length into twelve equal parts, and the increase in the number of strings on the lyre from seven to eight, establishing a precedence for seven-note scales (Partch, 363).

Music in the early part of the first millennium AD began with few standardized techniques. Musical traditions had to be passed down orally, which meant less information was lost in an isolated transmission. Without the hindrance of notation, parameters such as timbre, pitch, dynamics and rhythm can all be accurately passed on; "the ear has been capable of much more than has been required of it" (Partch, 146). Orally we can pass on far more complex sonic details than any notation system is capable. Pitch can be bending, wavering or "sounding between the cracks" of our equal tempered seven-note scale or twelve-note octave. The big drawback of the oral tradition is that "faulty recall, regional differences, and individual tastes" (Crosby, 141) do not insure accurate reproduction of the original performance. Certainly the body of plainchant codified by Pope Gregory would have been from a hodgepodge of traditions and sources, but for a Catholic Church interested in standardizing the religious practices of its practitioners, consistency was an important concern.

Ways were devised to aid monks in the recall of the numerous chants . For many years the monks simply used a system of heightened neumes drawn above the text of a chant "in campo aperto, 'in the open field' that is, without staff lines" (Crosby, 144), which vaguely referred to relative pitch. Eventually ledger lines were added, first one, then more, to make the system more exacting. This system was still only a memory device; a musician who did not already know the melody would presumably not be able to perform the chant.

Probing into performance practices of this era strongly suggests that Byzantine, Arabic and Jewish influence may have been present in early European music, but there is a good reason why those influences did not survive much past the Middle Ages. Early notation was intended as a memory aid to the monks who had to remember hundreds of different chants for different occasions of the church calendar. Just as today, performance practices that are habitual or taken for granted are not usually identified in the notation. For example, we can't know to what degree vibrato was applied a hundred years ago and only because of recorded music, not our notation system, will future generations know of our current musical habits. As music became less and less an oral tradition, what was heard as the most distinguishing aspects of these cultures' musics became the most difficult to notate in a quantified pitch and rhythm scheme. The very attributes by which we usually identify these musics are those that our current system is least capable of representing. "It is often proudly asserted that the modern Western musical system is sufficient to express everything, but the slightest true contact with other musical systems immediately proves the opposite" (Danielou, 122).

The early performers of plainchant would have orally learned the tradition of bending and tuning pitches but as these early notations evolved they also excluded the parameters that they took for granted. As musicians began to rely more on notation as a means of learning, the oral tradition's importance was lessened. When we look at early notation we can only guess at what the understood performance practices of the day would have been.

Up until the early middle ages, Europe survived with essentially a qualitative model of reality. Details, in the contemporary sense, were not as important for daily life. The Western world functioned without currency, maps were proportionately vague and time was not a linear progression of seconds and minutes as we think of today. "Our chronic difficulty with medieval and Renaissance time is that, like an octopus, its shape was no more than approximate. Time, beyond the individual life span, was envisioned not as a straight line marked off in equal quanta, but as a stage for the enactment of the greatest of all dramas, Salvation versus Damnation" (Crosby, 28).

With the Middle Ages and the Renaissance, Europe began to move away from a qualitative view of the world towards an obsession for quantification. Several factors led to this, one important influence on the western world being an increasing exposure to Arabic numerals. Roman numerals did not lend themselves to complex calculations whereas Arabic numerals were better suited for this. As Europeans gained skill with Arabic numerals, mathematics flourished and numbers took on a greater significance in many aspects of daily life. Every aspect of nature was susceptible to quantification, including space, time, temperature, and musical pitch and rhythm. With the invention of the clock, sometime around the end of the thirteenth century, Europe "had entered the age of quantified time, perhaps already too deeply to turn around" (Crosby, 80).

Polyphonic music evolved quickly with new developments in notation, but this also meant new limitations on what could be expressed. An approach so profoundly innovative systematically removed itself from its musical roots. "Modern Western music was able to develop its polyphonic system only by deliberately sacrificing the greater part of its possibilities and breaking the ties that connected it with other musical systems" (Danielou, 123).

The late medieval period saw a flourishing in all the arts of a fascination and superstition in numbers. The arts became something for God and only he could perceive the complicated details, hidden from the general public. The cathedrals were embellished with minute details hundreds of feet above the congregation. Composers began writing complicated musical games, such as canons with the instructions delivered in the form of a riddle. The melodies and rhythms were contrived with numerical significance. A rigidly codified musical system was developed and facilitated this *music for God*. "The West's distinctive intellectual accomplishment was to bring mathematics and measurement together and to hold them to the task of making sense of a sensorially perceivable reality, which Westerners, in a flying leap of faith, assumed was temporally and spatially uniform and therefore susceptible to such examination" (Crosby, 17).

As the system developed and the urge to quantify pitch increased, the nature of harmonic relations had to be understood more thoroughly. However, this was a complex task as even in a simple seven-note scale each note can serve several different functions. In a C major chord, the note 'E' is ideally tuned in a 5/4 frequency ratio to its root 'C', but in an alternate context the acoustical root of a different chord will demand that the pitch be adjusted to maintain harmonic purity. The twelve notes can only approximate their true acoustical functions, or else some relationships must be given preference over others. The usefulness of a notation system would demand that it be simple enough for musicians to learn and at the same time attempt to represent pertinent parameters. With simplicity came the limitations of our popular system.

The introduction of the keyboard further complicated the issue. With string ensembles and voices, the players can easily adjust a pitch to its proper harmonic relation. With the keyboard however, each key is set to a fixed pitch. With the introduction of the keyboard came some important theoretical and practical problems in tuning. Since the Western world was already thinking in terms of seven note scales and a twelve note octave, a system had to be devised that would address the fact that, especially with vocal and string music, individual notes have to serve several different functions.

By the early baroque period, many meantone temperaments had developed. In a major scale "purely" tuned to a 'C' root, two different sized major seconds occur. Between 'C' and 'D', the frequency ratio is 9/8. The ratio between 'D' and 'E' is 10/9. The difference in size between these two intervals is 81/80 (calculated as 9/8 divided by 10/9), this equals the syntonic comma and is 21.51 cents in size; meantone temperaments seek to average out this difference.

Without further specifications, meantone temperament generally implies a "quarter-comma" meantone. This means that all fifths are flattened by a quarter of the syntonic comma, except for one, which becomes "the wolf"; named as such for its roughness. This system results in many pure 5/4 major thirds, except for four that are wide, and 9/8 major seconds, with the exception of two. Other meantone systems can be similarly constructed by applying smaller fractions to the fifths. As the division of the comma gets higher, the out of tune fifths improve, the thirds get wider and the minor thirds become narrower. If a twelfth of the Pythagorean comma is applied, equal temperament occurs.

In the eighteenth century, theoreticians argued as to how the Western twelve-note octave should be tuned. In earlier years, the composer or head organist of a cathedral would be responsible for the tuning of their keyboard instrument. Purcell, Bach, Mozart and others all had their own preferred systems, but as we moved towards the rigid standardizations of the Industrial Revolution, a need was seen for a standard to be established. The debates continued for years, with equal temperament winning out somewhere around the end of the 19th century.

As partly a result of moving towards equal-temperament, and an expansion of chromatic melodic and harmonic possibilities, music was gradually becoming less firmly rooted in any one key. Modulation in the Classical and Romantic sense required a flexible tuning system. Many theorists felt that equal temperament was the best means for this. This method, however, fails to represent any pure harmonic relations and some intervals are significantly out of tune. The major third, which can be seen as one of the fundamental intervals of late Baroque through to Romantic era harmony, is actually mistuned by fourteen cents.

With the industrial revolution came a drive for standardization and economy of production. Industries wanted one way of completing a task and had to produce a product in as financially efficient a way as possible. "More" was (and still is) the economic imperative. This took workers out of the holistic type cottage industries and put them into the factories to work on smaller portions of the final product, essentially an assembly line mentality. These ideals penetrated the mind set of the Western world. In music, a need for the standardization of instruments, tuning, and notation was strongly felt.

As the piano gained in popularity and as music became more of a middle class pastime, the standardization of instruments was a necessity for music to be performed by people with less intense musical training. The Industrial Revolution not only allowed this kind of accessibility but was the impetus for the standardization of many elements of society. Music became a household endeavor as pianos and the printed page became more and more economically available to the working and middle classes. Teaching music had to become standardized to efficiently train the middle class as did the instruments they were playing on. No longer was tuning a function of the composer's will; decisions were now being made for the masses.

A prescriptive technology such as the Western musical system (which includes the notation system, instrument construction, the performer, etc.) we have just briefly explored "leaves little latitude for judgment" and are "designs for compliance" (Franklin, 23). As our system became more standardized, we increasingly ignored the many possibilities that our notation and tuning systems limit. "Prescriptive technologies eliminate the occasions for decision-making and judgement in general and especially for the making of principled decisions. Any goal of the technology is incorporated a priori in the design and is not negotiable" (Franklin, 25). It is a monumental task to attempt to address all the musical parameters with a single system, but without addressing these issues we limit our means of musical expression.

The Solution

By the middle of the twentieth century, it seemed that we had exhausted most of the potential for European music's evolution. While not directly concerned with music, Ursula Franklin shares this view. "It is my conviction that we are at the end of a historical period in which processes and approaches that initially had been exceedingly constructive and helpful have run their course and are now in many ways counterproductive" (61). With the twelve-tone serialists Western music was taken to its harmonic and melodic extreme where tonality was abandoned in favour of atonality. At the other end of the spectrum, the minimalist movement stripped Western music to many of its foundations, exploring limited facets of its key elements. The only place left to go is to explore the rudimentary elements of sound; to redefine and expand what music can be. Because music is still primarily produced prescriptively, only those who are willing to take on the broadest role of what a musician is can explore these possibilities, and this initially must be done via a holistic approach.

Our most innovative modern composers have essentially had to produce music holistically to realize their vision. Harry Partch is one of these people concerned with pure harmonic relationships. Primarily he looked at the limitations of the 12-note tempered scale as well as turning to the regular flow of speech for rhythmic inspiration. By looking at frequency ratios of an 11-limit as opposed to the traditional 5-limit he developed a 43-tone scale in which the smallest interval is 14.4 cents in size. All pitches are generated from pure harmonic relationships (i.e. whole number ratios).

To implement his system, Partch essentially had to take over the entire musical process, beginning with building his own and adapting common instruments. His new system also required him to train his own musicians, conduct them, perform and finance his own productions. The holistic approach, for the time being, is essential to the development of any new system.

A Compromise

A purely holistic approach to making music is a huge undertaking and limits the potential for works to be performed by anyone unassociated with the composer. To facilitate the accessibility of the compositions presented in this thesis, traditional Western instruments (with one exception) and standard notational conventions are modified in a minimally invasive manner. Specifically, the guitar is adapted only through the retuning of strings and through the adaptation of conventional guitar notation.

A unique system of temperament based on a fifteen-cent increment is employed that closely approximates almost all of the intervallic relationships found in the harmonic series up to and beyond the thirty-second partial. While just intonation purists may argue that any temperament is an abomination, the fifteen-cent temperament approximates intervals found in a 31-limit system to a degree that is almost imperceptibly out of tune. This expansion of resources allows a composer or performer to work in a harmonic system that far exceeds the available materials of Western harmony.

Chapter One: Introductory Tuning Theory

This chapter introduces the basic materials of tuning theory required for an understanding of the discussion of the fifteen-cent temperament for guitar tuning presented in chapter four, and as background to further issues that arise in chapters detailing individual compositions.

Ratios

A "ratio" is a means of expressing the frequency relationship between two pitches. This thesis will use the following convention: the numerator represents the number of cycles that the higher frequency will generate over the same period that the lower frequency takes to generate the number of cycles indicated in the denominator. For example, in the ratio 5/4, the higher frequency generates 5 cycles for every 4 cycles of the lower frequency (actual frequencies can also be used in place of these terms - e.g., 550 Hz / 440 Hz). This ratio is a pure major third and is smaller than an equal-tempered major third.

In general, all frequency ratios will be expressed in the simplest terms and reduced to an interval smaller than or equal to an octave (2/1) except when voicing is an important concern. For example, the intervals (or ratios) 10/4, 5/2, and 35/28 all reduce to 5/4. Ratios are reduced by factoring out any prime coefficient common to both the numerator and denominator, and are condensed to within an octave by either halving the numerator, or by doubling the denominator until the ratio is reduced to an interval smaller than 2/1 (an octave). The terms "ratio" and "interval" will be considered synonymous for the purpose of this thesis.

Harry Partch uses this convention of expressing all ratios in terms smaller than a 2/1. "A system of music is determined for one 2/1; the system is then duplicated in every other 2/1, above or below, that is employed. Consequently, symbols–ratios in this exposition–are used to denote the degrees of one 2/1, and the symbols are repeated in every 2/1 of the musical gamut" (Partch, 79). Partch's convention allows for the use of a ratio to describe a "pitch class" or an interval smaller than 2/1, and avoids the inconvenience of having a different set of symbols (ratios) for each octave of a musical system.

To calculate the cumulative size of two "stacked" intervals, for instance, a 5/4 built upon another 5/4 (a just augmented triad), the ratios are multiplied. 5/4 x 5/4 = 25/16. Or, conversely, to find the difference between the magnitude of two ratios, expressed as a ratio, divide. To divide ratios, invert the smaller interval, multiply, and simplify.

$$(25/16) / (5/4) = (25/16) \times (4/5) = 100/80 = 5/4.$$

Any ratio added to its reciprocal (inversion) results in an octave. 5/4 x 8/5 = 40/20 = 2/1. Any ratio subtracted from an octave or unison results in the reciprocal of that ratio after the result is simplified.

$$(x/y)(y/x) = 2/1^*$$

$$(2/1) / (x/y) = y/x^*$$

*these formulas are only relevant to the conditions of tuning theory established above

Cents

In the appendix of his translation of Hermann Von Helmholtz' On the Sensations of Tone, Alexander J. Ellis provides a means for measuring interval size. The basic unit is the "cent", which is a logarithmic unit equal to 1/100th of an equal-tempered semi-tone, dividing the octave (2/1) into 1200 parts. "Cents provide a logarithmic device which enables the theorist to add and subtract numbers representing the respective magnitudes of the various ratios, which he cannot do with the ratios themselves" (Partch, 83).

Expressing intervals in terms of their size in cents provides a good means for comparing just intervals to their closest equal-tempered equivalent, and to each other. While it is somewhat backward to think in terms of "out-of-tuneness" from equal temperament (because any temperament is a compromise between practicality and intonation), the notion of "cents-deviation" provides a good introduction to microtonality for anyone primarily familiar with the Western chromatic scale. Each equal-tempered semi-tone is 100 cents in magnitude and therefore, any twelve-tone equal-tempered interval is represented by a magnitude measured in multiples of 100 cents: a major third is 400 cents in size, a fifth is 700 cents, etc. But, as Harry Partch concludes, it is much more efficient and practical to develop a familiarity with just-intervals, in and of themselves, than in relation to equal-tempered intervals. The notion that the piano is "right" and that just-intervals are "out of tune" needs to be reversed.

To calculate the size in cents of an interval, the $\log_2$ of the ratio is taken and multiplied by 1200 (representing 1200 equal parts to the octave).

$$\log_2 (f^1/f^2) \times 1200 \text{ where } f^1 > f2$$

Many scientific calculators do not allow for logs to a base other than 10. To calculate the above formula on such a calculator, take the $\log_{10}$ of the ratio, multiply by 1200 and divide the result by the $\log_{10}$ of 2 (if your calculator has a log key without a base indicated, it is assumed that the base is 10).

$$\frac{\log_{10}(f^1/f^2) \times 1200}{\log_{10}(2)}$$

I find that it is easiest to commit $\log_{10}(2)$ to memory to make the calculation quicker.

$$\log_{10}(2) = 0.301029995$$

For example, plugging the interval 5/4 into either formula results in a size-in-cents value of 386.3 (rounded to the nearest 10^{th} - the convention this paper will adopt). The closest equal-tempered interval is the major third (400 cents). A simple subtraction of the first value from the second reveals that the 5/4 interval is about 13.7 cents smaller than the tempered major third. The 5/4 interval is known as the "pure", or "just" major third and the equal-tempered major third is generally considered an approximation of the 5/4, in the context of a major triad. More just intervals will be described and given a size-in-cents value as they appear in further discussions of other aspects of tuning theory.

To convert a known size-in-cents value to a ratio, take the 1200th root of two and multiply by the size-in-cents value.

$$\sqrt[(1200/x)]{2}$$

or

$$2^{(x/1200)}$$

where x = size in cents

The Harmonic Series

The "harmonic series"[1] is very often poorly represented in the classroom and in music theory textbooks. Most explanations use a twelve-note equal-tempered system, without qualification, to represent the harmonic series. This is completely inadequate, as equal-tempered pitches grossly misrepresent harmonic frequencies. In some cases the deviation is greater than 50 cents (1/4 tone) and is therefore not even the closest

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[1] The common term "overtone" will not be used in this thesis as it is often a source of confusion. The terms "harmonic" and "partial" are equivalent and refer to the same component of the series, overtones do not. The first overtone is equivalent to the second partial or second harmonic. Therefore, the numbering of overtones will always be off by one from the numbering of harmonics or partials.

approximate equal-tempered representation of the harmonic (Often the 13^{th} partial is represented as a major 6^{th} rather than a minor 6^{th}, neither of which are particularly appropriate).

The *harmonic series* is an array of tones that occur over a given fundamental tone. The frequency of these subsequent tones (harmonics) are integer multiples of a fundamental frequency. The relationship between a fundamental note and its harmonics will be the same as with any other fundamental tone and its harmonics. The relative weakness and strength of individual harmonics varies with each instrument and is one of the distinguishing features of timbre.

The fundamental frequency is the first partial, or first harmonic, and is referred to by the ratio 1/1. If the fundamental frequency is 220 Hz, then the first partial is expressed as 220 Hz / 220 Hz or 1/1. The second partial is described by the ratio 2/1, in this case 440 Hz / 220 Hz. This equals a purely tuned octave (Note: in equal temperament, only octaves are purely tuned).

The third partial is in a 3/1 relationship to the fundamental or 3/2 from the second partial. This equals 660 Hz for this example, an octave and a fifth (14th) above the fundamental or a perfect fifth above the second partial. An equal tempered fifth sounds approximately 2.0 cents flat of a pure 3/2 fifth.

An equal-tempered approximation of the overtone series continues in this fashion:

4/1 = 2 octaves	or	4/3 = P4^{th} above the 3^{rd} partial
5/1 = 15^{va} + maj 3^{rd}	or	5/4 = maj 3^{rd} above the 4^{th} partial
6/1 = 15^{va} + P5^{th}	or	6/5 = min 3^{rd} above the 5^{th} partial
7/1 = 15^{va} + min 7^{th}	or	7/6 = min 3^{rd} [2] above the 6^{th} partial

As the series continues upward, the interval between consecutive partials becomes increasingly small, so descriptions such as "major third", "minor third", "major second", etc. become less and less appropriate, and the language of ratios becomes more appropriate. For example: The ratio 9/8 produces an interval that is 203.9 cents in size, about four cents wider than an equal tempered major second. The ratio 10/9 produces an interval 182.4 cents in size, 17.6 cents narrower than an equal tempered major second. These two intervals have historically been referred to as the large major second and the small major second respectively, but many other ratios can also be referred to as major seconds (8/7, 17/15, 19/17, etc.). Using descriptions in terms of traditional interval names can quickly become inefficient, inaccurate, or inappropriate. Many microtonalists[3] prefer simply to refer to intervals by ratio rather than by the conventions of twelve-tone equal-temperament (TET). At first this can seem intimidating but the benefits are quickly realized as one's familiarity with ratios increases. However, since many musicians are not familiar with the language of ratios, this thesis will allow for comparisons related to TET.

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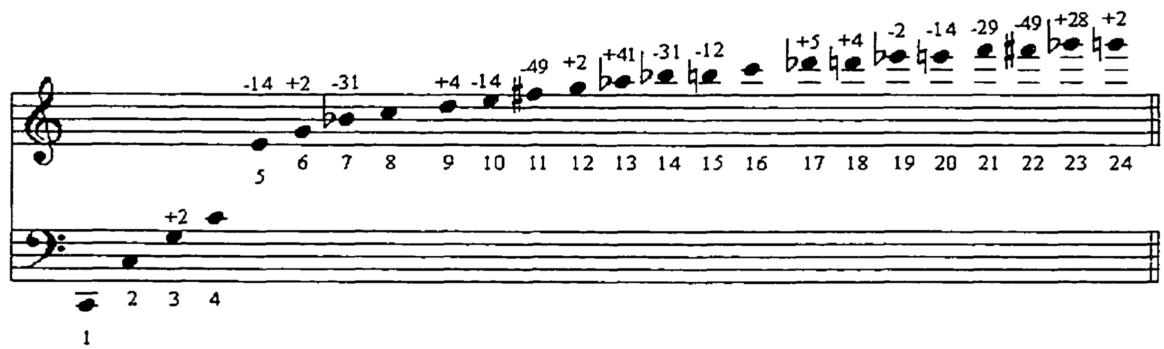
[2] The 7/6 is a smaller minor third than the 6/5 minor third.

[3] The term microtonal refers to any tuning that deviates from twelve-tone-equal temperament. Other temperaments, just-intonations, and arbitrary detunings are subsets of the larger microtonal umbrella. In this thesis, the term will usually imply microtonalists interested in just intonation.

With the above in mind, below is a representation of the overtone series in relation to equal temperament. It does not matter what frequency we begin with, as long as it corresponds to an equal tempered reference point. If A_2=110 Hz. is the fundamental frequency, the relationship will be maintained between equal temperament and pure ratios. Note that my choice of representing the thirteenth partial with an 'F' rather than an 'F♯' is not a common textbook choice.

Frequency Ratio	***Size in Cents*** 1200 x $\log_2(f^1/f^2)$ = size in cents	***Approximate E.T. pitch*** with A_2=110 Hz as fund.		***In Hertz*** (partial x fund.)
16/1	9600.0 cents	A_6		1760 Hz
15/1	4688.3 cents	$G\sharp_6$	- 11.7 cents	1650 Hz
14/1	4568.8 cents	G_6	- 31.2	1540 Hz
13/1	4440.5 cents	F_6	+ 40.5	1430 Hz
12/1	4302.0 cents	E_6	+ 2	1320 Hz
11/1	4151.3 cents	$D\sharp_6$	- 48.7	1210 Hz
10/1	3986.3 cents	$C\sharp_6$	- 13.7	1100 Hz
9/1	3803.9 cents	B_5	+ 3.9	990 Hz
8/1	4800.0 cents	A_5		880 Hz
7/1	3368.8 cents	G_5	- 31.2	770 Hz
6/1	3102.0 cents	E_5	+ 2	660 Hz
5/1	2786.3 cents	$C\sharp_5$	- 13.7	550 Hz
4/1	2400.0 cents	A_4		440 Hz
3/1	1902.0 cents	E_4	+ 2	330 Hz
2/1	1200.0 cents	A_3		220 Hz
1/1	0.0 cents	A_2		110 Hz

Below is the harmonic series to the 24th partial, built on C_2. Notice that the cents-deviation for the intervalic relationship from C_2 is the same as in the above chart from A_2.



While it is disputed whether or not tonal melodic/harmonic materials developed as a result of human sensitivity to the harmonic series, it can be seen that all of our basic harmonic relationships are found in the series. The octave is the relationship of the second partial to the first (2/1). The perfect fifth is the relationship between the third and second partials (3/2). The perfect fourth: 4/3, major second: 9/8 or 10/9, major third: 5/4, minor third: 6/5, major sixth: 5/3, minor sixth: 8/5.

It should also be noted that calculating the frequency of any harmonic over a given fundamental tone is straightforward. Simply multiply the partial number by the fundamental frequency. The fifth harmonic over A_2 =110 Hz equals 550 Hz (110 x 5).

$$f^1 \times p = f^p$$

Limit

The term *limit* is used to define the largest prime number used in the generation of ratios in a given system. For example, the intervals that make up a typical just major scale - 1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8 - are all constructed with prime numbers of five or less. The interval 25/16 (augmented 5^{th} (-27.4cents)) is also a 5-limit interval because the numerator is a power of 5 ($5^2 = 25$) and the denominator is a power of 2 ($2^4 = 16$).

The Pythagorean Scale

Very briefly, I would like to introduce two basic scales that will be important reference points for future discussions, the *Pythagorean* scale, and the *Just* chromatic scale (the discussion of the Just scale follows the section on complexity).

> [...W]e may take the opportunity to see how we select out of the whole range of audible frequencies those privileged frequencies to be used for practical musical purposes. It is in this area that mathematicians and numerologists (the two terms are not necessarily exclusive) have been most diligent. Numerology--that branch of the occult arts dealing with the magic of numbers--has a fascination for many, and unfortunately musicians are not exempt from its influence. This is best demonstrated by the history of musical scales (Backus, 134).

For the Pythagoreans, the numbers 1 (unity), 2 (duality), and 3 (harmony) had significant importance with regards to the soul, politics, the cosmos, and virtually every element of life including music. This predilection is reflected in the Greek musical system for the generation of five-note scales which was later adopted by Medieval theorists for seven-note scales. The Pythagorean scale is constructed entirely of ratios of a 3-limit. The Pythagorean scale is built on the 3/2 (the perfect fifth) and its inversion, the 4/3 (the perfect fourth). By stacking 3/2's and condensing the results into one octave, a seven-note scale is achieved (which is an arbitrary stopping point). Assuming that 1/1=C, the following pitches can be generated (the exponents indicate the cents-deviation from TET).

1/1	3/2	9/8	27/16	81/64	243/128	729/512
C	$G^{+2.0}$	$D^{+3.9}$	$A^{-5.9}$	$E^{+7.8}$	$B^{+9.8}$	$F\sharp^{+11.7}$

Rearranged into scalar form, the result is a type of C lydian (arbitrarily assuming that C is the generating tone):

1/1	9/8	81/64	729/512	3/2	27/16	243/128
C	$D^{+3.9}$	$E^{+7.8}$	$F\sharp^{+11.7}$	$G^{+2.0}$	$A^{+5.9}$	$B^{+9.8}$

For a mixolydian scale, two 4/3 steps are taken from 1/1 giving a perfect fourth ($F^{-2.0}$) and a minor seventh 16/9 ($B\flat^{-3.9}$) which replace the F♯and B.

If a twelve-note Pythagorean scale is desired (this was not typically a historical concern), the minor third is the inversion of the interval C to $A^{+5.9}$ (27/16) which is an $Eb^{-5.9}$ (32/27), the minor sixth is the inversion of C to $E^{+7.8}$ (81/64) producing an $Ab^{-7.8}$ (128/81), the minor second is the inversion of the C to $B^{+9.8}$ (243/128) giving $Db^{-9.8}$ (256/243). These can all be verified by calculating the interval between any two degrees of the original scale. For example, $A^{+5.9}$ to $Bb^{-3.9}$ is the Phrygian minor second, its frequency ratio is (16/9) / (27/16) = (16/9) x (16/27) = 256/243.

Complexity

One of the definitions of consonance and dissonance that Helmholtz and Partch provide is based on the relative levels of complexity (determined by the size of the numbers involved in both the numerator and the denominator of the ratio). A 2/1 (octave) involves low numbers and is therefore considered highly consonant. This is confirmed historically by the fact that the components of this interval are given the same name, which is a reflection of the aural experience in that it is somewhat difficult to distinguish the two tones due to the similarity of harmonic information.

The most common explanation for predicting the level of dissonance in a ratio involves the interference that occurs between any two harmonic components, of two complex tones, that sound within a critical bandwidth[4] of each other.

> Because of the pattern of harmonics, when two complex tones are sounded, their component frequencies will coincide to the extent that their fundamental frequencies of related by simple integer ratios. For example, if two tones, one an octave higher than the other, are sounded simultaneously, all the harmonics of the higher tone will be present as harmonics of the lower tone. As the integers needed to express the ratios of the frequencies increase, the number of mismatches between harmonics of the two tones will also increase (52).

As the complexity of the interval (or ratio) increases, so does the perceived dissonance. Krumhansl distinguishes "tonal consonance" and "musical consonance" as two separate ideas.

> Tonal consonance refers to the attribute of particular pairs of tones that, when sounded simultaneously in isolation, produce a harmonious or pleasing effect. Although the precise definition of this property varies in its many treatments in

[4]The term "critical bandwidth" is defined by Dowling and Harwood as "the frequency region over which stimuli interact in producing sensations of loudness" (81). It is generally difficult for the ear to process separately two tones sounding simultaneously within that frequency range.

> the literature, there is general consensus about the ordering of the intervals along a continuum of tonal consonance. Musical consonance, on the other hand, refers to intervals that are considered stable or free from tension, and constitute good resolutions. This kind of consonance depends strongly on the musical style and also the particular context in which the interval is sounded. Thus, musical consonance may bear only a rough correspondence to tonal consonance (51).

While it is beyond the scope of this paper to define definitively what constitutes a consonance or dissonance, it will address *relative* consonance and dissonance through the terms *simplicity* and *complexity* (A 5/4 is relatively more complex than a 3/2 because 5/4 involves larger numbers). As well, Krumhansl uses the subjective terms, "harmonious" and "pleasing" which I find somewhat problematic. Perhaps the idea that simpler (more consonant) ratios are more easily processed should replace any notions of preferential treatment. "We are confident, then, that melodic intervals with simple frequency ratios are inherently easier to process than those with more complex ratios" (Trehub, 116).

Relative complexity will only be discussed with regards to intervals fully simplified and reduced to within the size of an octave, i.e.; no comparisons of compound intervals to simple intervals are contemplated. I also generally assume that the limit of a given ratio is not significant in contributing to complexity. Higher limit ratios, however, may be responsible for a sense of unfamiliarity or exoticism[5] in certain situations.

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[5] I use this term with reservation but have included it because of its regularity of use in the responses of people hearing certain intervals for the first time. The term "bluesy" should probably also be included for this reason.

Looking at the Pythagorean scale, it should be noticed that the interval of a major third is represented by a very complex ratio - 81/64. This is the result of constraining the system to a 3-limit. Mathematically, the scale seems justified; however, it does not necessarily represent the inclinations of the human aural experience. While it is easy to tune a stringed instrument to Pythagorean intonation, it seems unlikely that an unaccompanied voice would sing this third in most harmonic contexts.

The Just Scale

If the Pythagorean 3-limit is increased to a 5-limit, new thirds and sixths can be generated that are less complex. In the overtone series, a major third occurs between the fifth and fourth partials, a minor third between the sixth and fifth partials, a minor sixth between the eighth and fifth partials, and a major sixth between the fifth and third partials; 5/4, 6/5, 8/5, 5/3 respectively. By substituting these simpler five-limit ratios for the more complex ratios of the Pythagorean scale, the following chromatic scale is generated:

1/1	16/15	9/8	6/5	5/4	4/3	(45/32)	3/2	8/5	5/3	9/5	15/8
C	$D\flat^{+11.7}$	$D^{+3.9}$	$E\flat^{+15.6}$	$E^{-13.7}$	$F^{-2.0}$	$(F\sharp^{-9.8})$	$G^{+2.0}$	$A\flat^{+13.7}$	$A^{-15.6}$	$B\flat^{-3.9}$	$B^{-11.7}$

This is one of the most common forms of a 5-limit just-intonation scale. There is, however, some flexibility, especially in the choice of tritone and minor seventh. In the version indicated above, the tritone is generated by a pure major third built off of the major second (9/8 x 5/4 = 45/32) and the minor seventh is calculated as a minor third from the perfect fifth (3/2 x 6/5 = 9/5).

Immediately it can be seen that many anomalies arise when the root modulates. For the most part, all the ratios are relatively simple in relation to the root 'C', but if the tonal centre changes, the relationships are in many cases no longer ideal. For example, a minor chord built on the second degree is fairly complex. Assuming 'D' is now 1/1, the chord built from it is 1/1 - 32/27 - 40/27. The bottom interval of the triad, 32/27, does not sound particularly foreign to most ears as it is a minor third that is close to an equal tempered minor third (294.1 cents), however it does sound "out of tune". 32/27 to 40/27 sounds in tune, as it is simply a 5/4 major third ((40/27) / (32/27) = 40/27 x 27/32 = 1080/864 = 5/4). The interval formed by the outer tones is 40/27, and sounds like an out of tune 3/2 (flat by about 19.6 cents). The resultant chord simply sounds out of tune due to the overall complexity of the ratios involved in the sonority.

Further problems and issues of scale generation and tuning will be approached after a more efficient means of discussion and comparison has been developed in chapter two.

Commas

In the generation of a ratio built from a succession of twelve 3/2's, it can be proven that a pure octave is never fully achieved. The "Pythagorean comma" is 23.5 cents in size, described by the ratio 531441/524288, and is the difference between the ratio $(3/2)^{12}$ and an octave. The mathematical fact that no power of any single ratio will result in a pure octave (2/1), or a compound of 2/1, has been at the root of tuning issues since its discovery.

A "comma" generally describes the difference between any two close intervals of different limit values. The difference between a 3-limit major third and a 5-limit major third is $(81/64) / (5/4) = 81/80$. The 5/4 is smaller than 81/64 by 21.5 cents and this is generally known as the "syntonic" comma.

This is obviously not a complete account of tuning theory but it does cover most of the issues involved in understanding the discussion of the compositions of this thesis. Further issues will be addressed in the following chapters. The texts that have contributed most to my understanding of the above are: Harry Partch's Genesis of a Music, Hermann Von Helmholtz' On the Sensations of Tone and its appendices by Alexander J. Ellis, and Arthur H. Benade's Fundamentals of Musical Acoustics. I refer the interested reader to these important texts.

Chapter Two: Models of Harmonic Relationship (Lattices)

This chapter presents several models that aid in the demonstration of harmonic relationships and further clarify several theoretical tuning issues. These models have been important to my own understanding of tuning theory and have influenced the development of my hybrid model, which is presented later in this chapter.

Early Melodic Diagrams

Several approaches to dealing with tuning systems and scales have been presented over the centuries. Although Pythagorean tunings only involve one class of interval (in addition to 2/1), the 3/2, graphical representations are historically relatively convoluted. This is partly due to the fact that numerological relationships had greater significance than their simple harmonic/melodic function. Musical intervals represented many aspects of the natural universe, the soul, and politics for the Greeks. Accordingly, graphical representations had to portray more than just abstract musical meaning. Ernest G. McClain has authored two books, The Pythagorean Plato: *Prelude to the Song Itself* and The Myth of Invariance: *The Origins of the Gods, Mathematics and Music from the Rg Veda to Plato*, which extensively investigate the relationships between music and Greek thought. In these texts, McClain tries to reconcile the mathematics and symbolism of Pythagorean and Platonian numerology.

Pitch Height Diagrams

One of the simplest models for the demonstration of scales or pitch sets is on a vertical or horizontal axis with points indicating the pitch height, or distance, of each given interval in the system within a 2/1. Unlike the division of a string, the division of the pitch height axis is achieved logarithmically so that ratios display comparative sizes based on the perception of pitch rather than frequency.

Pitch height models are effective in demonstrating the size of scalar steps within a given system but provide no information about the inherent relationship between any two pitches. The models presented below provide harmonic relations without regard for pitch height information, with the exception of James Tenney's, which provides a solution to representing both concerns. Because my primary interests in composition are harmonic rather than melodic (the two are not entirely independent), my models have gravitated towards the display of harmonic relationship over pitch height.

Harmonic Models

Graphical representations of tuning systems help to visualize common harmonic properties within a system and also may reflect symmetrical relationships, a continuum of relative harmonic complexity, and the occurrence of various types of commas. As will be seen, models become increasingly complex with the introduction of new generating numbers. Several composer/theorists have devised their own graphical systems, and the following provides a lineage of models that have contributed to my own understanding of tuning theory and to the development of my own harmonic models.

With the Pythagorean scale, harmonic relationships are easily described with a linear, horizontal model, portraying a succession of 3/2 relationships, as shown in chapter one. With the addition of five-limit intervals, a new dimension is required showing both 3/2 relationships and 5/4 relationships. Most models tackle this by describing 3/2 relationships horizontally and 5/4 relationships vertically. With the addition of higher limit intervals comes the problem of demonstrating multiple "dimensions" of harmonic relationship. "For a given set of pitches, the number of dimensions of the implied harmonic space would correspond to the number of prime factors required to specify their frequency ratios with respect to the reference pitch" (Tenney, 1983, 15).

This is true of the following except that most models are generally not concerned with octaves and therefore all of the following models have one less dimension than the number of prime factors involved in the system, i.e. the prime number two is basically ignored. A "pc projection space" is James Tenney's term for lattices that demonstrate pitch class relations through the exclusion of the octave, or the prime number 2.

The following are some of the more influential twentieth century models of tuning relationship and represent a conceptual lineage leading to the models I have adopted for the realization and discussion of my own work.

J. Murray Barbour

J. Murray Barbour uses a system, which he describes in the introduction to his text Tuning and Temperament, devised by K.A. Eitz[1] in which exponents are used to relate intervals connected by their prime coefficient. Pitches connected by fifths have an exponent of zero (^{0}), the fundamental has no exponent and an exponent of -1 indicates a detuning by the syntonic comma from its fifth related equivalent. A major third is indicated as $C^0 - E^{-1}$, This indicates that the E is to be tuned 21.5 cents flat of E^0 which, in relation to a C root is approximately 7.8 cents sharp of its equal tempered equivalent. 7.8 cents minus 21.5 cents equals –13.7 cents, which is the correct tuning for a pure 5/4 interval [$1200 \times \log_2(5/4) = 386.3$].

Barbour arranges pitches according to intervalic relationship. The major third (5/4) is situated at a 45-degree angle above the root. Relations of fifths (3/2) and fourths (4/3) are arranged horizontally:

$$\begin{array}{ccccccccc} A^{-1} & & E^{-1} & & B^{-1} & & F\sharp^{-1} & & C\sharp^{-1} \\ & C^0 & & G^0 & & D^0 & & A^0 & \end{array}$$

Minor third (6/5) relations are indicated by an exponential difference of +1. It can be seen in the above example that these relations are already present. For example, B^{-1} to D^0 is a minor third (6/5), with an exponential difference of +1. Any other pair situated in a similar relative spatial relationship will represent a 6/5. Chains of 5/4 and 6/5

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[1] Barbour does not provide any references to Eitz in Tuning and Temperament except with the mention of his name in the introduction. I have not been able to find any further information on this individual.

relationships are simply represented in increasing increments of the syntonic comma. For example, a 25/16 is represented as:

$$G\sharp^{-2}\ (25/16)$$

$$E^{-1}\ (5/4)$$

$$C^{0}\ (1/1)$$

Barbour's text, Tuning and Temperament: *A Historical Survey*, is concerned mainly with the discussion of historical tuning systems and therefore concentrates on 5-limit tuning systems (European music is historically concerned with tertian based harmony which is in turn almost exclusively based on 5-limit intervals[2]). Because of this preference, Barbour's method of indicating the difference of the syntonic comma is only applicable to 5-limit tuning systems, and therefore not useful on its own in systems of higher-limit ratios.

Barbour also uses fractions to indicate portions of a syntonic comma, which is useful in his discussions of historical mean-tone tunings. For example $x^{-1/4}$ indicates a detuning of 1/4 of the syntonic comma, approximately 5.5 cents. In Barbour's own words, the preceding can be summarized by his definition of *exponents* in the glossary. "In tuning theory exponents are used to indicate deviations from the Pythagorean tuning, the unit being the syntonic comma. Plus values are sharper and minus values flatter than the corresponding Pythagorean notes. Fractional exponents indicate subdivisions of the

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[2] This is not to say that higher limit tuning systems were never contemplated prior to the twentieth century. The theorists Giuseppe Tartini, Jean Adam Serre, and Leonard Euler all proposed 7-limit systems in the mid-eighteenth century (Partch, 386-388).

comma, as in meantone and many irregular temperaments" (ix).

Figure 2.1 demonstrates the occurrence of the syntonic comma in the common chord progression: I – vi – ii – V – I. In this example, the proper intonation for each chord is notated, and each voicing maintains as many common tones as possible. Beginning with the tonic chord, C major shares two pitches with the A minor chord ('C' and 'E'), which shares an 'A' with the D minor chord, which shares 'D' with the G chord, which shares 'G' with the new tonic chord C major, now tuned one syntonic comma lower than the original tonic.

This figure demonstrates two important points; that all similar harmonic structures share a common shape, in this case major chords are found in upright triangular formations, and minor chords by an inverted triangular formation. Also, this figure demonstrates James Tenney's observation that traditional harmonic progressions with strong root movements tend to progress upwards and to the left in "harmonic space" (in similarly arranged lattices[3]) (Tenney 1987, 72).

Harry Partch - Tonality Diamonds

Probably the most influential figure in contemporary tuning theory is Harry Partch. His models stand alone in modern tuning theory and are not ones that are directly reflected in my own models, but they have contributed to my understanding. His work

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[3] James Tenney's lattices are similarly arranged, with 3/2 (fifth) relations represented horizontally to the right and 5/4's (major thirds) vertically upward, although his lattices do not share the 45-degree positioning of the 5/4.

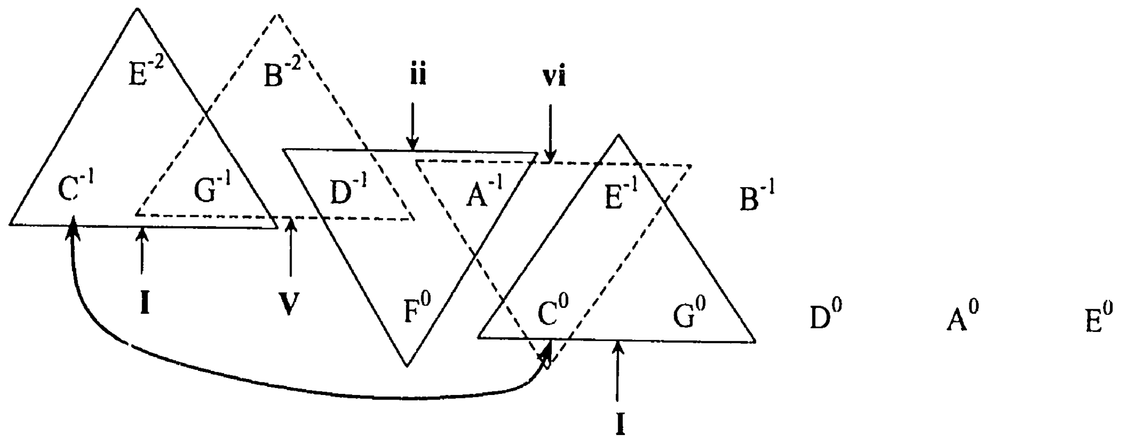

Figure 2.1: Demonstration of the syntonic comma using Barbour's system

warrants explanation due to its innovation and popularity, and also because I have chosen to adapt a few of his terms to label variations of my system.

In Genesis of a Music, Harry Partch introduces his readers to the concept of 'tonality diamonds'. Harmonic relationships are arranged in a diamond shaped lattice, with the reference pitch, which Partch calls the *prime unity*, located in the bottom corner. Ascending upward and to the right is a series of frequency ratios, in scalar order, based on the odd numbers up to the limit number of the system in use, or a power of one of those numbers. In the example (figure 2.2), an 11-limit system is presented, which generates the scale: 1/1 – 9/8 – 5/4 – 11/8 – 3/2 – 7/4. Partch calls these pitch sets "Otonalities". "O" for "overtone", and these represent the first eleven overtones of 1/1, excluding octave equivalence[4].

Ascending to the left from 1/1 is a descending scale based on the inversions of the Otonality scale: 1/1 –16/9 – 8/5 – 16/11 – 4/3 – 8/7. Partch calls these pitch sets "Utonalities", "U" for the theoretical "undertone". From each of these ratios, an ascending scale is built based on the ratios of the first Otonality. The result is a lattice with the generating tone running upwards through the middle of the diamond, expressed by a ratio where the numerator and denominator are the same: 1/1 – 9/9 – 5/5 – 11/11 – 3/3 – 7/7. The number involved in each defines the *identity* of each scale that it is a part of, and the generating tone is called the *Numerary Nexus*. Looking at the 9/9, it can be seen that 9 is the denominator of each interval of the Otonality scale that it is a part of and

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[4] **1/1** – 2/1 (octave of 1/1) – **3/1** (3/2) – 4/1 (double octave of 1/1) – **5/1** (5/4) – 6/1 (octave of 3/1) – **7/1** (7/4) – 8/1 (third octave of 1/1) – **9/1** (9/8) – 10/1 (octave of 5/1) – **11/1** (11/8).

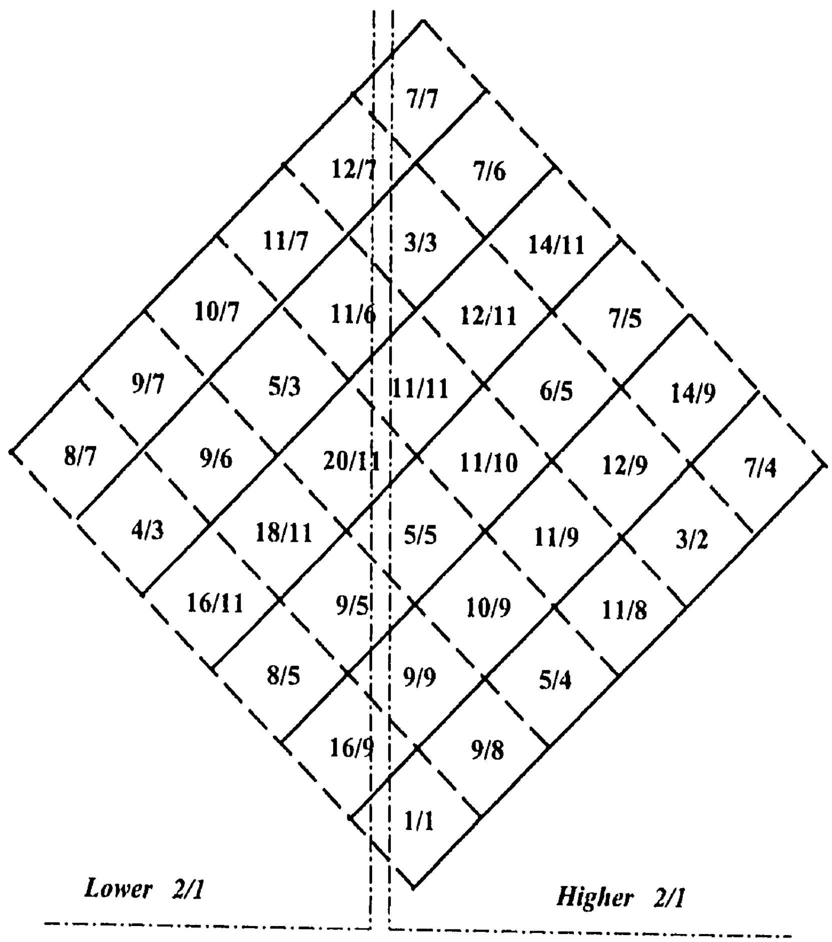
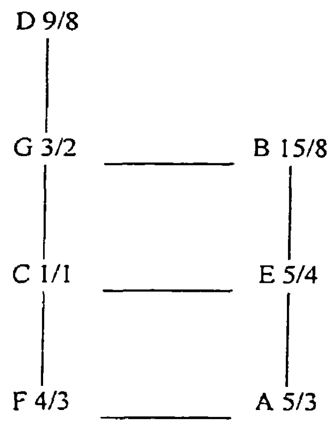


Figure 2.2: "The Expanded Tonality Diamond" (Partch, 159)

that 9 is the numerator of each interval of the Utonality it is a part of. This tonality diamond efficiently represents frequency ratios based on the first eleven overtones for each tone of the undertone series, representing 29 distinct pitch classes.

Ben Johnston

Ben Johnston uses a system in some ways very similar to Barbour's, but expanded to include higher-limit intervalic relationships. Johnston's starting point is slightly different from Barbour's in that Johnston assumes all unmarked pitches to correspond to the 5-limit just scale shown below. In Johnston's lattices, relations of a 5^{th} (3/2) are represented vertically, major thirds (5/4's) horizontally to the right. All intervals are described in relation to a single fundamental pitch, 1/1.



Instead of using exponents as Barbour does, Johnston uses the symbol "+" to indicate that a pitch is to be raised by an increment of the syntonic comma (81/80), approximately 21.5 cents in size, and the symbol "-" to lower a pitch by the same increment (Von Gunden, 62). A pitch is to be detuned by as many syntonic commas as there are symbols.

Sharps and flats have a unique, yet accurately descriptive, interpretation in Johnston's system. A sharp or flat changes the pitch by approximately 70.7 cents which is the interval 25/24 (Von Gunden, 62) and describes the difference between a 5/4 and a 6/5. In other words, Through the use of the symbols "♯", "♭", "-", "+", Johnston is able to notate any extended 5-limit tuning system.

To represent higher limit intervals, additional planes or dimensions are required. Johnston maintains the 5-limit configuration and adds a third dimension indicating relations of a 7/4 (pure seven-limit minor seventh) back and to the right, and its inversion, 8/7, towards the front and left. To notate a 7-limit system, the additional use of the symbols "7" and "L" (an up-side-down seven) are required to raise or lower a pitch by approximately 48.8 cents (nearly a ¼-tone). This comma, 36/35, is the difference between a 7/4 minor seventh and a 5-limit minor seventh 9/5. A 7/4 'B♭' from 'C' would be notated "B7♭". (B = 15/8 = 1088.3 cents - 70.7 cents - 48.8 cents = 968.8 cents).

Using a three-dimensional model, Johnston can indicate harmonic relationships in a seven-limit system. 45-degree angles indicate 7/4 and 8/7 relations, vertical positioning indicates 3/2 and 4/3 relations, and the horizontal plane represents 5/4's and 8/5's. Figure 2.3 is an example of a 3-5-7 lattice diagram, copied from Von Gunden, pg. 129.

Johnston also allows for higher limit systems. Each new prime coefficient requires a new symbol. The prime number eleven requires the addition of the symbols

"↑" and "↓". Prime numbers above and including 13 simply use the number as the symbol, and an upside down version for the inversion. Johnston also combines these symbols into special configurations. For example, often an arrowhead is attached to one of the arms of the sharp signs or the stem of a flat.

For lattice diagrams requiring more than three dimensions, Johnston uses parallel diagrams, each in three dimensions, usually separated by a 5/4 relation. For example, an 11-limit system is diagramed in several sets of 3-7-11 spaces related by 5/4's. A small 11-limit space is presented in figure 2.4.

The symbols used in Johnston's lattices are the same used in his musical notations. While Johnston's system is very elegant and theoretically efficient, the amount of decoding required of a musician can be quite extensive. It is for this reason that I have not adopted Johnston's notations. However, his system has been extremely helpful in my understanding of tuning theory. Elements of Johnston's lattices are apparent in my own diagrams, which are presented later in this chapter.

A lot of information is contained in any tuning lattice. Two pitch sets represented by the same shape, with the same directional orientation, have the same interval structure. For example, in the lattice in figure 2.3, a major triad always appears as a vertical step upward from the fundamental and a horizontal step to the right of the fundamental.

James Tenney - Harmonic Space

James Tenney's lattices represent not only harmonic relationships but also pitch

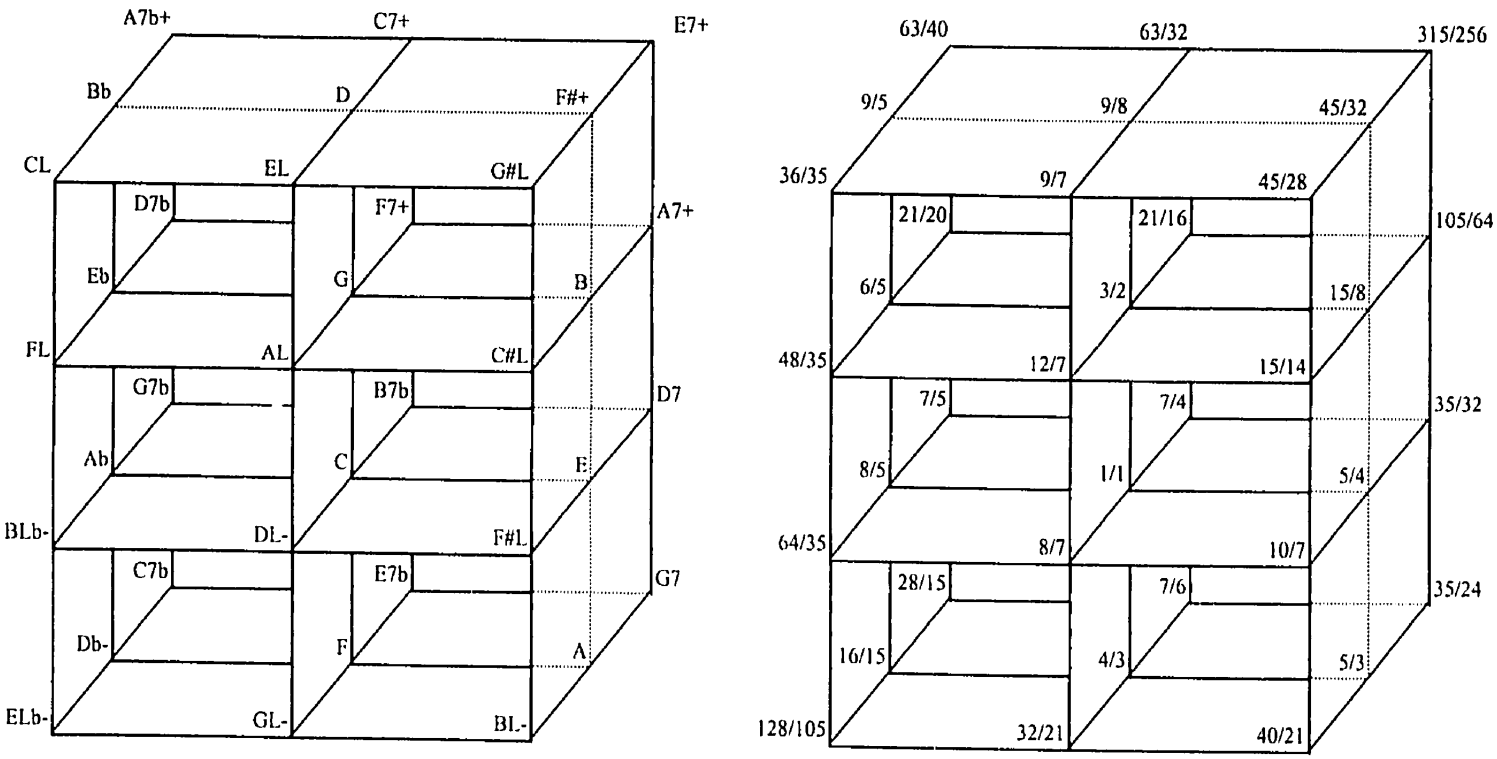


Figure 2.3: Johnston 3-5-7 Lattice (adapted from Von Gunden, 129)

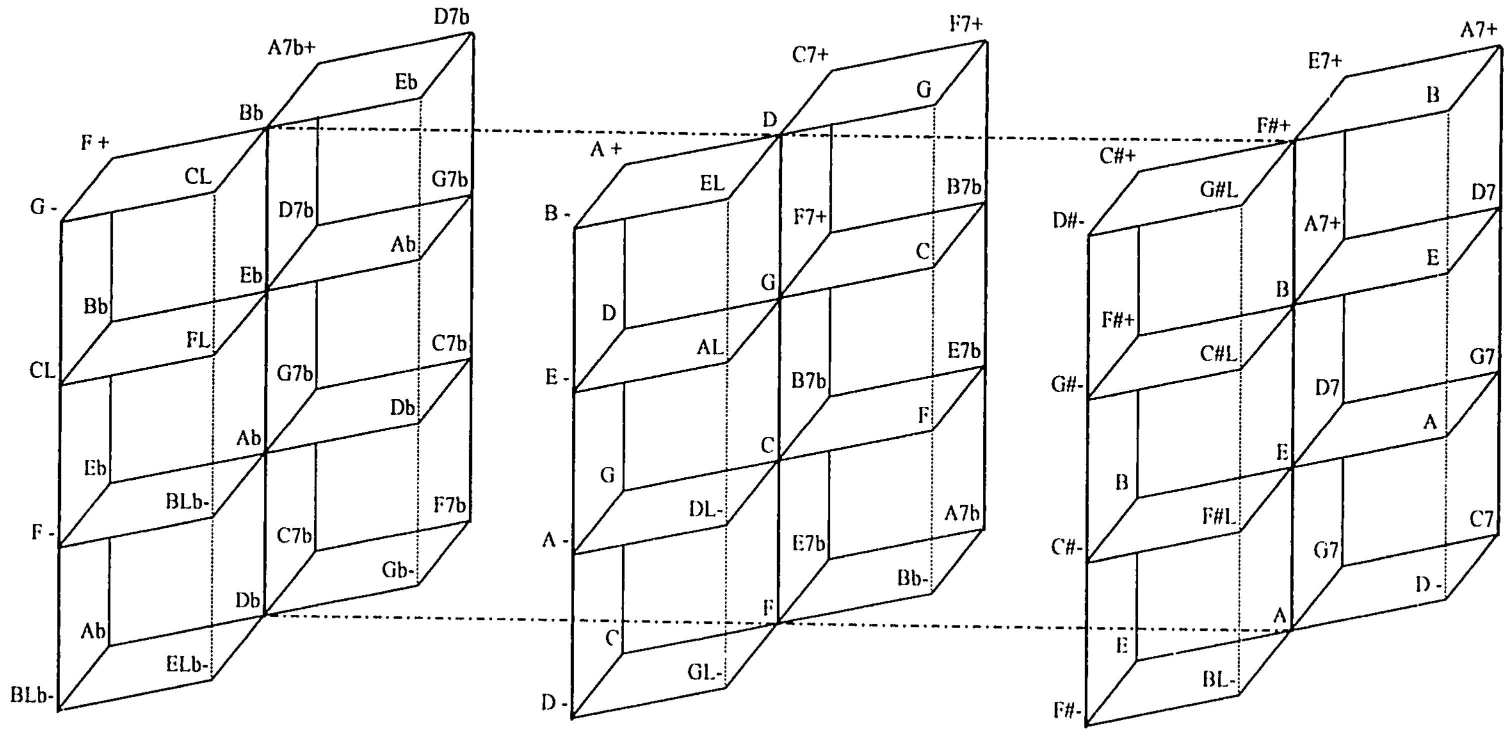


Figure 2.4: Johnston 3-5-7-11 Lattice (adapted from Von Gunden, 131)

height, the combination of which contributes to Tenney's concept of *harmonic space* (defined below).

> Cage has always emphasized the multidimensional character of sound-space, with pitch as just one of its dimensions. This is perfectly consistent with current acoustical definitions of pitch, in which – like its physical correlate, frequency – it is conceived as a one-dimensional continuum running from low to high. But our perception of relations between pitches is more complicated than this. The phenomenon of "octave-equivalence," for example, cannot be represented on such a one-dimensional continuum, and octave-equivalence is just one of several specifically harmonic relations between pitches – i.e. relations other that merely "higher" or "lower." This suggests that the single acoustical variable , frequency, must give rise to more than one dimension in sound-space – that the "space" of pitch-perception is itself multidimensional. This multidimensional space of pitch-perception will be called harmonic space. (James Tenney, 1983, 69).

In lattices that consider octave compounds, a central axis represents pitch-height. For each generating prime number, a new axis is generated that denotes intervallic relationships extending outward and upward from a reference tone on the pitch-height axis. The position of an individual frequency ratio represents its relationship to surrounding tones *and* its pitch height in relation to that "central axis of projection". The position of that point of projection is calculated logarithmically in relation to the fundamental frequency. As well, the *harmonic distance* between any two points in the lattice can be calculated by taking the logarithmic distance of the shortest path connecting the two tones[5]. The compactness of a set of pitches is indicative of the set's relative complexity. The lattice can also be collapsed to within a 2/1 to yield "a reduced pitch-class projection space with one fewer dimensions" (18). Figure 2.5 Shows a couple variations of James Tenney's *harmonic space* in which the first example preserves pitch-height and the second does not.

I have also adopted one of James Tenney's notation devices, a system that indicates the cents deviation from equal temperament above the note heads in a score via a plus or minus numeric value. This notation is used for all instruments that are capable of adjusting their intonation while playing (most winds and bowed strings). This notation represents an ideal towards which the musician is to strive.

Lattices Used in the Discussion of Compositions

My own lattices borrow heavily from those of Ben Johnston and James Tenney. For the most part, mine are simplified versions that represent harmonic relationships without respect to accurate harmonic distance. They do allow for the representation of multiple "dimensions" associated with higher limit intervals. For the most part, my system serves as a compositional aid rather than an analytical tool, which is why I have allowed for a simplified representation.

Where Tenney and Johnston represent higher limit materials with additional planes or dimensions, I have used the angle of a trajectory from a central reference to

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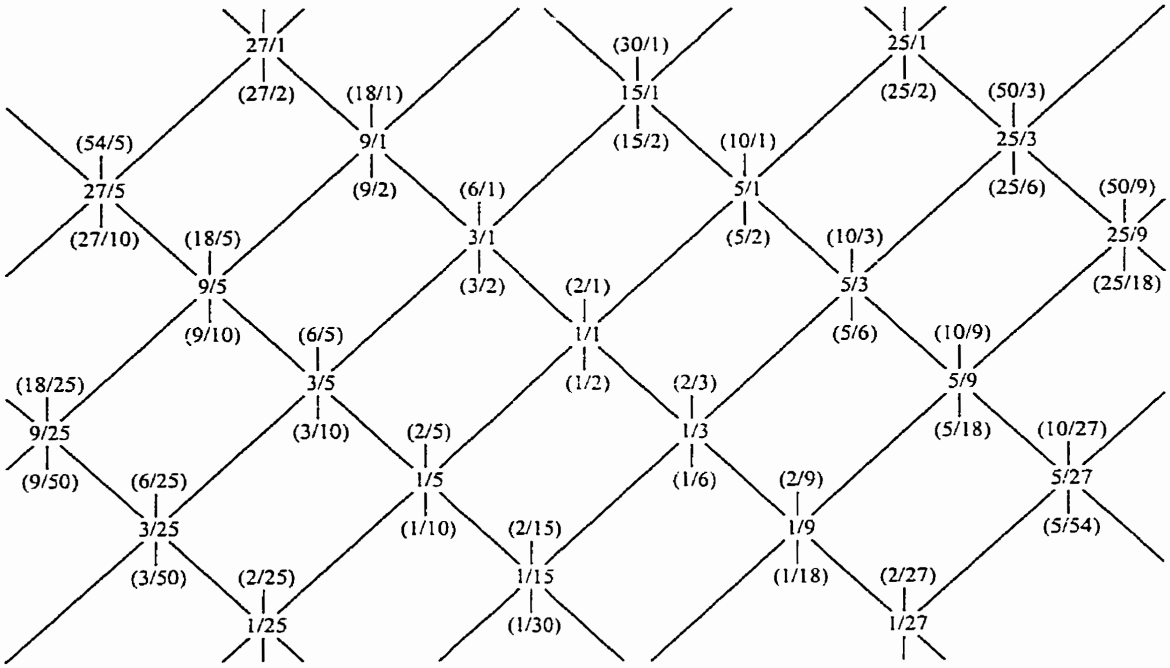
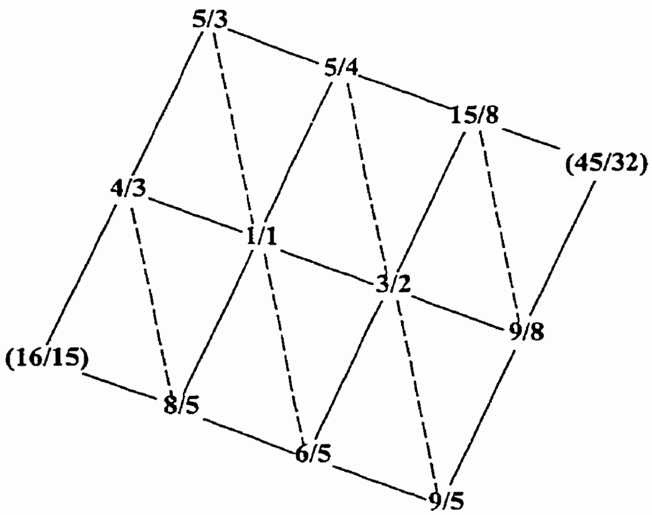
[5] See *John Cage and the Theory of Harmony* for a more complete explanation of *harmonic distance* and its calculation.



The 3,5 plane of harmonic space as a pitch-class projection plane within 2,3,5 space.



Primary harmonic relations within the chromatic scale

Figure 2.5: Examples of James Tenney's Harmonic Space (Tenney, 72-74)

differentiate ratio limits. The horizontal plane represents 3/2 relations and vertical planes, 5/4's. The lattice can theoretically extend infinitely in any direction, but for my purposes, I usually limit the *space* to a fairly compact size.

Assuming a 13-limit system, 7-limit materials extend at a 15-degree angle from the 5-limit reference. 11-limit materials are positioned a further 15-degrees, 30-degrees off the vertical axis. 13-limit materials are positioned at a 45-degree angle. Using this system, Figure 2.6 shows all of the intervals found in the first sixteen partials of the harmonic series.

This model clearly shows the relationship between all the intervals involved. For example, it can be seen that 14/9, 7/6, and 7/4 are related by 3/2's from their horizontal positioning with respect to one another. Or that 11/8 is a 5/4 from 11/10, reflected in their vertical relationship.

It should also be noted, as is the case in Johnston and Tenney lattices, that inversions occur in symmetrical geographic positions. For example, trace a path from 11/6 to its inversion 12/11 through 1/1. The path to 1/1 is symmetrical to the path from 1/1 to 12/11.

Problems arise when a ratio extends from a high-limit reference (seven and up). If a series of intervals were to be projected from one of these ratios, the lattice would become quite complicated and messy. This is solved by the initiation of a separate lattice in which the new reference is connected by a dashed line to its position in the source lattice. An example of this is demonstrated in the generation of a 7/4, 11/8, and 13/8

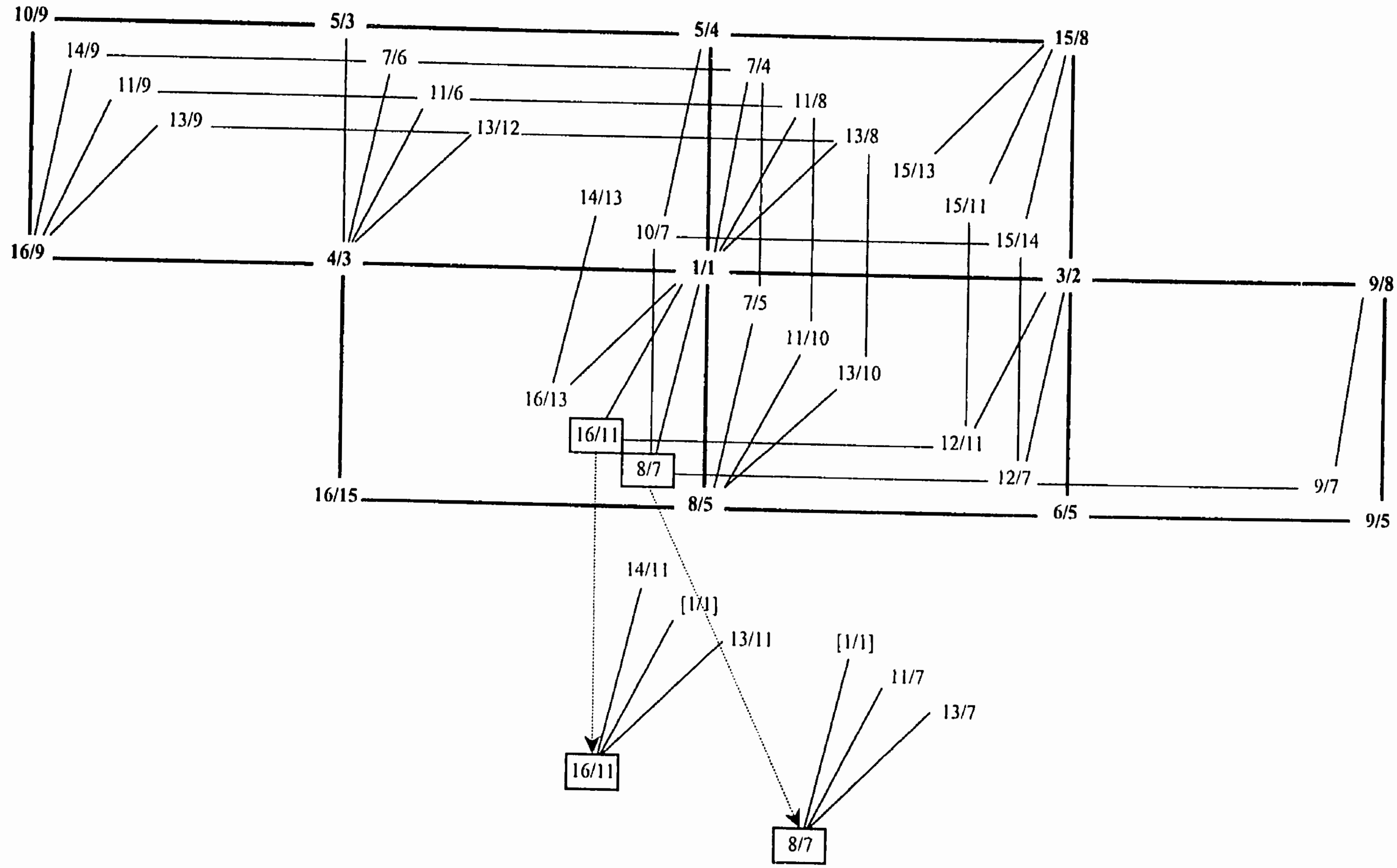


Figure 2.6: Intervalic Relationships Found in the Harmonic Series (to the 16th partial)

from 16/11 and 8/7.

In future chapters, this model will be used to describe the materials of individual compositions. When the space is relatively simple, the exact angle associated with different limit-ratios may be ignored for convenience.

Chapter Three: Temperament

In Genesis of a Music Harry Partch states that "the word temperament originally meant simply a system of tuning, any system, but in modern usage it applies specifically to a system which deliberately robs its intervals of their purity in order to implement the idea of every-tone-in-several-senses" (74). *Equal temperament* means specifically that an octave is divided into any number of equal parts, usually compromising the intonation of pure ratios for the convenience of fewer notes and the facilitation of modulation. The most common equal temperament is twelve-tone equal temperament (TET) in which the system is intended to approximate just 5-limit scales and harmonies.

The weakness of any temperament is the fact that the purity of the ratios is compromised. The power of a temperament is that each pitch is potentially ambiguous allowing for easy modulation and the ability to function in many different harmonic contexts. As well, tempering is a way of dealing with the anomalies that arise with various commas; for example, in TET three stacked major thirds equals an octave, this is not the case with just major thirds (5/4 x 5/4 x 5/4 = 125/64) which fall short of the octave by 41.1 cents.

With the rise of interest in just intonation, two camps have emerged: those that accept only absolutely pure ratios, and those that support convenience, good approximations, and a finite set of pitches with which to work (of course many crossover figures exist). Several equal temperaments have become popular in the twentieth century, most common are: 24-tone[1], 72-tone[2], 31-tone, and 19-tone[3]. All of these, to a varying extent, can be seen as an improvement with regards to tuning accuracy over twelve-tone equal-temperament in that all but one are capable of approximating 5-limit and higher ratios with greater accuracy. The 24-tone temperament is the exception in that it provides no improvement upon 3- and 5-limit materials. It does however approximate basic 7-limit ratios slightly better than TET and 11-limit ratios are almost perfect.

A comparison of these temperaments will be considered at the end of this chapter after the pertinent parameters have been established.

Defining a temperament

Several factors should be taken into consideration in establishing an effective temperament. Fundamental to any proposed temperament should be a consideration for: a) the threshold of pitch discrimination; b) the establishment of an allowable range of tuning that maintains harmonic function; c) the establishment of an allowable range of "in tuneness" that maintains harmonic "fusion" or "smoothness"; d) an acknowledgment of the capabilities of the intended performers; and e) an understanding of how these boundaries will change with the harmonic and melodic intention of any given composition[4].

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[1] More commonly known as 1/4 tone temperament.

[2] More commonly known as 1/12 tone temperament.

[3] Joseph Yasser was the twentieth century theorist who championed 31- and 19-tone temperament, reviving the 17th century theories of Christian Huygens.

[4] This last concern will be addressed in the discussions pertaining to individual compositions.

The Threshold of Pitch Discrimination

Several authors in the field of musical acoustics have established values for the threshold of pitch discrimination. In Information Theory and Esthetic Perception, by Abraham Moles, the author approximately defines three thresholds of pitch perception. "...for pitch, one has (a) a lower threshold on the order of f min = 16 cps; (b) an upper threshold on the order of f max = 16,000 cps; (c) a difference threshold (which varies greatly with f) averaging 0.5 per cent or 1 comma. As a result, there are about 1,200 distinct pitch levels" (12).

Looking at (c), the difference threshold, Moles provides three different numerical approximations. If the average, 0.5 per cent, is taken, a size-in-cents difference can be calculated for any frequency at 8.63 cents.

Eg. $\log_2(402 \text{ Hz} / 400 \text{ Hz}) \times 1200 = 8.63$.

The second approximation is established as being in the neighborhood of the size of a comma. Moles does not say to which comma he is referring, but regardless, it is easy to see that a 22-cent syntonic comma or a 23.5-cent Pythagorean comma is more than twice as large as the difference that the 0.5 per cent approximation reveals.

The third approximation claims that there are "about 1,200 distinct pitch levels". If his two thresholds of pitch perception are taken, 16 Hz and 16,000 Hz, then a difference threshold can be calculated at approximately 10 cents. The difference between 16 Hz and 16,000 Hz is approximately 10 octaves. 1,200 divided by 10 equals 120 increments per octave; 120 divided by 12 equals 10 increments per semi-tone (i.e. 10

cents). This value comes much closer to the 0.5 per cent value of 8.63 cents. It should be noted however, that these increments are not equal across the range of hearing.

In The Acoustical Foundations of Music, John Backus claims pitch discrimination levels to be on the order of 0.5 per cent for frequencies above 400 Hz (.3 per cent for exceptional ears or 5 cents), and quite a bit larger for lower frequencies (as great as 10 percent for a frequency of 30 Hz [Log_2 (33/30) = 165 cents!]). It should be noted that Backus' methodology involves the perception of the frequency *modulation* of a tone rather than the perception of the consecutive sounding of two distinct pitches.

Carl E. Seashore establishes an average difference threshold of 1/17 of a tone in Psychology of Music. "The average threshold for an unselected group of adults is about 3~ [5]at the level of international pitch, 435~. This is 1/17 of a tone, but a very sensitive ear can hear as small a difference as 0.5~ or less, which, at this level, is less than 0.01 of a tone" (56). This puts his calculations in line with the other two authors. 1/17 multiplied by 200 cents (a tone) equals 11.76 cents. The sensitive ear can distinguish a change on the order of 2 cents at 435 Hz.

A generous value based on the findings of these three authors is 5 to 10 cents for a difference threshold of pitch perception. If it is assumed that most subjects were non-musicians in these studies, than a slightly lower threshold is reasonable for the capabilities of the average musician. Seashore claims that although pitch perception does not improve with training, people with good pitch perception are more likely to be drawn

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[5] The symbol "~" is the equivalent of "cycles per second" or "Hz".

to music as a profession; therefore the pitch discrimination of musicians should be finer than that of the general population (58-59).

The Sense of Harmonic Function

For a temperament to work effectively, the function of a pitch in relation to other pitches must be maintained. In twelve-tone equal temperament, although the system closely resembles a Pythagorean tuning, the way in which the pitch materials are commonly used often implies a tuning system based on higher-limit ratios.

In a C7 chord[6], for example, a 'G' is well approximated in TET, being only two cents out of tune from a pure 3/2. The 'E' is tuned more closely to an 81/64 Pythagorean major third but its context strongly implies a 5/4 major third. The tempered 'E' sounds approximately 7.8 cents flat of an 81/64 and 13.7 cents sharp of a 5/4. The seventh, B♭, is tuned 6.1 cents flat of a 16/9 minor seventh and 31.2 cents sharp of a 7/4 minor seventh. James Tenney argues that the 7/4 minor seventh is the implied interval and the one to which a listener tries to adjust. If the chord is extended further, function and tuning become increasingly more difficult to reconcile. A major ninth (D) does not present any real problem, as it is a well-approximated 3-limit ratio (9/8), out by 3.9 cents. An augmented eleventh added to this chord, however, is extremely problematic. Does the listener hear the F♯ as a 5/4 built from the 9th (45/32 from C) sounding 9.8 cents sharp?

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[6]This explanation applies specifically to a chord sounding in isolation; i.e. without regard for the occurrence of a chord through circumstances of voice-leading where the "ideal" tuning of the sonority may be different from its simplest rationalization in harmonic space, the subject of which is beyond the scope of this paper.

Is it heard as an eleventh partial of the root 'C' (11/8) sounding 48.7 cents sharp? Or is it heard as a 7/5 augmented eleventh, sounding 17.5 cents sharp? A thirteenth added to the chord creates similar problems with regards to implied harmonic function. The voicing of a chord will also contribute to the understanding of harmonic function.

For a temperament to be truly effective, ambiguities should not exist when a pitch is used in a specific context. Deciding on a practical margin of tolerance is difficult. Obviously, our European predecessors felt that a 14-cent margin of error was acceptable for 5-limit intervals. However, the majority of 18th century tuning theorists were not thinking in terms of 7-limit systems so it would be unfair to speculate that a 31-cent margin of error for dominant seventh chords was supported.

The second problem in choosing a tolerance range is that, as harmonic complexity increases so does the fineness of tuning required to maintain the sense of harmonic function. James Tenney states on page 15 of his article *John Cage and the Theory of Harmony* that:

> Since our perception of pitch intervals involves some degree of approximation, these frequency ratios must be understood to represent pitches within a certain tolerance range – i.e. a range of relative frequencies within which some slight mistuning is possible without altering the harmonic identity of an interval. The actual magnitude of this tolerance range would depend on several factors, and it is not yet possible to specify it precisely, but it seems likely that it would vary inversely with the ratio-complexity of the interval. That is, the smaller the integers needed to designate the frequency ratio for a given interval, the larger its tolerance range would be.

An octave can be quite out of tune before it is considered something other than an octave. An 11/8 interval, however, must be tuned much more precisely in order for it to be recognized as such. But context also influences the sense of function, sometimes reducing the need for fine-tuning. The triadic example of European music (1600-1900) proves this point to a certain degree, but conditioning must also be taken into consideration.

Another phenomenon that may help to define an allowable margin of error for the maintenance of harmonic function is explained by a series of experiment conducted by Moore, Peters, and Glasberg described in *Thresholds for the detection of inharmonicity in complex tones*. "The object of the experiment was to determine the amount by which a partial in a complex tone had to be mistuned from its harmonic value in order for inharmonicity to be detected" (1862). The results of this experiment state that:

> Thresholds for inharmonicity, expressed as percent mistuning of the partial concerned, decreased progressively with increasing harmonic number and with increasing fundamental frequency… The pattern of results appears somewhat different if the thresholds are expressed simply as mistuning in Hz. Then thresholds vary little with harmonic number or fundamental frequency, covering a range from 2.4-7.3 Hz (1863).

Using a mean value of 4 Hz, the second partial of a tone generated at 400 Hz has a tuning threshold of 8.6 cents. The third, fifth, seventh, eleventh, and thirteenth partials have thresholds of 5.8, 3.5, 2.5, 1.6, and 1.3 cents respectively.

The actual thresholds defined above are not directly useful in this discussion as the experiment was conducted with very short tones of less than 1610 ms, however the tolerance supports a range of less than 10 cents. The conclusion that the thresholds decrease (when measured logarithmically) with harmonic number seems analogous to the supposition that the intonation threshold for harmonic function decreases as ratio complexity increases.

The Maintenance of Harmonic Fusion

We saw above that as frequency ratios become more complex, tuning must become more precise in order to maintain harmonic understanding. The reverse is true in order to maintain harmonic fusion (or "smoothness")[7]. While an out of tune octave is easy to understand as such, the roughness of a mistuning is more difficult to reconcile. This notion is supported (although conditioning is likely a major factor) through the general acceptance of twelve-tone equal-temperament: Octaves are tuned pure (ignoring the fine points of piano tuning – i.e. the stretching of the octaves, etc.); 5ths, 4ths and 9ths (3-limit ratios) are approximately 2 to 4 cents out of tune; 3rds, 6ths, major 7ths, and minor 2nds (5-limit ratios) are 12 to 16 cents out of tune; and minor 7ths (7-limit ratios) are 31-cents out of tune. This shows a historical precedence for the finer tuning of simple ratios and greater latitude for more complex ratios.

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[7] These are subjective terms used to describe how well the components of an interval or chord blend with each other. The terms are somewhat inversely related to "roughness" or "beating".

Twelve-tone equal temperament demonstrates a certain level of tolerance with regards to tuning, although the only intervals that maintain harmonic fusion or smoothness to my ears are octaves, 5ths, 4ths, and 9ths. As well, though only through coincidence, 17 and 19-limit intervals are well enough approximated by TET to maintain harmonic fusion. For example, a chord voiced above middle C: Root – maj.9th – min.10th (1/1 – 9/4 – 19/8) or Root – maj.2nd – min9th (1/1 – 9/8 – 17/8) sounds relatively smooth. These are purely subjective responses, and admittedly, more sensitive ears claim not even to accept equal-tempered 5^{ths} and 4^{ths}.

A possible, though not proven, theory for tuning tolerance is provided by Arthur H. Benade in his text, Fundamentals of Musical Acoustics. (Pg. 271-277). With complex tones, the number of partials that line up in a harmonic relationship is most complete with simple intervals. Each partial has the potential to beat when out of tune with any other partial of the other tone when it is within a close frequency range[8]. The strongest heterodyne[9] frequencies are also considered in these measurements. The group of frequencies that fall within this close range are known as "indicators", the number of which occur determines the type of indicator; either "double", "triple", "quadruple", etc.

Benade uses two tones with four harmonics in each to calculate the number of tuning references between the two tones. In a 2/1, he identifies one triple indicator, four

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[8] The difference of the two frequencies must be less than the lower threshold of hearing, otherwise the result is a hyterodyne tone (see below).
[9] Heterodyne frequencies are caused by non-linearities in the ear producing sum and difference tones between the partials of one or more source tones. A sine tone can also induce a perceived complex tone. Heterodyne frequencies are also referred to as subjective tones or combination tones.

quadruple indicators, and three quintuple indicators. Compare this to a 7/6, for example, which only has three double indicators (274). The following deduction can be made from this: as the complexity of a ratio increases the number of indicators for tuning decreases.

The Physiological Limits of Intonation

Although it is not an issue with the guitar itself, a consideration for the capabilities of performers on other instruments may be useful in defining a reasonable margin of error for an effective temperament. This criterion is most pertinent when fixed-pitched instruments (such as the guitar) are used in combination with variable-pitched instruments (violins, voice, winds).

Carl Seashore states the obvious in that "naturally one cannot control pitch any finer than he can hear it" (74). Therefore, the physiological limit (i.e. muscular control) can only ever approach the cognitive limit, defined by the fidelity of the ear.

In Fundamentals of Musical Acoustics, Arthur Benade identifies three groups of tuning tendencies in musicians playing common intervals.

> A tabulation comparing...the location of a given equally tempered note with the places it would need to be to permit perfectly tuned transitions to it from *any other note* in the scale shows that the most-needed settings gather themselves roughly into three groups. One group extends over a range of about 7 cents clustered at a point about 12 cents below the equally tempered setting; a similar group collects around a setting that is 12 cents above equal temperament, and a third collection of settings is found in the immediate neighborhood of the equal-tempered note (295).

It would seem natural to assume that the clusters at plus/minus 12 cents pertain to 5-limit ratios and that the clusters in the equal tempered vicinity pertain to 3-limit intervals; however, in The Acoustical Foundations of Music, John Backus states that in testing violinists, the tendency for tuning major thirds is towards sharpening the pitch, and the tendency for minor thirds is to flatten. This, he claims, indicates a tendency towards Pythagorean intervals. While it seems unlikely to me that musicians are predisposed to Pythagorean intonation in classical musical settings, and that voice-leading seems a more likely explanation for this phenomenon, it is beyond the scope of this thesis to investigate for what reasons these tendencies exist.

Benade also notes that if a musician is asked to play a little sharper or a little more flat, the tendency is to change the pitch by approximately 10 cents (196). It is not clear whether this value suggests a physiological or cognitive threshold, but it does lend additional support to other measures of pitch discrimination.

The Five-Cent Margin of Error

Although the information presented above all point to a reasonable margin of error for an effective temperament at roughly less than ten cents, I had already adopted a five-cent margin for my own work prior to this research. James Tenney, whose work often uses this same tuning margin, originally suggested this number to me.

James Tenney uses a 5-cent margin to explain how accurately a pitch must be tuned to ensure that it is interpreted as the ratio intended. For example, to ensure that a 7/6 minor third is differentiated from a 32/27 minor third and from a 6/5 minor third, a 5-cent resolution will help to clarify. A 7/6 is 33 cents flat from an equal-tempered minor third. Any pitch tuned between -27 and -38 cents should be interpreted as a 7/6. A 32/27 is approximately -6 cents from its equal-tempered equivalent and therefore must be tuned between -1 and -11 cents, however as this is a relatively complex ratio, even an absolutely pure tuning may not clarify its function when taken out of context. A 6/5 is 16 cents sharp of an equal-tempered third and must be tuned between +11 and +21 cents to be interpreted as such.

Between these tuning "radii", some confusion will exist as to the function of the interval. While context has a great deal to do with interpretation and increases the acceptable limits, the five-cent deviation limit helps to clarify function. An area of ambiguity lies between intervals that occur in close intonational proximity, although it is not likely that a 75/64 would be the interpreted interval over a 7/6 with a pitch tuned -25 cents from equal-temperament. Generally, the simpler of two intervals that are closely in tune with each other will be the preferred interpreted function.

A Comparison of Popular Temperaments

For the sake of comparison, all of these temperaments will be measured for effectiveness in approximating the following expanded 13-limit just scale: 1/1 - 16/15 - 9/8 - 8/7 - 7/6 - 6/5 - 16/13 - 5/4 - 4/3 - 11/8 - 7/5 - 10/7 - 16/11 - 3/2 - 8/5 - 13/8 - 5/3 - 12/7 - 7/4 - 16/9 - 15/8. This scale is built from the 5-limit scale presented in chapter one, to which is added missing reciprocal 5-limit intervals, some common 7-, 11-, and 13-limit ratios and their inversions, and 7/4 replaces the 9/5. The harmonic space of this scale is presented in figure 3.1. It can be assumed that any temperament will approximate the inversion of any ratio as well as the original ratio. Therefore, the following comparison will only consider the simpler of any two reciprocal ratios.

To calculate the frequency ratios involved in a temperament, the n^{th} root of two is taken where n = the number of pitches per octave. This provides a frequency ratio for the distance between any two adjacent tones in the system. For twelve-tone equal-temperament, the 12^{th} root of 2 is taken giving a ratio of 1.059463094 for a semitone. The size-in-cents of this ratio is, predictably, 100.

$$^{n/1}\sqrt{2}$$

The frequency ratio for a step in a given equal-temperament to the power of x, where x = the number of steps in an interval, gives a ratio for any interval found in the scale.

$$(^{n/1}\sqrt{2})^{x}$$

or

$$2^{(x/n)}$$

The following table presents the ratio involved in adjacent degrees of each of the temperaments mentioned above, and a size-in-cents value for each.

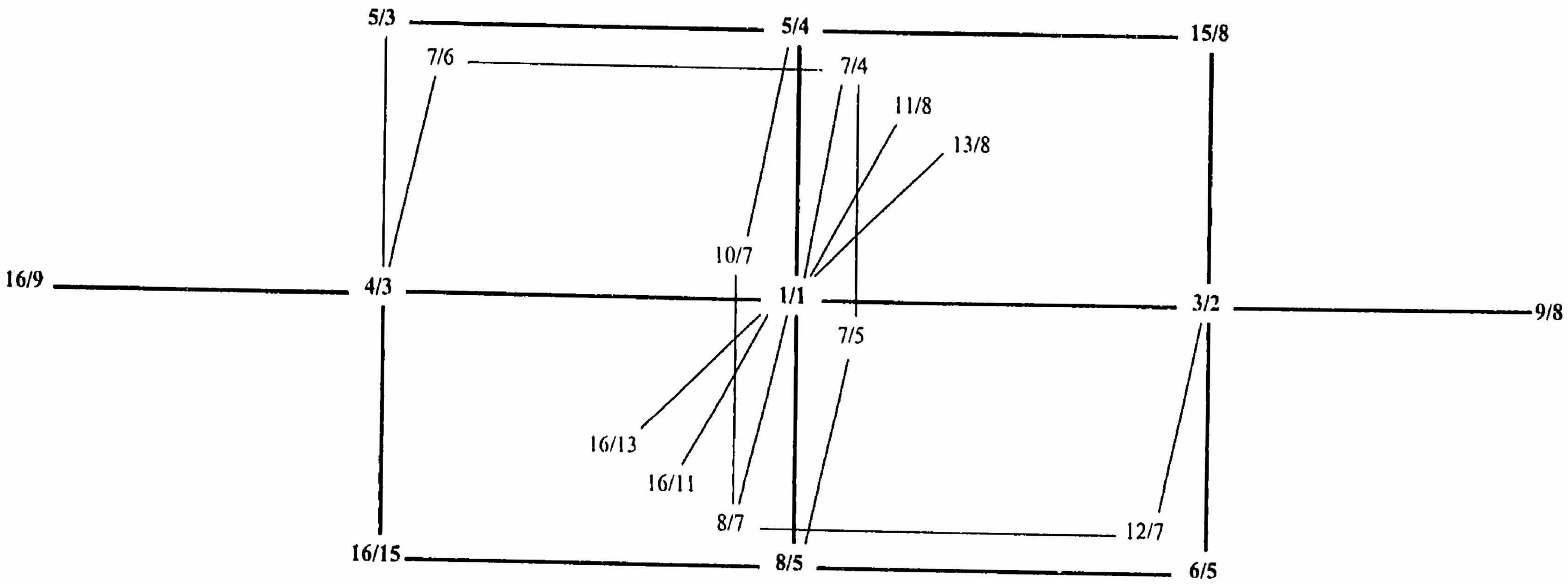

Figure 3.1: 13-Limit Just Scale

	TET	*24-tone*	*72-tone*	*19-tone*	*31-tone*
freq. ratio	1.059	1.029	1.010	1.037	1.023
size-in-cents	100	50	16.67	63.16	38.71

We can now compare how well each of these temperaments approximates the ratios of the just scale laid out above by finding the closest interval available from each temperament.

	9/8	*7/6*	*5/4*	*4/3*	*11/8*	*7/5*	*13/8*	*5/3*	*7/4*	*15/8*
JI	203.9	266.9	386.3	498.0	551.3	582.5	840.5	884.4	968.8	1088.3
TET	200	300	400	500	600	600	800	900	1000	1100
24	200	300	400	500	550	600	850	900	950	1100
72	200	266.7	383.3	500	550	583.3	833.3	883.3	966.7	1083.3
19	189.5	252.6	378.0	505.3	568.4	568.4	821.1	884.2	947.4	1073.7
31	193.5	271.0	387.1	503.2	541.9	580.6	851.6	890.3	967.7	1083.9

By taking the average error of each temperament, we can begin to consider which temperaments most effectively approximate the just intervals. These errors are slightly weighted toward to the more prominent 3- and 5-limit ratios and away from the lower occurrence of higher-limit ratios.

Average Error	
TET	21.8 cents
24-tone	12.7 cents
72-tone	2.66 cents
19-tone	13.1 cents
31-tone	5.4 cents

These measurements only convey the accuracy of intonation for each system. The practicality issue has been ignored. Obviously, the more tones per octave, the smaller the maximum possible error. If the smallest interval in the 72-tone temperament is 16.7 cents, we cannot expect the average error to be any greater than half; 8.35 cents. The maximum possible error is 1/6th the size of TET's.

Although the 31-tone temperament, at half the pitches, appears to be as accurate as the 72-tone temperament, closer inspection reveals a flaw. While 5-limit ratios are well approximated, 3-limit ones (especially the 9/8) are poorly approximated. In terms of accuracy of intonation, it is clear that the 72-tone temperament is the superior one. However, other factors should be taken into consideration, such as the number of tones and the limit number of the system it is intended to represent. All the above temperaments have their strengths. In general an economy of pitches is sacrificed as tuning accuracy increases.

For the purposes of this thesis, the 72-tone temperament is the one to which the proposed 15-cent temperament will be compared in the next chapter, as the approximation of just-intervals, not an economy of means, is the most important issue to the compositions involved.

Chapter Four: The Fifteen-Cent Guitar

Conventions and Notation

To indicate the fingering for specifically tuned pitches and chords, the following convention is used; guitar strings are numbered VI through I, beginning with the low E-string (VI) and ending on high E-string (I), and frets are numbered using Arabic numerals, zero indicating that an open string is to be struck. For example, a basic G major chord is indicated VI-3, V-2, IV-0, III-0, II-0, I-3. This is consistent with conventional orchestration methods but is reversed from typical pop guitar conventions in which the frets are typically indicated by Roman numerals rather than Arabic.

On a staff, only Roman numerals are used above each note head. If more than one pitch is sounding at once then the Roman numerals are placed vertically above the chord or interval in a placement respective to the voicing of the sonority. The performer must play the note at the fret where the indicated pitch occurs on that string.

If a string has been detuned more than 50 cents, the score will be transposed so that the performer plays as if tuned to the standard equal-tempered pitches. As well, the notation maintains the standard of being written an octave higher than it sounds.

The "Oguitar"

The 15-cent temperament for guitar tuning is a flexible system which sprouts from this basic setup: Starting with the A-string, each subsequent (higher) string is detuned in increasing decrements of 15 cents from their normal equal-tempered tuning. The A-string remains tuned to 'A'=110 Hz., the D-string is flattened by 115 cents to $C\sharp_3^{-15}$, the G-string is flattened by 30 cents to G_3^{-30}, the B-string by 45 cents to B_3^{-45}, and the high E-string by 60 cents to $D\sharp_4^{+40}$. For Partch fans, this basic setup can be thought of as the "Oguitar" (from Partch's "Otonality"), in that each string represents a higher prime- numbered harmonic from the previous. If the open A-string represents the fundamental, then the open C♯-string has a 5/4 relationship to A_3, the open G-string is in a 7/4 relationship, an $E\flat_4$ played on the B-string has an 11/8 relationship, and an $F\sharp_4$ on the high E-string has a 13/8 relationship (note that the 'F♯' actually sounds closer to an equal tempered 'F'). This chord is fingered V-0, VI-0, III-0, II-4, I-2.

Relationship Between Strings

Looking at this basic set up, several observations can be made: Between any adjacent strings (ignoring the low E-string for now), the lower pitched string can represent the fundamental and a series of tempered fifths and fourths in a 3-limit system. The next higher string represents the 5-axis. The 5-axis string also represents a string of tempered fifths and fourths along that 5-axis. This relationship is true of any two adjacent strings.

Any two adjacent strings can actually represent any two adjacent prime number axes in a 13-limit system: 3 and 5 (as above), 5 and 7, 7 and 11, and 11 and 13.

Between any two non-adjacent strings separated by one string (i.e. between V and III, IV and II, or III and I), the lower string can represent the 3-axis and the higher string the 7-axis (in a string of tempered 5ths and 4ths). Or any non-adjacent axes separated by one prime number, up to 13; 3 and 7 (as above), 5 and 11, or 7 and 13.

Non-adjacent strings separated by two strings represent the relationship between a 3-axis and an 11-axis, or a 5-axis and a 13-axis. Separated by three strings, a 3-axis and a 13-axis are represented.

Complex Harmonic Relationships

Harmonic relationships of higher complexity can also be represented in this system. An augmented triad can be represented by a pitch on a string serving as a 3-axis (1/1), a 5/4 relationship above that and another 5/4 interval from the second pitch (or 25/16 from the fundamental). Although 25/16 is a 5-limit ratio, it appears on a separate axis above the 5-axis in harmonic space (the 25-axis) but is available on the 7-axis string. Therefore, three adjacent strings could represent the 3, 5 and 25-axes of harmonic space. In harmonic terms, an augmented triad built on 'A' is A (1/1), C$\sharp^{-13.7}$ (5/4), and E$\sharp^{-27.4}$ (25/16), fingered V-12, IV-12, III-10. It can be seen that all these pitches fall into the allowable 5 cent tuning range (C♯ is 1.3 cent out of tune and E♯ is only 2.6 cents out of tune from pure).

Other complex spaces can also be approximated. Imagine a chord built from an 'A' fundamental: 1/1 - 5/3 - 35/24 (A - F$\sharp^{-15.6}$ E$^{-46.8}$), fingered V-0, IV-4, II-4. The 35/24 is a

7/4 from the 5/3. Again, the largest deviation from pure is well within the allowable limit, 5/3 is out by 0.6 cents and 35/24 is 1.8 cents out.

A pure sounding tritone can sound on adjacent strings. The 7/5 is 582.5 cents in size, which is 17.5 cents flat of a tempered tritone. In harmonic space, this interval occurs at a 7/4 from 8/5 and is fingered V-0, IV-2.

On occasion, I will use the Oguitar in a very slightly modified version based on the low E-string instead. The A-string will be flattened by 15 cents, the D-string by 30 cents, the G-string by 45 cents, the B-string by 60 cents to $B\flat_3^{+40}$. The High E-string can be treated in two different ways. Detuned by 75 cents, it can be used in any 13-limit or lower relationship with any other string other than the low E-string. Most often, I will tune this high E-string normally (i.e. with no cents deviation). This allows for voicings of 3-limit materials above the lower strings; it also approximates 17- and 19-limit ratios very well.

The "Uguitar"

The "Uguitar", as the name implies, facilitates the easy execution of intervals based on the theoretical undertone series, which is derived by integer divisions of a fundamental frequency. For the purpose of this thesis, these ratios are simplified as specified in chapter one. For example, a set of ratios based on the first six prime numbers is 1/2, 1/3, 1/5, 1/7, 1/11, 1/13. These simplify respectively to 1/1, 4/3, 8/5, 8/7, 16/11, and 16/13. On the Uguitar, each subsequent string relates to increasing divisions of the generating tone. The A-string is tuned to A_2=110 Hz., the D-string is tuned 15 cents sharp, the G-string is 30 cents sharp, the B-string is 45 cents sharp, and the E-string is 60 cents sharp (or F^{-40}). It should be noted that both types of intervals are available on either guitar, except that the particular setup makes one type more easily accessible on the corresponding guitar. For example, a 10/7 can be sounded on the Uguitar with IV-5 and III-6, or a 7/5 on IV-11 and III-0.

Pitch-Height Projection and Tuning Diagram

Figure 4.1 shows a method I've developed for showing the cents-deviation of a given ratio in a 15-cent temperament from twelve-tone equal-temperament, and its pitch height within an octave. Each radius extends in 15-degree increments from the central vertical plane, any group of which can be thought of as adjacent strings on either the Oguitar (looking from left to right), or the Uguitar (looking right to left). The central vertical plane represents the string on which the generating tone occurs. On each radius, a group of intervals are indicated and positioned according to where they fall in respect to the octave, scaled logarithmically. All intervals are projected onto the central vertical plane to show the relationship of all the pitches in scalar form. This diagram, like the pitch-height diagram explained in Chapter Two does not endeavor to portray harmonic relations

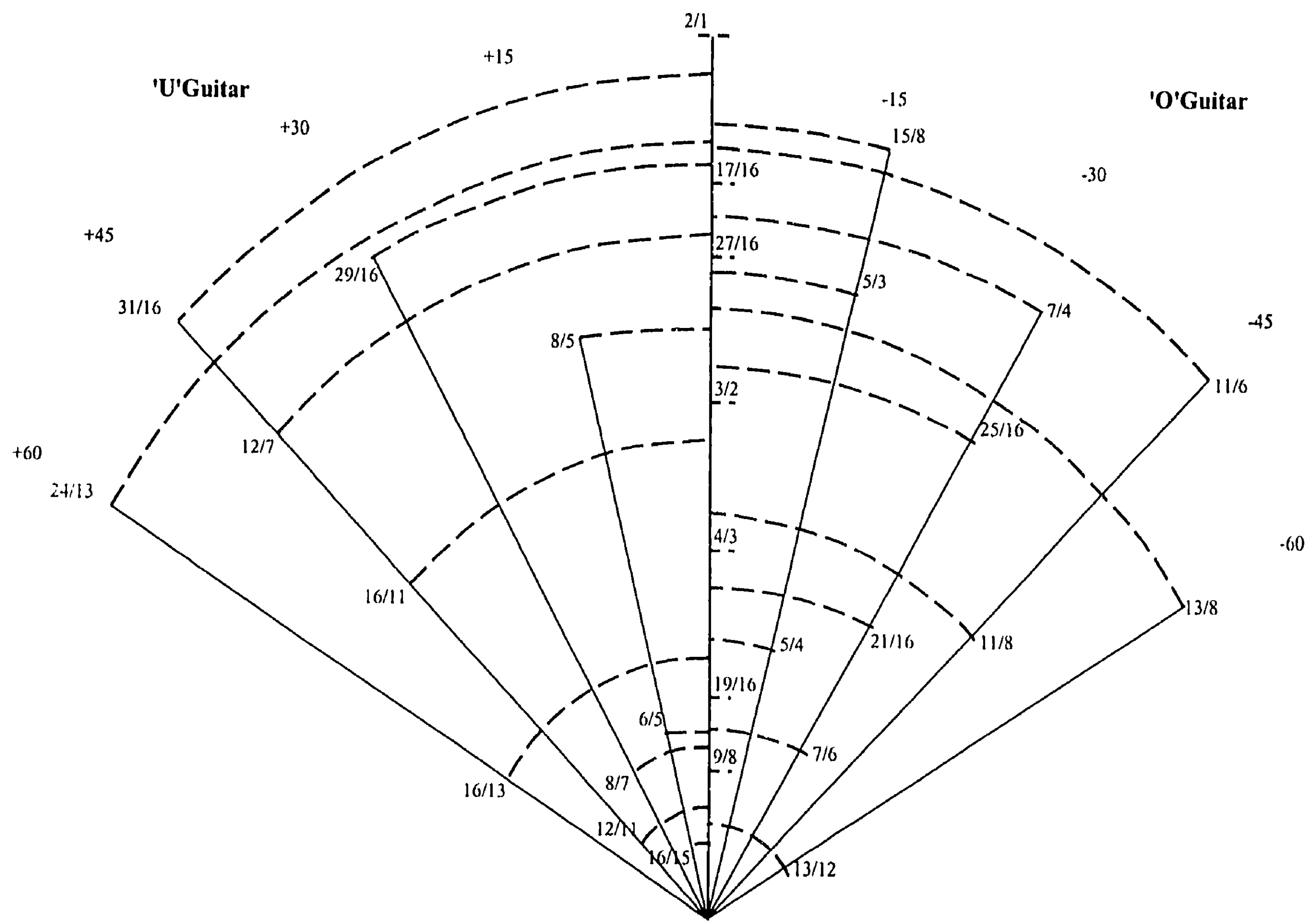

Figure 4.1: Pitch Projection of Ratios Available on the Fifteen-Cent Guitar

Implication of the Fifteen-Cent Increment

While the decay portion of the amplitude envelope of a plucked guitar string can be tuned within a relatively precise measure, the nature of plucked strings limits how accurately the fifteen-cent temperament can be implemented, since the frequency of the attack phase of an excited string will tend to sound sharper than its decay frequency. Intonation challenges are even greater for instruments of unfixed pitch. The fifteen-cent increment represents an ideal towards which the musician strives (we can expect from the discussion in the preceding chapter that no musician can consistently achieve an intonation better than within five cents). However, the true goal of the fifteen-cent temperament is to push the musician toward a class of interval, or an area of implied harmony, which his or her instincts would not necessarily go in certain harmonic, or especially melodic, situations.

The fifteen-cent increment allows for the quick processing of interval class by its association with certain limit numbers. In a simple 13-limit tuning system, plus or minus 15 implies a 5-limit interval (5/4, 5/3, 15/8, 8/5, 6/5, 16/15), plus/minus 30 implies basic 7-limit materials (7/4, 7/6, 21/16, 8/7, 12/7), etc.

Justification of a 13-Limit

My initial interests in just intonation were primarily concerned with a 13-limit system. For the most part, this is a logical stopping point. Although it is arguable that the TET can be thought of as an approximation of a 7-limit system, representing the first four prime numbers, each of these prime numbers has one interceding integer (with the exception of 2-3), all of which are even numbers. Harry Partch made the logical step of expanding to the next higher prime number eleven, which is separated from the prime number seven by two even numbers (8 and 10) and the first non-prime odd number (9). While Partch admits that eleven was an arbitrary stopping point, the expansion of harmonic relationships is quite large and represents the first jump to vastly unfamiliar pitch materials. As well, the 11-limit closely resembles the efforts of other microtonalists not concerned with just intonation (Ives, Varese) who were using quarter tones in the early twentieth century.

Once the leap to eleven has been made, the expansion to thirteen seems to naturally fall into place because of its numeric proximity to the number eleven. Although this is a subjective observation, prime numbers often occur in pairs in the lower digits; 1 and 3, 5 and 7, 11 and 13, 17 and 19, 23 (an exception), 29 and 31, 37 (exception), 41 and 43, etc. It seems natural to me that the expansion of harmonic materials based on new limit-numbers should occur in these pairs.

The approximation of a thirteen-limit system was my initial goal, primarily motivated by the simple fact that Partch had stopped at eleven. However, once the fifteen-cent temperament had been laid out, the possibilities for higher-limit systems became apparent. I arbitrarily decided to stop at the prime number 31 for compositional work but the fifteen-cent temperament is very effective well beyond the 31st partial.

A Comparison to the 72-Tone Temperament

Below is a comparison between the 72-tone temperament and the 15-cent temperament in their ability to approximate a selection of intervals occurring in the first 31 partials of the harmonic series, listed first in limit-number order and then by relative complexity (again, reciprocals are not considered).

Ratio	*Just*	*72-tone*	*Error*	*15-cent*	*Error*
4/3	498.0 cents	500 cents	2.0 cents	500 cents	2.0 cents
9/8	203.9	200	3.9	200	3.9
5/3	884.4	883.3	1.1	885	0.6
5/4	386.3	383.3	3.0	385	1.3
9/5	1017.6	1016.6	1.0	1015	2.6
15/8	1088.3	1083.3	5.0	1085	3.3
25/16	772.6	766.7	5.9	770	2.6
7/4	968.8	966.7	2.1	970	1.2
7/5	582.5	583.3	0.8	585	2.5
9/7	435.1	433.3	1.8	430	5.1
21/16	470.8	466.7	4.1	470	0.8
11/8	551.3	550	1.3	555	3.7
11/9	347.4	350	2.6	345	2.4
11/7	782.5	783.3	0.8	785	2.5
13/8	840.5	833.3	7.2	840	0.5
13/7	1071.7	1066.7	5.0	1070	1.7
17/16	105.0	100	5.0	100	5.0
19/16	297.5	300	2.5	300	2.5
31/16	1145.0	1150	5.0	1145	0.0
Average Error:			**3.2 cents**		**2.3 cents**

Both systems are really quite effective according to the terms defined in Chapter Three and without weighting, the average error is not very telling on its own. Although the two systems are not significantly different, they each seem to be stronger where the other is weakest (compare 13/8 and 11/8). Importantly, the 15-cent temperament is strongest with low number ratios, the notable exception being the 9/7, and only one other interval has an error above or equal to the 5-cent margin; the 17/16, which the 72-tone system similarly approximates. It can be concluded that the 15-cent temperament is marginally superior with respect to tuning accuracy, however, the clean logic and recurring semitones of the 72-tone temperament is an advantage that cannot be ignored.

Tuning the 15-cent Guitar

Assuming the basic Oguitar set-up [E – A – C♯ $^{-15}$ – G^{-30} – B^{-45} – E^{-60}] where the open A-string is tuned to 110 Hz, the following relationships can be tuned by ear, facilitated through the counting of beats that occur between the fundamental tones of stopped and open strings.

On the A-string, stop the C♯$_3$ at the fourth fret. The frequency of this tone is 138.59 Hz, an equal-tempered major third above A 110 Hz. This is calculated by multiplying 110 Hz by the 12th root of 2, four times, or:

$$110 \text{ Hz } (2^{4/12}) = 138.59 \text{ Hz}$$

This tone is compared to the open 4th-string, a C♯$_3$ which is detuned by 15 cents. To calculate the frequency of each fretted pitch on a single string, multiply the frequency of the open string by the 12th root of two as many times as the value of the fret number:

$$f^{(\text{open string})} \times 2^{x/12} = f^{(\text{fret } x)}$$

(where x = fret number (or number of semitones above open string))

To calculate the frequency of an open string in relation to the next lower pitched string (which will be narrowed by 15 cents), multiply the frequency of the lower open string by the 12th root of 2, as many times as there are semitones between the two open strings, and divide the result by $2^{15/1200}$ ($2^{15/1200}$ is the ratio equivalent of 15 cents)[1].

$$f^{(\text{open string 1})} \times 2^{x/12} / 2^{15/1200} = f^{(\text{open string 2})}$$

(where x = number of semitones between string 1 and string 2)

This formula is easily adapted to the Uguitar tuning by multiplying the first half of the equation by $2^{15/1200}$ instead of dividing:

$$(f^{(\text{open string 1})} \times 2^{x/12}) \times 2^{15/1200} = f^{(\text{open string 2})}$$

To calculate the number of beats that will occur when the fretted lower string is sounded simultaneously with the next open string of the same note name, subtract the open string frequency from the fretted string frequency. The difference is the number of beats per second that should be heard when properly tuned in a 15-cent temperament. For example, to tune the open C♯-string we need to know the frequency of a C$\sharp_3$ stopped at the fourth fret of the A-string;

$$110\text{ Hz} \times 2^{4/12} = 138.5913\text{ Hz}$$

the desired frequency of the open C#-string;

$$(110\text{ Hz} \times 2^{4/12} = 138.5913\text{ Hz}) / 2^{15/1200} = 137.3957\text{ Hz}$$

---

[1] $2^{15/1200}$ is equivalent to the 1200th root of 2 to the power of 15. 1200 in the equation represents 1200 cents to the octave (2).

and the difference between the two frequency values;

$$138.5913 \text{ Hz} - 137.3957 \text{ Hz} = 1.1956 \text{ Hz}$$

Figure 4.2 shows the frequency of every pitch on the 15-cent 'U'-guitar fretboard up to the twelfth fret.

Tuning using the above method can be verified in several ways. All adjacent strings (excluding the E-string) should produce very pure major thirds and sixths and should sound equally in tune. As well, I test the tuning by fingering a pure chord based on the first six prime numbered partials of the harmonic series: A (1/1) – C♯$^{-15}$(5/4) – G^{-30}(7/4) – D♯$^{-45}$(11/8) – F^{+40}(13/8), Fingered: V-0, IV-0, III-0, II-4, I-2.

Disadvantages of the 15-Cent Guitar

One of the disadvantages of the fifteen-cent guitar, or likewise a guitar tuned in a 72-tone temperament, is that melodic passages are cumbersome and in some cases impossible. Unlike a standard TET tuning, or on a refretted JI guitar, the performance of a simple scale requires the musician to jump around from string to string and from position to position to execute a step-wise progression of tones. In fact, if the pitch set has not been designed specifically with the fifteen-cent guitar in mind, some scales may be impossible to perform on one guitar.

I did however have the opportunity to test the system for melodic passages in a somewhat objective manner. Composer David Lidov presented me with a microtonal

	E	A	C#-15	G-30	B-45	E-60
Beats/sec.*:			**1.2**	**1.7**	**2.1**	**2.8**
Open:	**82.41**	**110.00**	**137.40**	**192.63**	**240.61**	**318.40**
Fret #	F	Bb	D	Ab	C	F
1	87.31	116.54	145.57	204.09	254.91	337.33
	F#	B	Eb	A	C#	F#
2	92.50	123.47	154.22	216.22	270.07	357.39
	G	C	E	Bb	D	G
3	98.00	130.81	163.39	229.08	286.13	378.64
	Ab	C#	F	B	Eb	Ab
4	103.83	138.59	173.11	242.70	303.14	401.16
	A	D	F#	C	E	A
5	110.00	146.83	183.40	257.13	321.17	425.01
	Bb	Eb	G	C#	F	Bb
6	116.54	155.56	194.31	272.42	340.27	450.28
	B	E	Ab	D	F#	B
7	123.47	164.81	205.86	288.62	360.50	477.06
	C	F	A	Eb	G	C
8	130.81	174.61	218.10	305.78	381.94	505.43
	C#	F#	Bb	E	Ab	C#
9	138.59	185.00	231.07	323.96	404.65	535.48
	D	G	B	F	A	D
10	146.83	196.00	244.81	343.23	428.71	567.32
	Eb	Ab	C	F#	Bb	Eb
11	155.56	207.65	259.37	363.64	454.20	601.06
	E	A	C#	G	B	E
12	164.81	220.00	274.79	385.26	481.21	636.80

*number of beats that occur when the open string is tuned to the same stopped note of the next adjacent lower string

Figure 4.2: Tuning by Beats

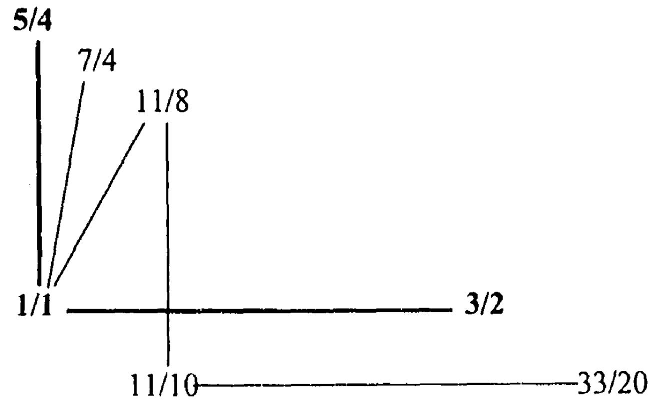

Figure 4.3: Harmonic Space

for "Polar Dawn" by D. Lidov

piece entitled *Polar Dawn* (for guitar, flute and cello) for which he had sketched out the scale:

$$G_2(1/1) - G_3(1/1) - A_3^{-31}(11/10) - B_3^{-14}(5/4) - C\sharp_4^{-49}(11/8)$$

$$- D_4^{+2}(3/2) - E_4^{-31}(33/20) - F_4^{-31}(7/4) - G_4(1/1).$$

Although the use of string harmonics had to be employed, the following temperament worked quite well and allowed for fluid execution:

Open strings: E - G - D - G^{-30} - B^{-15} - $C\sharp^{-45}$

Figure 4.3 (on page 76) shows the harmonic space for Polar Dawn.

Some simultaneous tones may also be impossible in the 15-cent temperament. Two tones that exist on the same axis of harmonic space cannot be sounded together, unless two or more strings have been similarly tuned, and some tones with different generating numbers may exist in positions that are not physically possible to bridge. To solve these problems, many of the compositions presented employ more than one guitar.

Chapter Five: Development of the Fifteen-cent Guitar

This chapter traces the development of the fifteen-cent guitar through some of my earlier experiments. These short pieces coincide with my learning of tuning theory and represent my attempts to understand and hear just intervals and harmonic sonorities.

Study I

Study I represents my first attempt at composing in just intonation and utilizes a system similar to the fifteen-cent guitar in that each string is retuned by a precise amount that allows for the execution of specific intervallic relations. The important difference from the fifteen-cent temperament is that each string is tuned to produce a specific prime numbered harmonic; the multi-function of strings is not considered at all. The tuning of the open strings is $E^{0} – A^{0} – D^{-14} – G^{-49} – B^{-31} – E^{-14}$. These strings correspond respectively to the 3-axis ('E' and 'A'), 5-axis, 11-axis, 7-axis, and another 5-axis. This allows for the perfect intonation of an 11-limit harmonic chord. The tuning is also used to approximate adjacent ratios along any one of these axes. For example, along the 5-axis 5/3, 5/4, and 15/8 are available, the 5/4 is in tune, the 5/3 and 15/8 are each about two cents out of tune.

To build up sonorities, a delay and volume pedal is used. The volume pedal is used to hide the attack of the plucked string and the delay pedal creates a long sustain that rings beyond the excitation period of the string, creating a lasting sonority to which more tones can be added.

The piece is essentially a written out improvisation and is simply an exploration of what was then, to me, unfamiliar harmonic relationships. The first two bars consist simply of a 7/4 minor seventh interval to which is added a 15/8 major seventh in the third bar. Figure 5.1 shows the harmonic space of the first sixteen bars of "Study I".

Assuming A=1/1, the second system builds up a sonority consisting of the following pitch classes: 7/4, 7/6, 5/3, 1/1, and 33/32.

While there is nothing remarkable compositionally about this piece, the use of a single string in approximating several ratios related by 3/2's was an important step toward the realization of a more flexible tempered tuning system.

8-Track Improvisation I

A couple of early attempts at using the guitar as a microtonal instrument involved tuning a single chord and recording repetitive rhythmic patterns using those pitches. By overdubbing several related chords or intervals, a harmonic bed was created over which a simple melodic line was improvised. The initial generating chord is a pure dominant seventh – 1/1, 5/4, 7/4, 3/2 – to which was added a repeated rhythmic figure sounding a single tone based on the ratio11/8. The chord is extended further but the added tones were figured more intuitively and I no longer remember the exact relationships I had in mind.

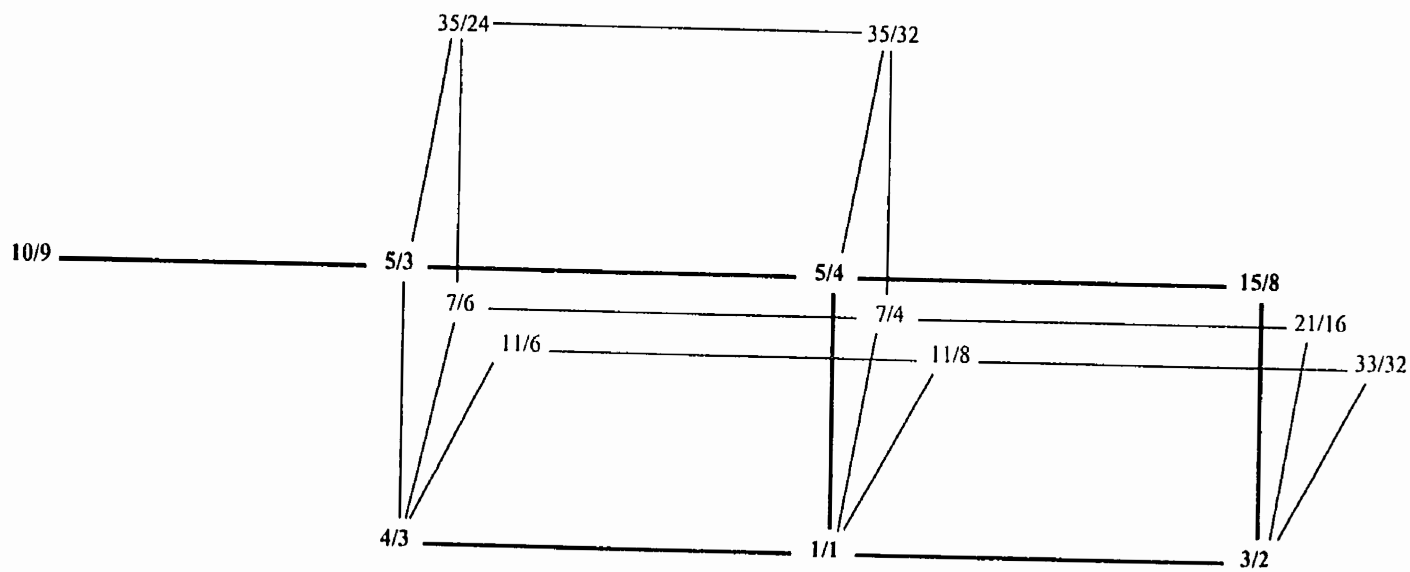

Figure 5.1: Harmonic Space for "Study I" bars 1 through 16

Using the G- and B-strings tuned a 5/4 apart, I bent pitches to their approximate correct intonation in the just scale to create a simple improvised melody. 5-limit ratios, and to a lesser extent 7-limit ratios, were relatively easy to tune by ear. However, at the time I was unable to be confident that I was accurately tuning the scale degrees based on the ratios 11/8 and 13/8.

8-Track Improvisation II

This second improvisation is similar to the first in that harmonic sonorities are built up by overdubbing several related chords. The ratios involved are slightly more complex than those of *Improvisation I*, and the improvisation additionally employs the use of string harmonics, which by their nature are strongly related to the open string involved. No melodic materials exist on the recorded track.

15-Cent Campfire Song and Study II

These two pieces are the first examples of the fifteen-cent temperament applied to the guitar. Both are concerned primarily with the execution of chords that are manageable by a single guitarist. Again, these are simple, written out improvisations and are more experimental than compositional in intent. They also represent the two extremes of simple and complex harmonic sonorities.

The *15-Cent Campfire Song* uses a familiar picking pattern to build up a sonority. Begining with an interval sounding on two adjacent strings, additional tones are gradually added until the chord is fully realized. Each additional tone is of a simple relation to the previous but as the chord extends through harmonic space, the relationship to the root becomes very complex. Figure 5.2 shows the harmonic space of the first few chords of this composition.

Study II is in a 7-limit system that is not harmonically adventurous. I was curious as to the system's ability to realize familiar traditional chord structures; for example: the first three pitches make a purely tuned E-major triad, the second three pitches create an A-major chord in second inversion. In bar two a pure dominant seventh chord, missing its fifth, is sounded in the second and third beamed groups: A (1/1) – C# (5/4) – G (7/4). The harmony continues in a relatively traditional harmonic language.

The issues that arose in the creation of these experiments are represented in the concerns of the larger compositions presented in the following chapters.

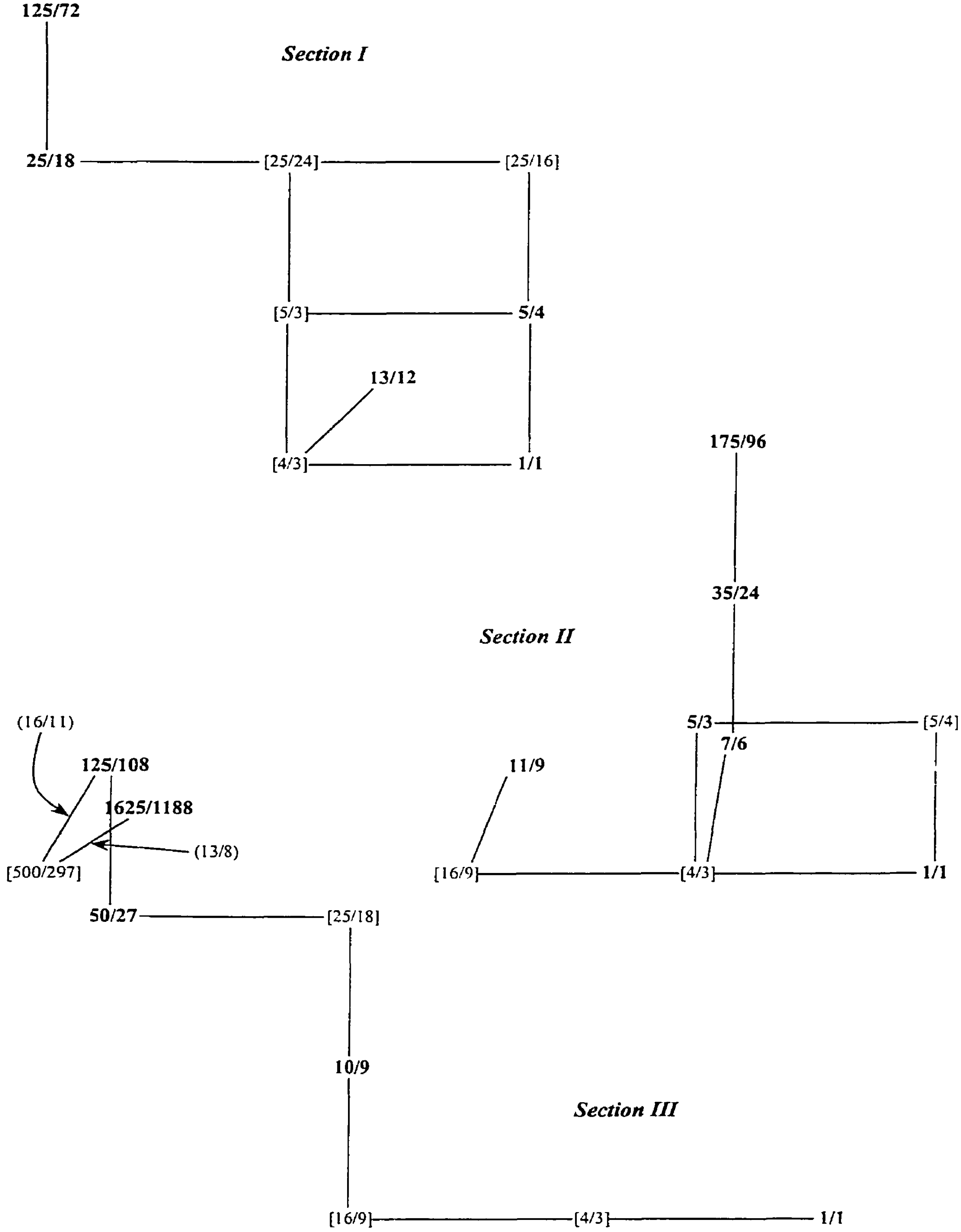

Figure 5.2: Harmonic Space for "15-Cent Campfire Song" (first three sections)

Chapter Six: 45-Cents Worth

(for three electric guitars)

This structured improvisatory piece is for three guitars; each tuned to a variation of the basic fifteen-cent temperament. Each of the three sections explores a unique and moderately complex "harmonic space" by providing an expanding set of available pitches from which the performers are expected to improvise[1].

Harmonic Materials

Each section begins with the most complex harmonic relationships in the set, to which are added tones that "contextualize", or give "relevance"[2] to. the preceding materials. The desired result is that the listener is presented with a series of pitches that appear to have no easily understood harmonic relationship to each other, but as new pitches are added, the relationships become increasingly comprehensible, although still rather complex.

The pitch materials are presented, in approximate order, from the most complex ratios to the most simple. My method for the ordered selection is to exhaust the highest

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[1] James Tenney uses the expression *available pitch process* to describe this type of structured improvisation. Critical Band and Forms are examples of his works that use this proceedure.
[2] Ben Johnston prefers this term.

limit ratios before going on to lower numbered ones and also moving left to right in the harmonic space generated by the ratios. For example, in section I the first group of classes are, in order: 13/8, 11/8, 7/6. Next, 7/4 is added and subsequently 5/3, 5/4, 1/1 and 3/2.

The pitch classes involved in the third section of the piece clearly point to the root 'D'. In the first two sections however, the root is more ambiguous. The implied root of both sections I and II is most likely a 'G', but it is neither provided nor strongly supported, especially in section II where only the 7/6 points to a 'G' (in which case the interval would be a 7/4 from G).

Each of the three guitars is tuned uniquely so that pitch materials are available in a variety of registers. I have chosen not to explore all registers, especially with high limit ratios where they are generally confined to higher registers. The system is however capable of a larger frequency range than indicated in the score.

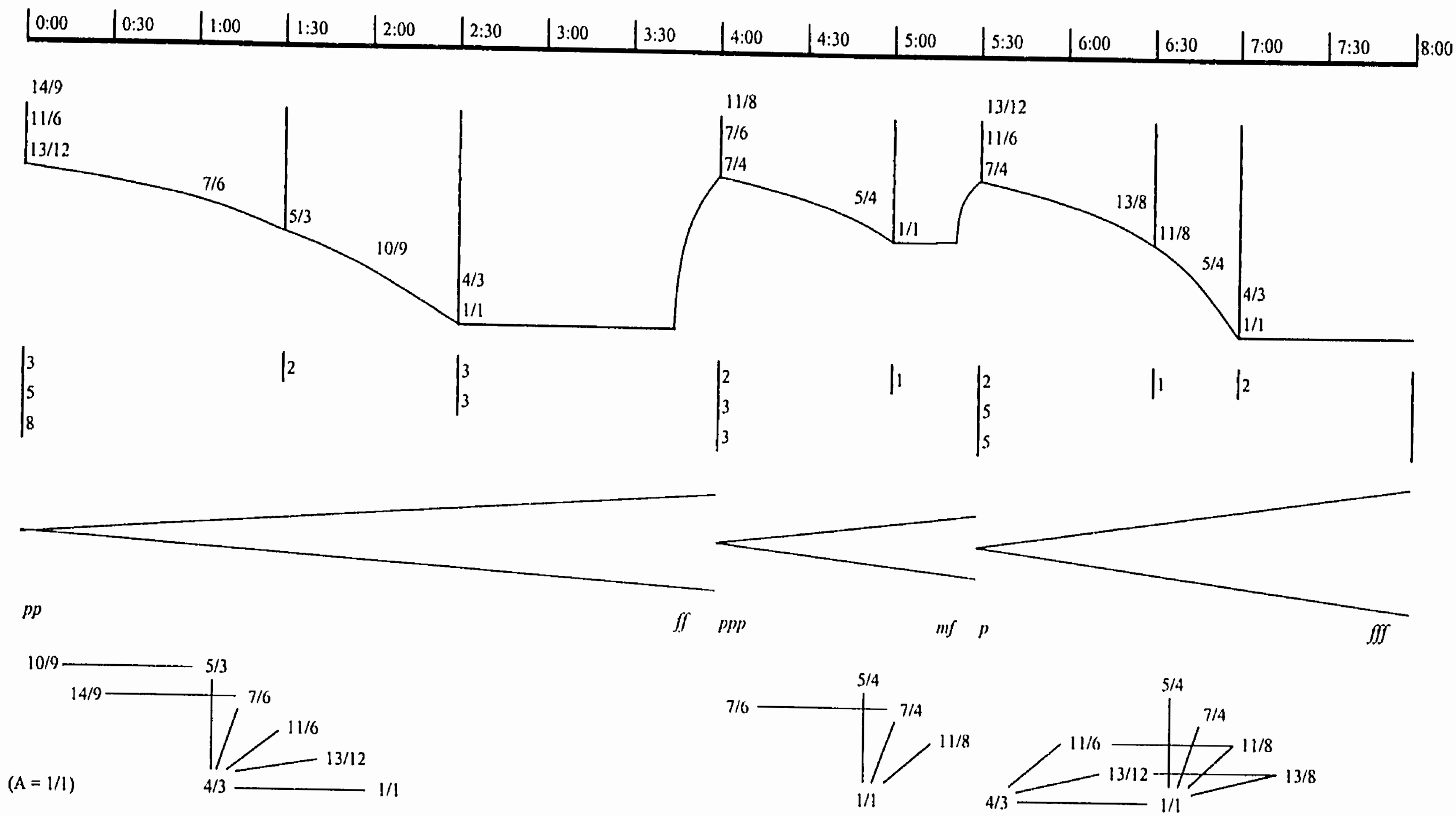
Form

The overall form is derived from the Fibonnaci series. Section I is divided into three parts with the relative durations 3:2:3. Section II is in two parts in a 2:1 relation. Section III is three parts, 2:1:2. The relative duration of the three larger sections is in the ratio of 8:3:5.

Dynamics

The dynamics also have a loose Fibonnaci relationship. The magnitude of the dynamic range is equal in each section but the average amplitude is in a relationship of 8:5:13. By roughly assigning a number to each dynamic indicator, section I has an average level of *mf* (*pp* to *ff*), II = *p-pp* (*ppp* to *mf*) and III = *f-ff* (*p* to *fff*). If *ppp* = 1 then *mf* = 4 and *p-pp* = 2.5 and *f-ff* = 6.5. If these values are doubled then a relationship of 8:5:13 is achieved.

Figure 6.1 outlines the important parametric profiles and structural concerns of *45-Cents Worth.*



Tuning:

	E	A	D	G	B	E
Guitar I	-30	0	-15	-30	-45	-60
Guitar II	-15	0	-45	-60	-15	-30
Guitar III	-60	-45	-30	-15	0	-45

Figure 6.1: Plan for "45-Cents Worth"

Chapter Seven: Music Box II

(trio for guitar, Columbine, and tunable keyboard)

This piece is primarily concerned with controlling vertical harmonic density over time. Three instruments each play two lines simultaneously, an ascending and a descending scale, creating six independent scalar lines. By mapping out the pitch trajectory of each line, the frequency range of the ensemble is precisely controlled, and consequently the harmonic density.

Form

All instruments begin on the pitch class 'C', each in a different octave. The total span is six octaves; C_1 to C_7. At the Golden Section, all the lines converge on a predefined harmonic sonority: 7/4 – 13/8 – 5/4 – 9/8 – 1/1 – 3/2, confined to a range of about a minor tenth. All lines reverse their position in the voicing so that the instrument beginning on C_7 is now the bottom voice, the voice beginning on C_1 is now the top voice, and all other inner voices are inverted. The voice reversal creates a dense, harmonic convergence around 45 seconds before the Golden Section, in which the bandwidth spans less than a major third.

For the second portion of the piece, the process is reversed, this time resolving on a harmonic sonority consisting of the ratios, bottom to top: 1/1 – 7/6 – 3/2 – 5/3 – 4/3 – 11/6, spanning about two and a half octaves. Again, the voices cross and reverse, converging at a moment around 40 seconds after the Golden Section, with a bandwidth slightly larger than a major second.

While the piece has two moments of harmonic resolution, at the end and at the Golden Section, the two moments of maximum vertical harmonic density vie for the climaxes of the piece. Fig. 7.1 diagrams the pitch projection of each voice.

Harmonic Materials

Each instrument has a unique set of pitches. The Columbine[1] is a 23 pitch-per-octave metallophone designed and built by Gayle Young. The Columbine uses three basic 5-limit just scales based on the pitches 'E' (1/1), 'G#' (5/4), and 'C' (8/5), with the addition of a major second 10/9 and a minor seventh 9/5 from each of these roots. Fig. 7.2 shows the entire harmonic space of the Columbine. The bold text indicates the primary just scale from which Gayle Young works.

For the purpose of this piece, I have defined 'C' as the generating pitch class, therefore, the score does not match that of Gayle Young's system. However, intervalically, the two are identical.

The keyboard part is ideally played on a Rhodes piano, or on an instrument with a similar amplitude envelope. The pitch set of the keyboard part is of a 13-limit and uses a repeating octave. Generally, the pitches fill in much of the harmonic space surrounding the fundamental. The scale set is: 1/1 – 33/32 – 8/7 – 7/6 – 9/7 – 21/16 – 11/8 – 3/2 –

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[1] "The instrument was named Columbine after the five-petaled flower of the same name, native to North America" (Young, 54).

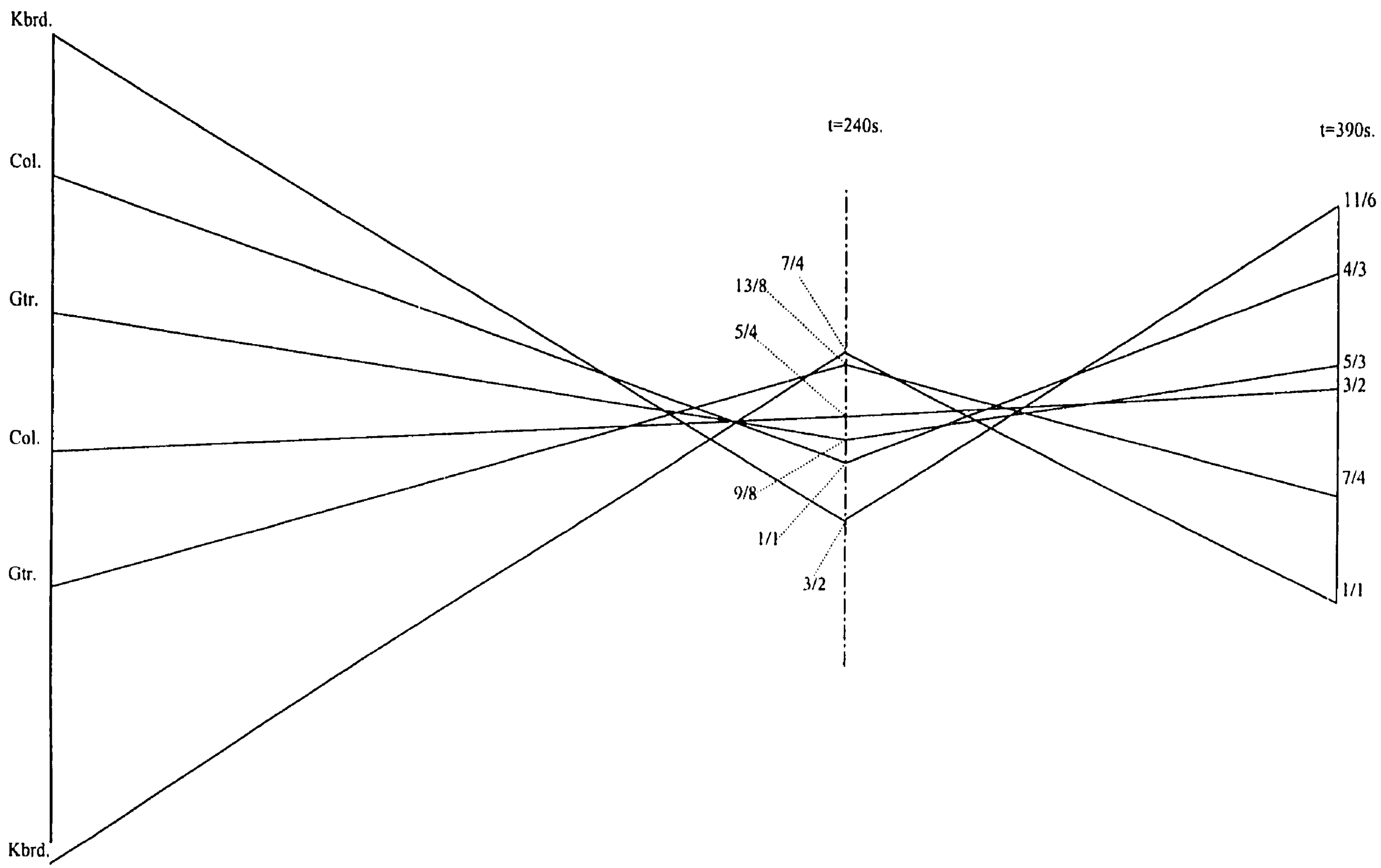

Figure 7.1: Pitch Projection Plan for Music Box II

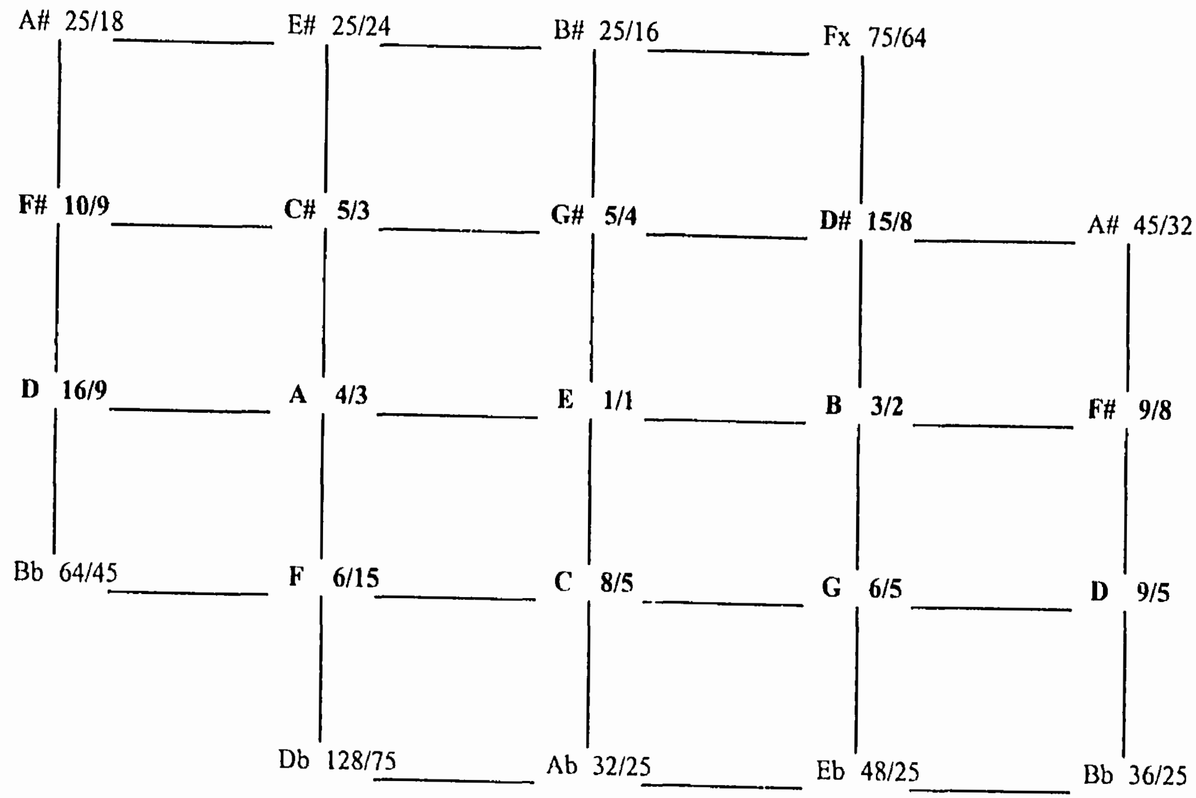

Figure 7.2: Tuning Lattice for Gayle Young's Columbine

13/8 – 12/7 – 7/4 – 11/6. The only pitch classes that occur in both the Columbine and the keyboard parts are the 1/1 and 3/2. Fig. 7.3 shows a concentrated area of the Columbine scale set and the 13-limit materials of the keyboard part, now in bold.

The guitar, unlike the other two instruments, does not use a repeating octave. Its pitch material spans two octaves and extends the harmonic space of the piece into ratios in which both the numerator and denominator can be of a high limit value. In ascending order, the pitch set is:

C_3 (1/1) - Db_3^{+15} (15/14) - D_3^{-30} (11/10) - D_3 (9/8) - $E\flat_3^{-30}$ (7/6) - E_3^{+15} (14/11) - F_3 (4/3) - $F\sharp_3^{+15}$ (10/7) - G_3 (3/2) - $A\flat_3^{-30}$ (14/9) - $A\flat_3^{-15}$ (11/7) - $B\flat_3^{-30}$ (7/4) - B_3^{-50} (11/6) - B_3^{-30} (13/7) - C_4 (1/1) - D_4^{-50} (12/11) - D_4^{-30} (11/10) - $E\flat_4^{-50}$ (15/13) - $E\flat_4^{-30}$ (7/6) - $E\flat_4^{-15}$ (13/11) - E_4^{-50} (11/9) - E_4^{+40} (9/7) - F_4^{-50} (13/10) - F_4^{+40} (15/11) - $F\sharp_4^{-50}$ (11/8) - $F\sharp_4^{-15}$ (7/5) - $F\sharp_4^{+40}$ (13/9) - G_4^{-50} (16/11) - $A\flat_4^{-15}$ (11/7) - $A\flat_4^{+40}$ (13/8).

It can be seen in figure 7.4 that the harmonic space of the guitar pitch set serves to further emphasize the harmonic region immediately surrounding C 1/1. Figure 7.5 shows the fret and string location of each pitch in the set; the darkened areas represent further available pitches that are not utilized in the piece.

The harmonic "cohesiveness" of any moment of the piece is most strongly defined by the pitches of the Columbine. If a pitch defined by a relatively complex ratio is sounding, likely the pitches of other instruments are also in a complex relationship to that tone. The overall, desired effect is that of a fluctuating harmonic relativeness, with a guaranteed maximum dissonance occurring at the two moments of high vertical harmonic

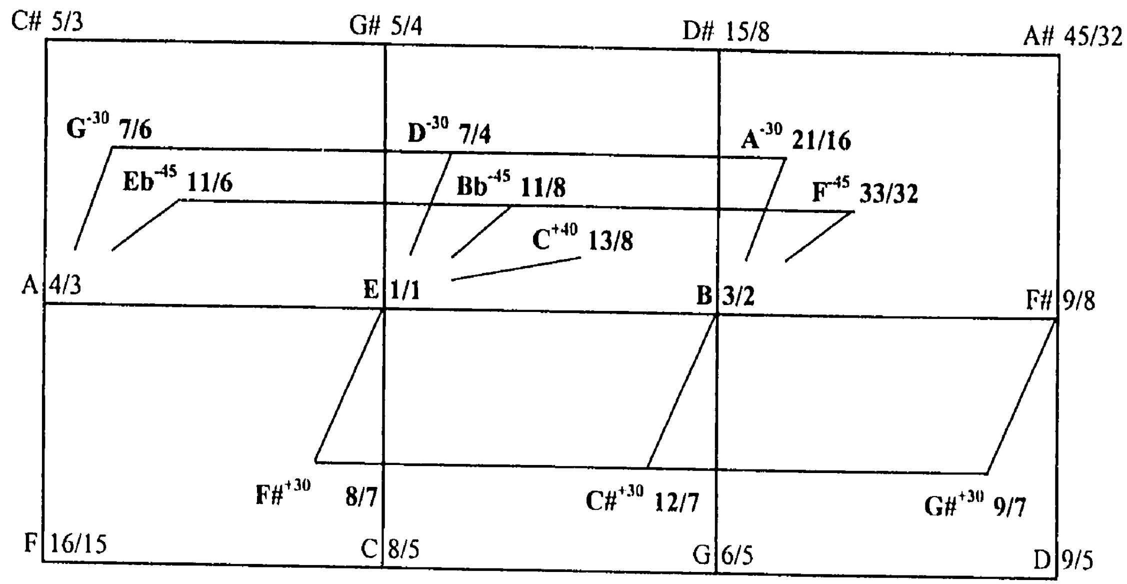

Figure 7.3: Harmonic Space for Keyboard Part of "Music Box II" with concentrated Columbine space

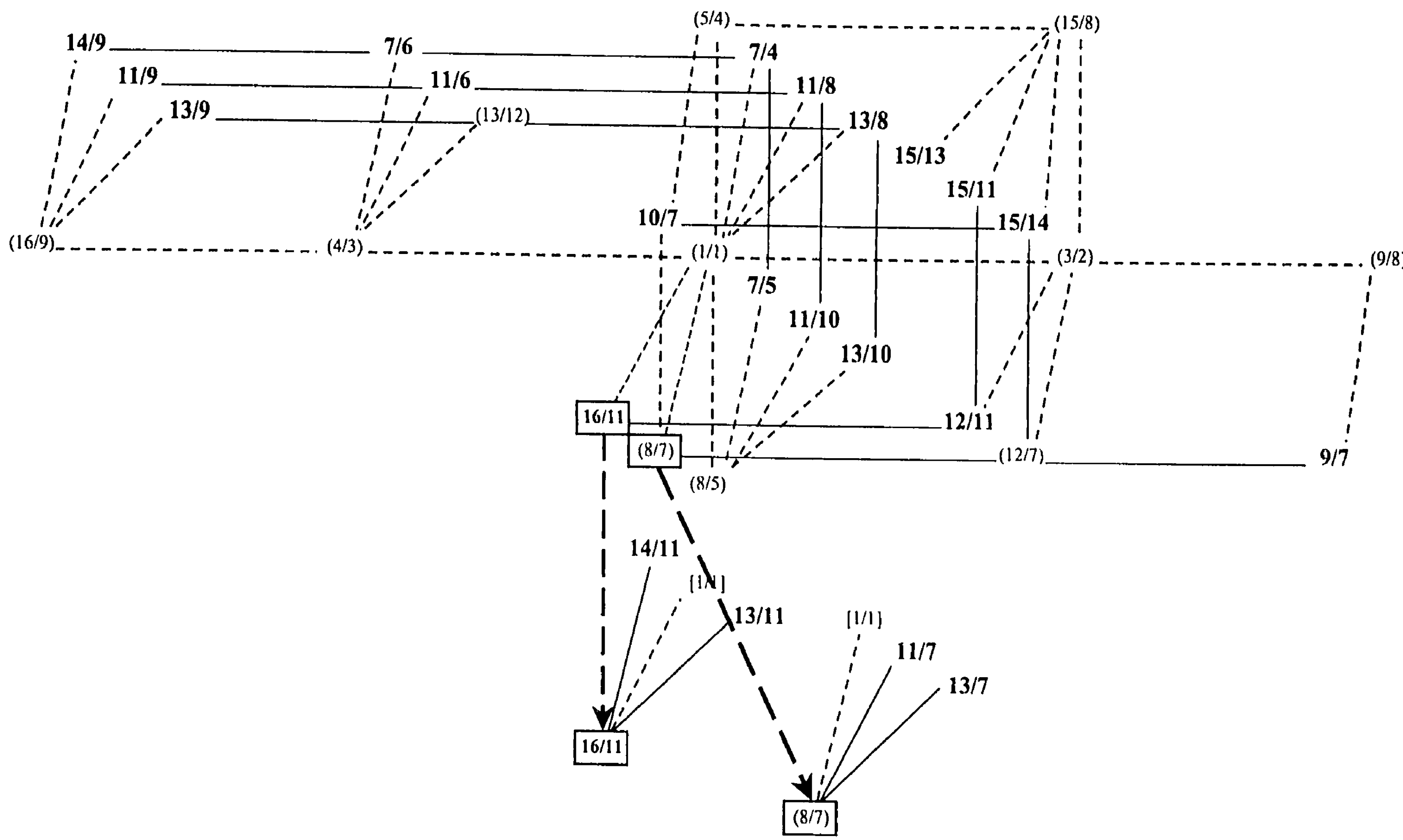

Figure 7.4: Harmonic Space for Guitar Part of "Music Box II"

E_2^{+15}	A_2	D_3^{-30}	G_3^{-15}	B_3^{-50}	E_4^{+40}	
		11/10		11/6	9/7	
		7/6	11/7		15/11	
					13/9	
	1/1			12/11		
				15/13	13/8	
	9/8			11/9		V
		14/9		13/10		
				11/8		VII
	4/3	7/4	13/11	16/11		
15/14		13/7				
	3/2					
			7/5			
14/11		11/10		11/6		XII
		7/6	11/7			
10/7						
	1/1			12/11		
				15/13		
				11/9		

Figure 7.5: Fretboard for Guitar in "Music Box II"

density and maximum consonance occurring at the beginning, end, and Golden Section. Otherwise, regardless of the mathematical predictability, the moment-to-moment complexity is of an unpredictable quality for the listener.

Rhythmic Concept

The rhythm of each part is also defined by the ratios involved in each instruments' scale set. In this case, ratios define a portion of time. To calculate the rhythmic units in each part, the total duration of each of the two sections is divided logarithmically according to the ratios of the scale. These proportions are approximated with a maximum resolution of a quintuplet eighth note (where a quarter note equals one second).

Chapter Eight: Music Box III

(for string quintet and two electric guitars)

In this piece two guitars are used as a tuning reference for the bowed instruments. Each string of each guitar is tuned to the average pitch of all the pitches that occur on any axis. This is a slight variation of the 15-cent guitar. 3-limit ratios are tuned to their standard twelve-tone equal-tempered equivalent. 5-limit ratios are tuned +/- 15 cents from their TET equivalent. 7-limit ratios are tuned to +/- 32 cents (a better approximation than the 15-cent guitar's 30 cents). 11-limit ratios are tuned to +/- 50 cents (as opposed to +/- 45 cents). 13-limit ratios are tuned to +/- 60 cents (+/- 40 cents). The 15-cent guitar tuning was adapted to this because a very specific pitch set that does not modulate is being utilized, which means that the multi-functional capabilities of each string are not required, therefore, it makes more sense to use better approximations than the 15-cent temperament allows. A twelfth-tone temperament was also considered, but the use of 13-limit ratios negated this possibility (as 13-limit ratios are out of tune by a margin greater than 5-cents).

Harmonic and Rhythmic Materials

Each stringed instrument has a unique scale set that is executed at an independent rate defined by a frequency ratio that relates all the parts. The following lists each instruments scale set and the relative frequency at which each pitch of the set occurs:

Vln. I	1/1	12/11	6/5	16/13	4/3	16/11	8/5	24/13
	C_5	$D\flat 5^{+50}$	$E\flat_5^{-15}$	E_5^{-40}	F_5	$F\sharp_5^{+50}$	$A\flat_5^{-15}$	B_5^{-40}
Vln. II	1/1	13/12	5/4	11/8	3/2	13/8	5/3	11/6
	C_4	$D\flat_4^{+40}$	E_4^{-15}	$F\sharp_4^{-50}$	G_4	$A\flat_4^{-40}$	A_4^{-15}	B_4^{-40}
Vla.	12/11	8/7	7/6	11/8	16/11	12/7	7/4	11/6
	$C\sharp_3^{-50}$	D_3^{-32}	$E\flat_3^{-32}$	$F\sharp_3^{-50}$	$G\flat_3^{+50}$	A_3^{+32}	$B\flat_3^{-32}$	B_3^{-50}
Vcl.	8/7	7/6	6/5	5/4	8/5	5/3	12/7	7/4
	D_2^{+32}	$D\sharp_2^{-32}$	$E\flat_2^{+15}$	E_2^{-15}	$A\flat_2^{+15}$	A_2^{-15}	A_2^{+32}	$B\flat_2^{-32}$
Bs.	1/1	6/5	5/4	4/3	3/2	8/5	5/3	
	C_1	$E\flat_1^{+15}$	E_1^{-15}	F_1	G_1	$A\flat_1^{+15}$	A_1^{-15}	

Although not designed to demonstrate any particular point, the scale sets for the viola, cello and bass warrant some attention. The scales for the viola and cello are symmetrical; the top half of each scale is derived from the inversions of the bottom half of the scales. As well, the bottom half of the cello and bass scales are made up of consecutive superparticular ratios (a superparticular ratio is a ratio in which the numerator is one integer larger than the denominator, these ratios occur at adjacent partials in the harmonic series and have historically been given special attention by tuning theorists).

It should be noticed that simple, low-limit ratios have been reserved for the lowest pitched instruments. The bass part consists of 5-limit ratios, all of which are found in our traditional scales. The cello part is within a 7-limit, the viola is an 11-limit scale, and the violins are in a 13-limit system. This bias exists because complex ratios tend to fuse most poorly in lower registers.

The Role of the Guitars

The primary function of the guitars in this piece is to provide tuning references for the string players. Initially, I had considered hiding the attack of the guitars through the use of a volume pedal, but allowing the attack added a timbral element that seemed more interesting and gave the guitars a function beyond a simple supportive role.

In several cases, it is impossible for the guitars to provide a tuning reference for every pitch in a given sonority, especially where the parts line up rhythmically. In several cases I opted to leave a tuning reference out for tones that were generated by a simple or familiar ratio, especially 3- and 5-limit ratios.

The two guitars are tuned as follows, one is an Oguitar variation, the other a Uguitar variation:

Gtr. I	(Oguitar)	E	A	D^{-15}	G^{-32}	B^{-50}	F^{-60} (E^{+40})
Gtr. II	(Uguitar)	E	A	D^{+15}	G^{+32}	B^{+50}	E^{-40} (D# $^{+60}$)

Rhythmic Concept and Form

If the rate of the bass is represented by the value 2 than the cello's rate = 4, the violas = 3:4, violin II = 5:4, and violin I = 7:4.

Violin II and the Cello begin the piece. At the first instance of rhythmically coinciding pitches, the viola begins its scale. When all three instruments coincide rhythmically, the first violin enters, and upon the next four coinciding pitches, the Bass enters. All instruments continue for seven more points of rhythmic unison.

Chapter Nine: Quartet for Six Guitars

This piece attempts to tackle an approach to dissonant counterpoint. A couple simple rules are set up defining the harmonic possibilities. For any given moment, at least one 5-limit dissonant[1] interval is present, a 5-limit tertian interval[2] from one of the two dissonant tones, and either a 7/4 minor seventh or a 7/6 minor third usually from the root (though some exceptions occur). This attempts to imitate and extend (through the introduction of 7-limit ratios) the harmonic tendencies of composers such as Carl Ruggles, Toru Takemitsu and the early works of Schoenberg, although in an entirely less sophisticated manner.

The counterpoint is generated by allowing only one or two pitches to change at a time, all other tones are common to both sonorities and maintain the conditions above. To facilitate the harmonic planning of this piece I resorted to diagrams that resemble those of Barbour's, described in chapter two. Placing the tertian relations at a 45-degree angle visually implies an equal harmonic relativeness for major and minor thirds and sixths.

Six basic 5-limit harmonic possibilities were pre-established, each with two or three 7-limit extension options. Figure 9.1 shows the basic shapes on the top half of the diagram, the bottom half indicates the frequency ratios involved in each shape. In the

---

[1] For the purpose of this discussion, *dissonant* simply implies the presence of a major seventh (15/8) or minor second (16/15) relation or a compound.
[2] Major or minor third or a major or minor sixth (5/4, 6/5, 5/3, 8/5 respectively).

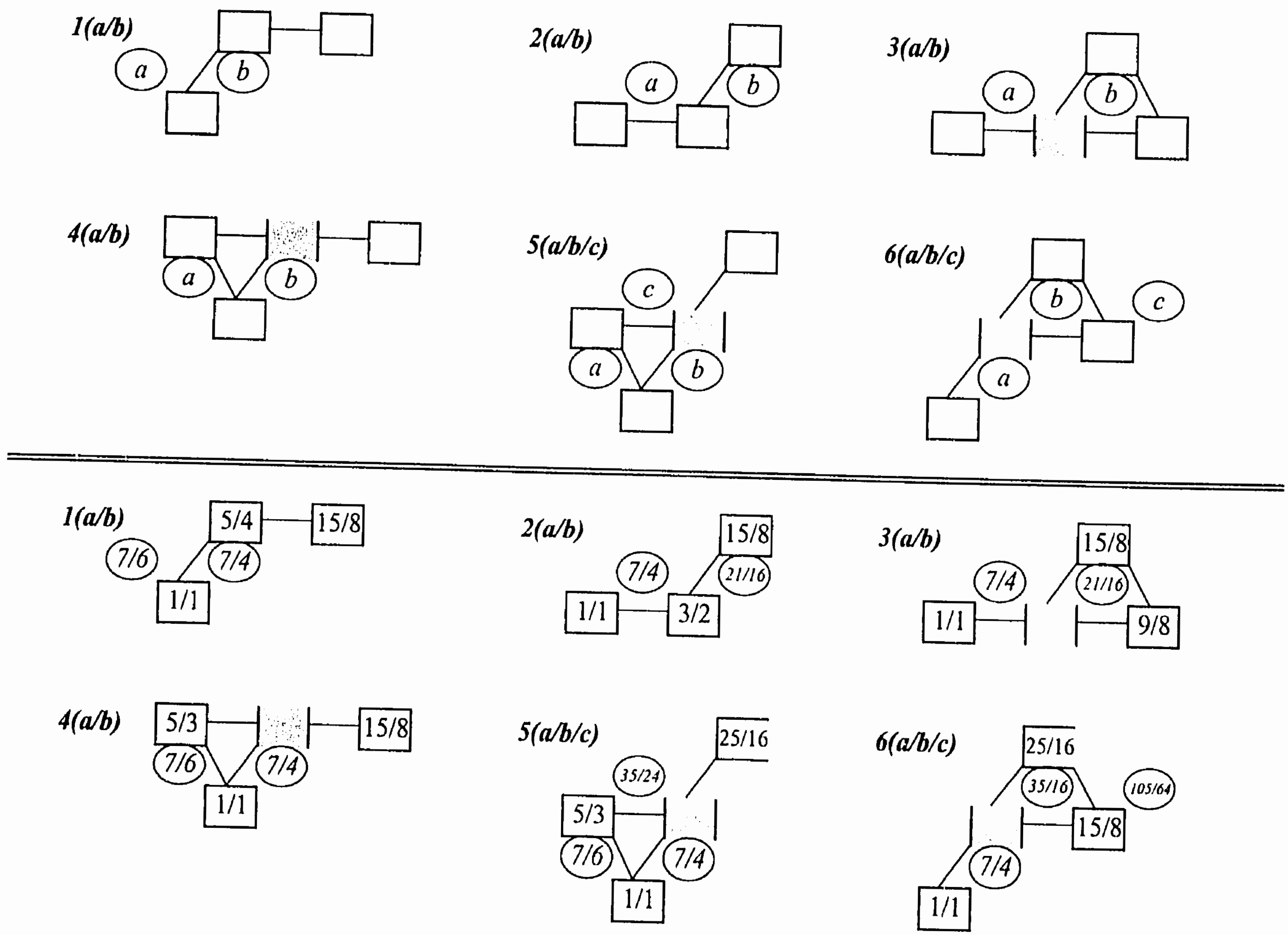

Figure 9.1: Basic Harmonic Relationships for "Quartet for Six Guitars"

first three shapes, the root is quite strong. This is reflected in the root's positioning in the bottom left corner of the lattice. Spaces 4 and 5 have a weaker foundation because the implied root is missing. I opted to establish the lowest position as the root to maintain a sense for the complexity of the ratios involved, facilitating comparison with the first three lattices. It should be noted that in space 5, the 15/8 relation occurs between 25/16 and 5/3. Space 6 has the potential to be the most complex sonority available, depending on the 7-limit ratio involved. The sense of root is slightly usurped by the absence of both a 5/4 and a 3/2.

It can be seen that in diagrams 3 through 6 at least one ratio is missing that would connect two otherwise distant ratios. A question arose regarding the capability of these sonorities for harmonic fusion. I conducted an informal experiment of which the results loosely correlated "complete" spaces with a stronger sense of harmonic fusion and "incomplete" spaces with a lower sense of fusion. With these particular sonorities, relative complexity must also be regarded as a factor. In general, spaces missing connecting pitch classes were less likely to convincingly fuse.

Figure 9.2 outlines a portion of the harmonic progression for this piece with the tuning for each horizontal plane indicated on the left, in a fifteen-cent temperament.

It was necessary to use six guitars to cover the four voices involved in the composition. By using six distinctly tuned guitars to play four voices, parts trade off when the available tuning is no longer available on a given guitar. Because all of the guitars are to have similar timbres, the overall effect is that of four contrapuntal melodic

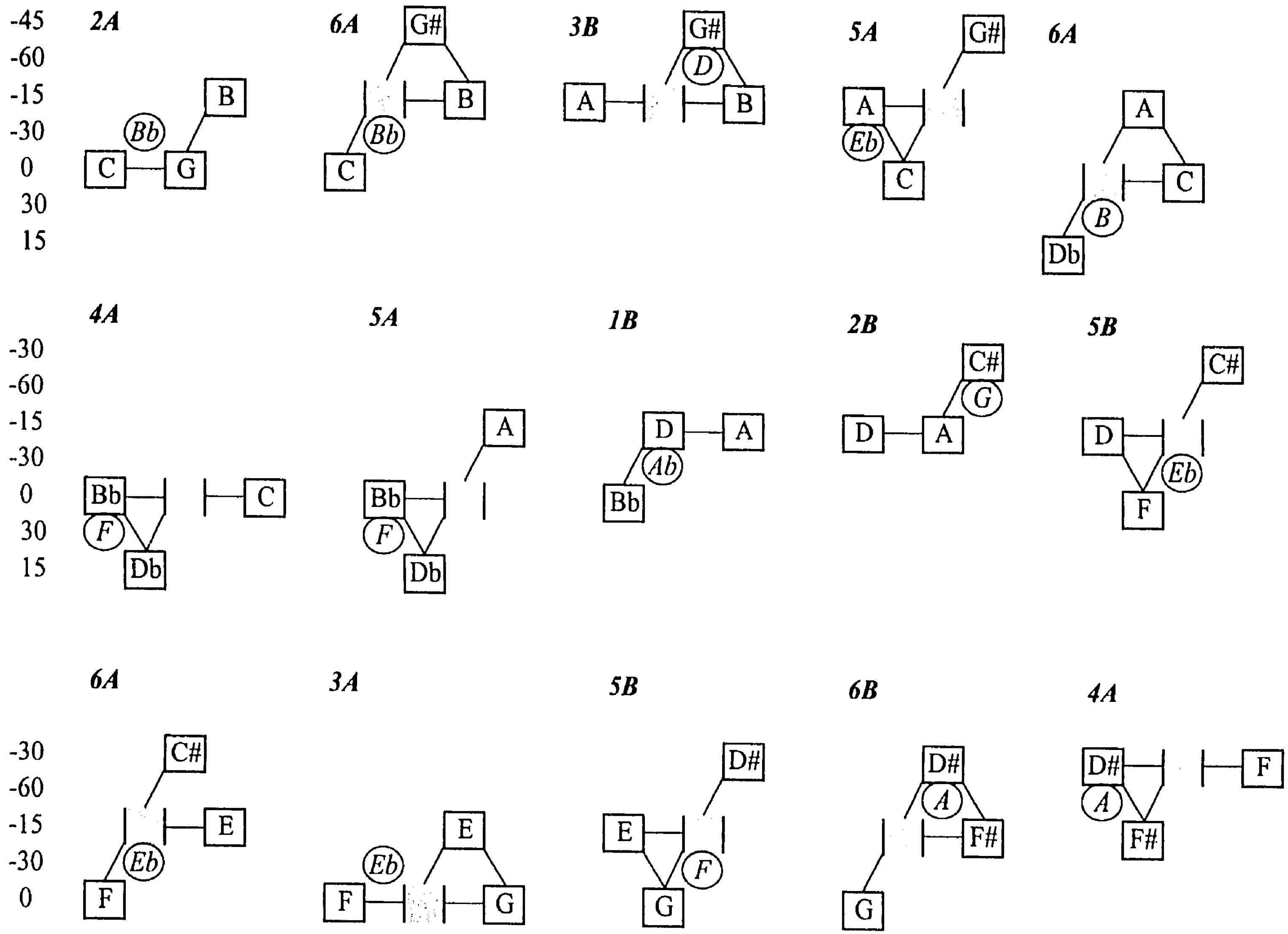

Figure 9.2: Harmonic Progression for Portion of "Quartet for Six Guitars"

lines.

Rhythmic Concept and Form

The piece is in five short sections of which the temporal density is manipulated on a logarithmic scale. each section begins with a higher level of rhythmic activity than the previous section, with the exception of the fifth, and slow through the course of the section to an average temporal density of one ictus per bar. The average number of independent attacks was predetermined but the specific rhythmic materials were intuitively generated to simulate a sense of indeterminacy.

Each section recycles the harmonic material of the previous section and adds to the progression, which is extended to fulfill the predetermined ictus requirements of the section.

Summary and Conclusions

The compositions presented in this thesis all reflect a similar aesthetic approach. The pieces are primarily about "harmony", harmony in the very basic concept that is *the relationship between two or more tones in any instant of time*; in other words, they are mostly about "the harmonic moment". The pieces are not about larger scale aspects of harmony which may involve modulation, melodic development, or structural harmonic approaches (although a harmonic process may be expressed). An effort has been made to emphasize the importance of the harmonic moment by intentionally downplaying other musical parameters. Specifically, the rhythmic component of each piece possesses a perceptually sporadic quality. It is believed that by generating rhythms that provide no sense of pulse, pattern, or predictability and are essentially statistically flat, that the attention of a listener will quickly move away from the rhythmic elements towards those that are changing, in this case the harmony. As well, the pieces are generally dynamically and texturally flat further reducing the attention given to non-harmonic elements.

For myself, the listening experience is somewhat meditative, requiring or forcing the listener to be in a state that exists somewhere between attention and unconsciousness. One can listen for the details of harmonic interaction, which change constantly but unpredictably, or one can simply accept the music on a larger scale which is often of a statistically flat, or a steadily changing quality.

The aesthetic aims of these pieces do not necessarily reflect those of my other compositions or intentions for future works. Where these works are conceived almost entirely in the precompositional stage, future works will consider and shift where the intuitive breaks from the structural. Where the works of this thesis are strongly linked to the fifteen-cent tuning system and micro-level harmonic conditions, future works will focus on large-scale compositional concerns of which the choice of tuning system is an important but secondary consideration.

The writing of this thesis has been a discovery process that has considerably improved my potential for using high-limit harmonic relationships in an informed manner. Many of the pieces presented in this thesis resulted from a question of which the composition was designed to answer.

A tuning system can influence or be influenced. Twelve-tone equal temperament gradually developed out of the tonal approach of 17th, 18th, and 19th century composers. In turn, the atonal compositions of the early twentieth century may never have come about without the establishment of TET. 24-tone equal temperament was a solution for composers who felt restricted by the limitations of twelve notes. However, it is unlikely that without TET would 24-tone ET have followed.

Harry Partch's 43-tone tuning system was very much a response against the restrictions and arbitrariness of TET. He suggested an approach that has been an example for many just-intonation musicians. His legacy includes Ben Johnston, James Tenney, and many others, all of whom have developed unique systems that tackle various compositional concerns. Most importantly, Johnston and Tenney have not limited their harmonic language by adopting a single "perfect"system for all their work. Rather, they have been examples of how a thorough understanding of harmonic proportion can lead to many diverse and satisfying answers to equally diverse compositional problems.

The need is still felt by some for a universal tuning system; many suggest that the 19- and 31-tone systems of Yasser should replace TET. But the replacement of one system for another is not a solution at all; it is as restrictive as the first and is a further "design...for compliance" (Franklin, 23). By suggesting an approach that simply informs the musician of the properties of harmonic proportion, the composer/musician is free to develop a system that best supports her or his music. In many cases, the desire may be to usurp pure harmonic relationships rather than encourage, in which case an understanding of tuning theory is still advantageous.

No temperament can claim universal superiority; the definition of a tuning system is inherently linked to the compositional process and should therefore change with intent. The fifteen-cent temperament that has just been explored is intended as an approach to a particular set of problems and is only one of many possible solutions. The purpose of this thesis is not to suggest that the fifteen-cent temperament is a system to be universally adopted, rather, it suggests an approach to developing tuning systems according to the aims of the composer or purpose of individual compositions.

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SCORES

Study I

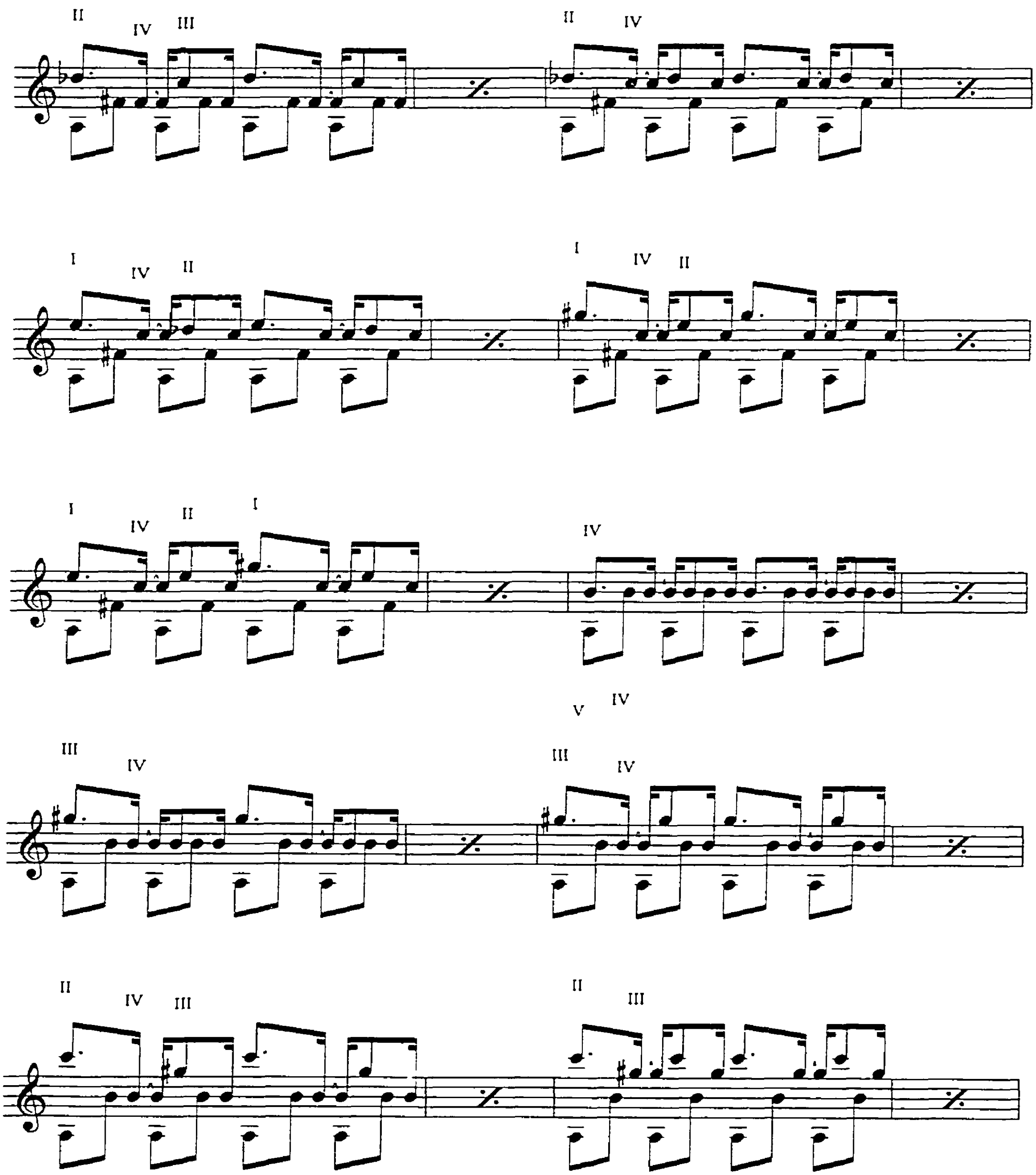
Tuning (cents deviation):
$E^{0}, A^{0}, D^{-14}, G^{-49}, B^{-31}, E^{-14}$

Perform with medium delay, several repeats, long decay. Each note swells from 0 with volume pedal (no attacks are heard).

Bar lines indicate phrases, breath marks indicate longer pauses. Rhythmic values are only vague quideines for note durations.

Roman numerals indicate string to be used.

15-Cent Campfire Song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II IV
I IV II
I IV II
I IV II I
IV
III IV
V IV
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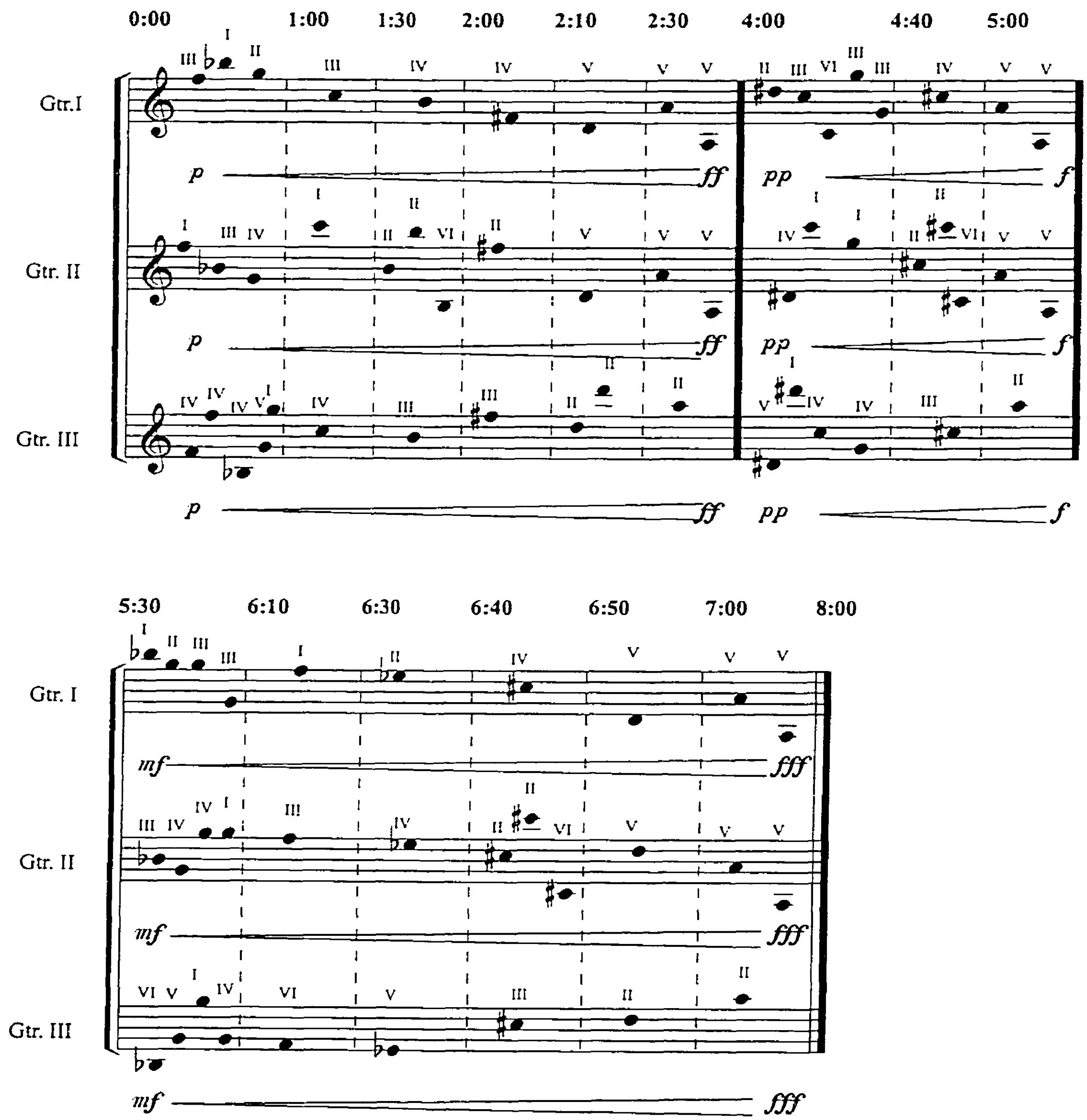
II III

Study II

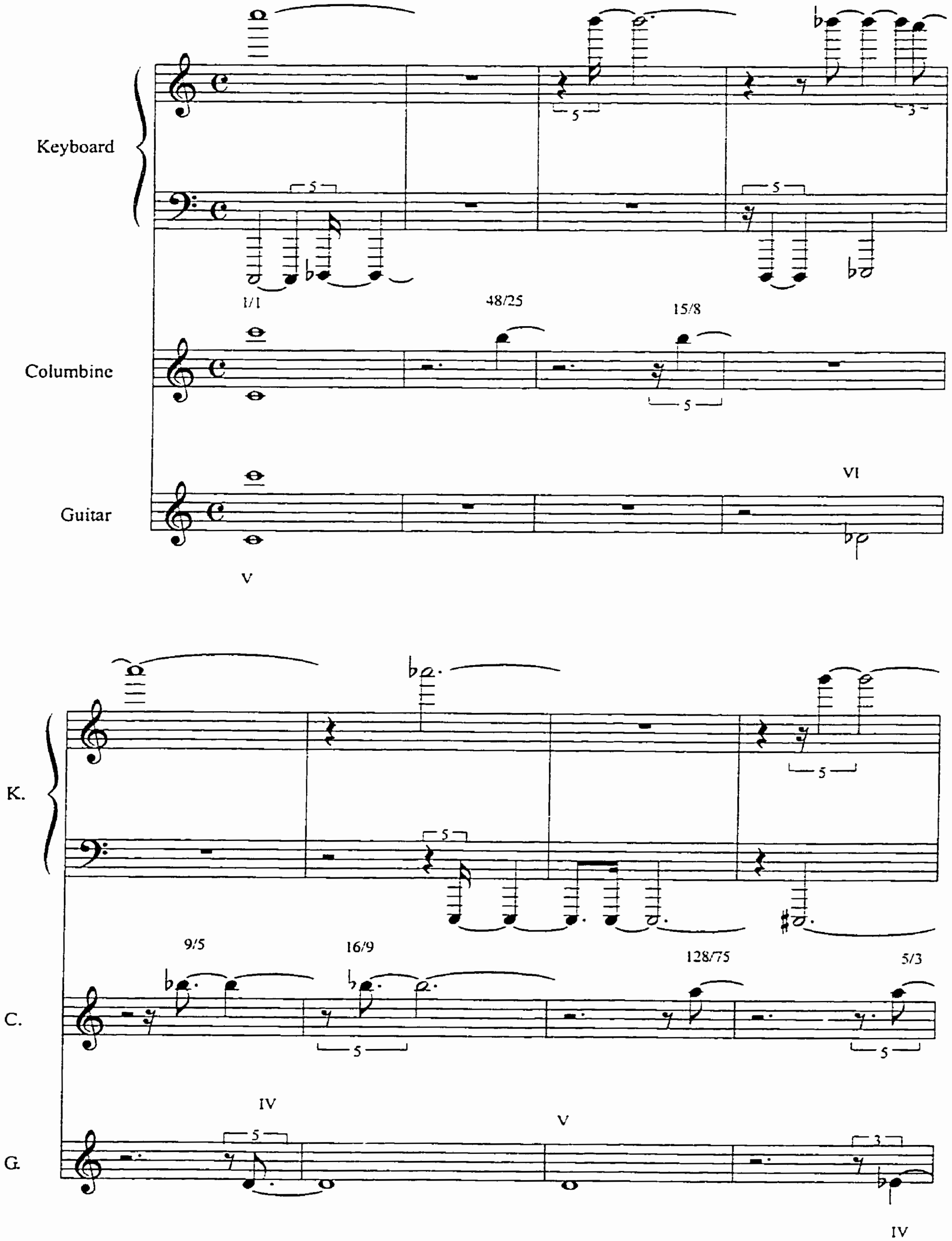
E^{0}, A^{-15}, D^{-30}, G^{0}, B^{-15}, E^{0}

45 Cents Worth: for three guitars



Tuning	Gtr. I	Gtr. II	Gtr. III
1) E	-60	-30	-45
2) B	-45	-15	0
3) G	-30	-60	-15
4) D	-15	-45	-30
5) A	0	0	-45
6) E	-30	-15	-60

Music Box II (trio)

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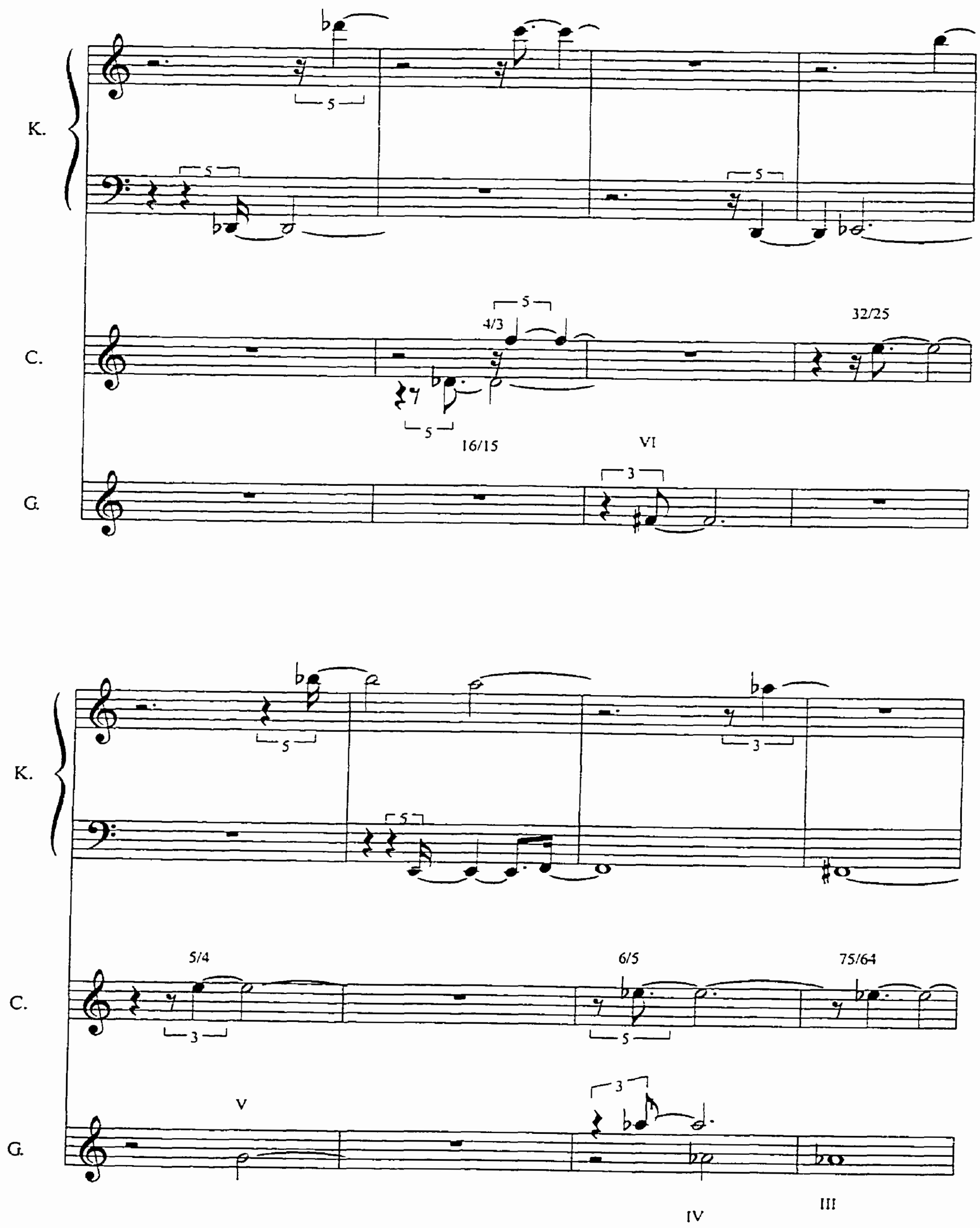
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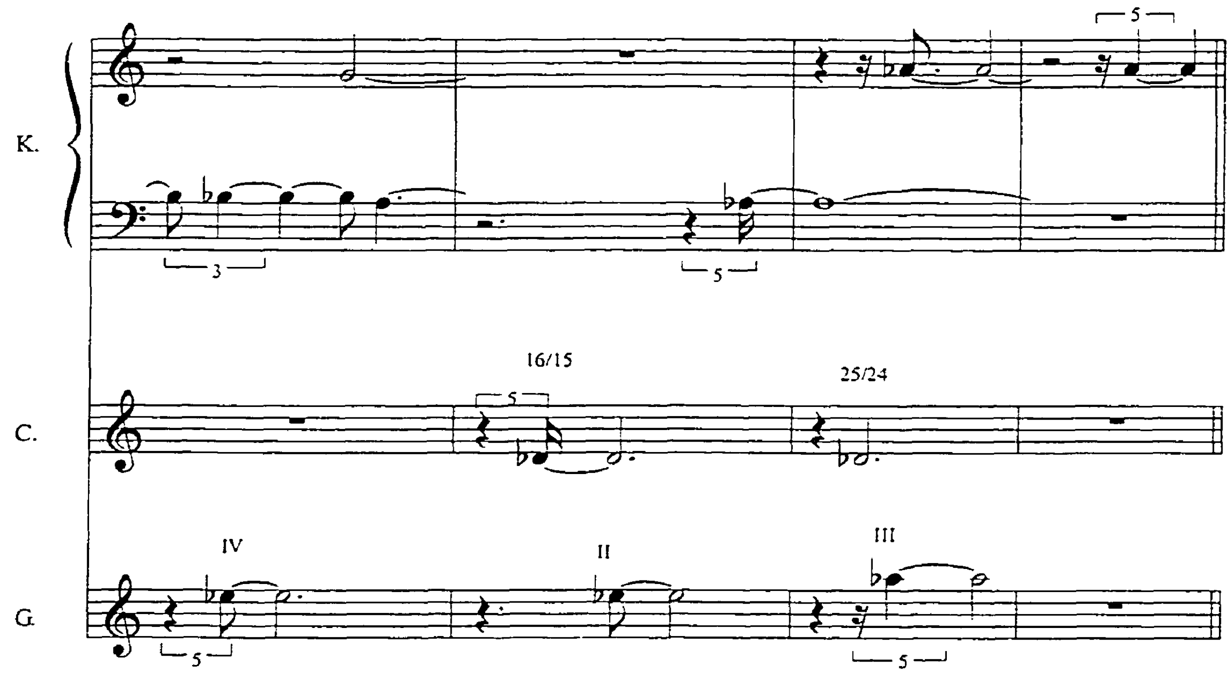
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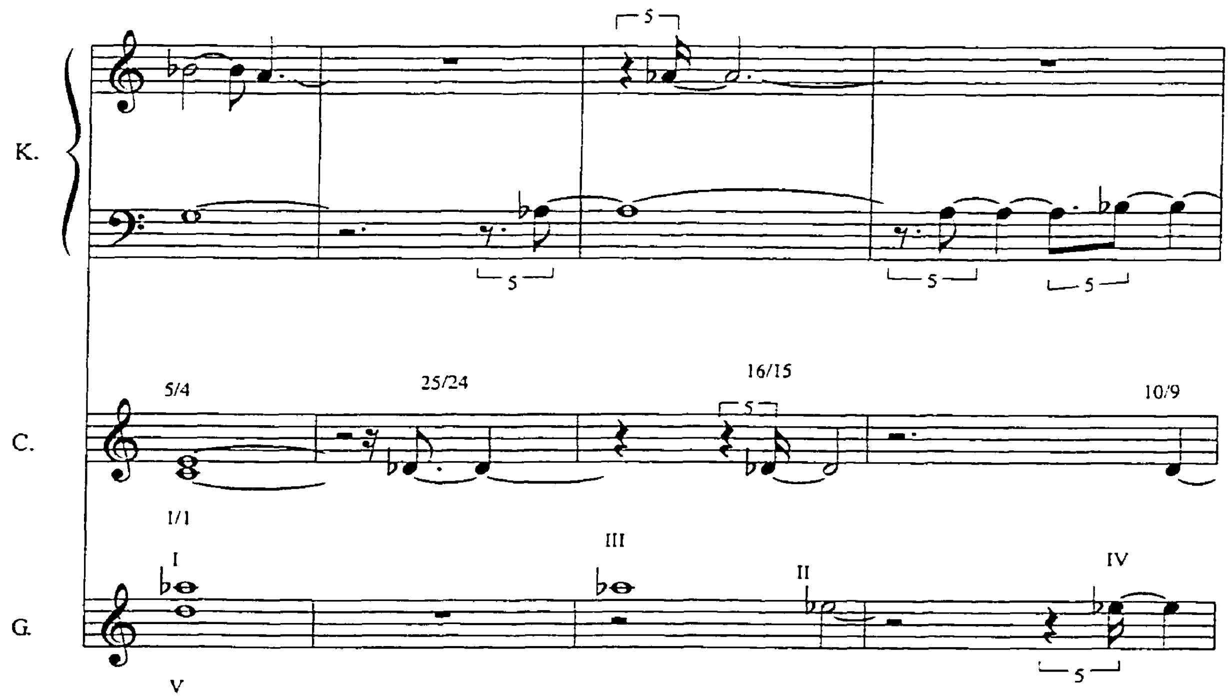
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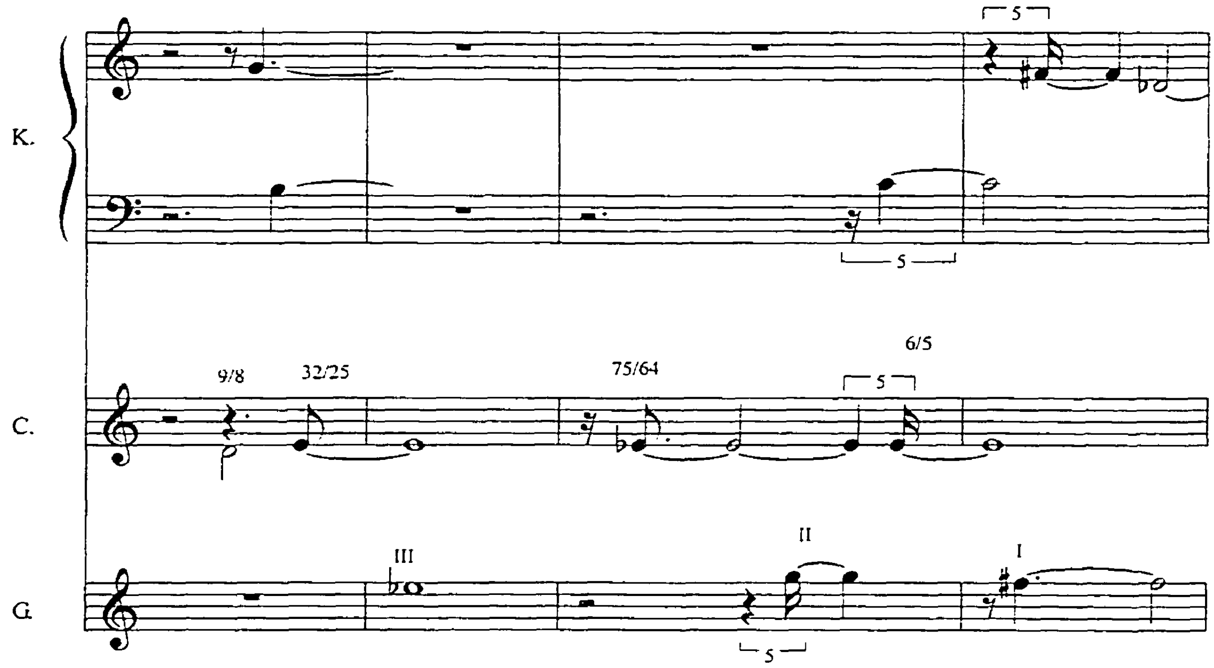
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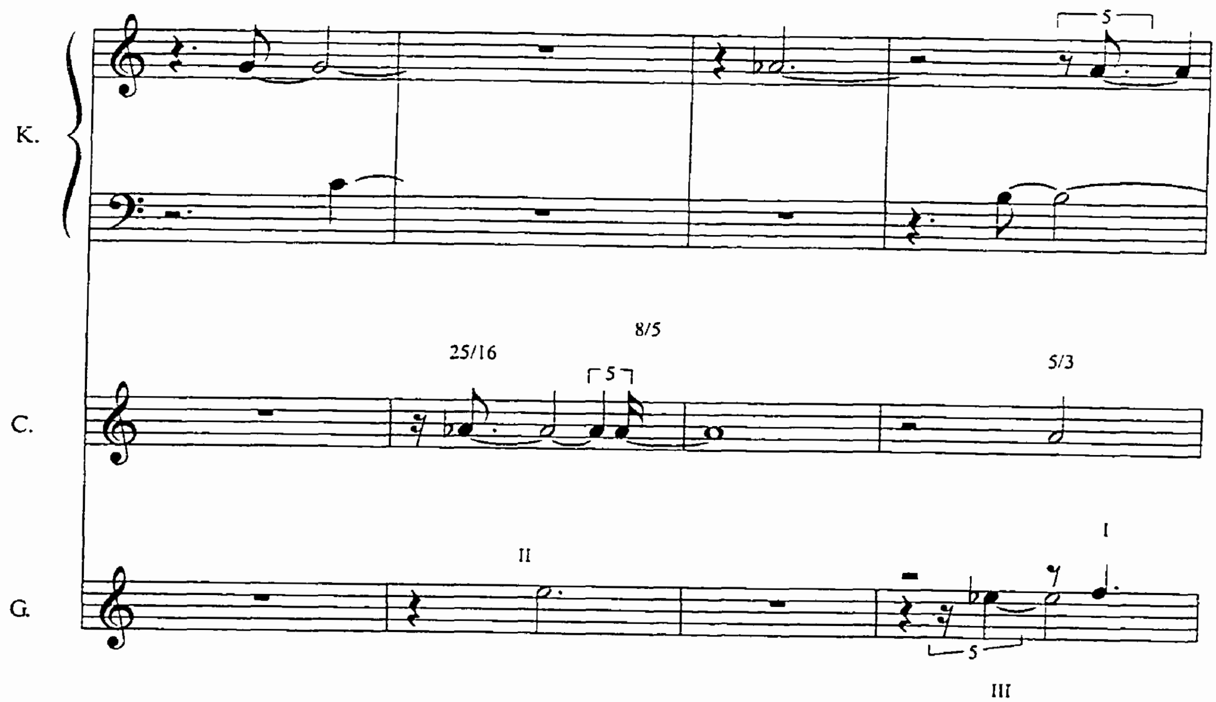
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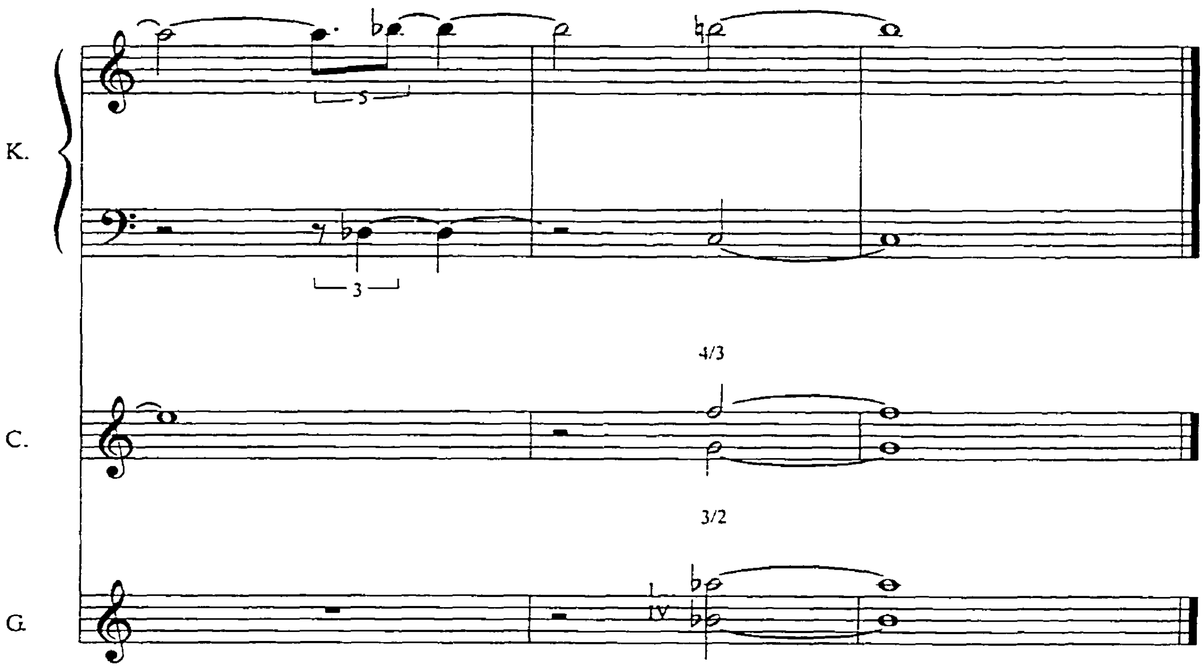
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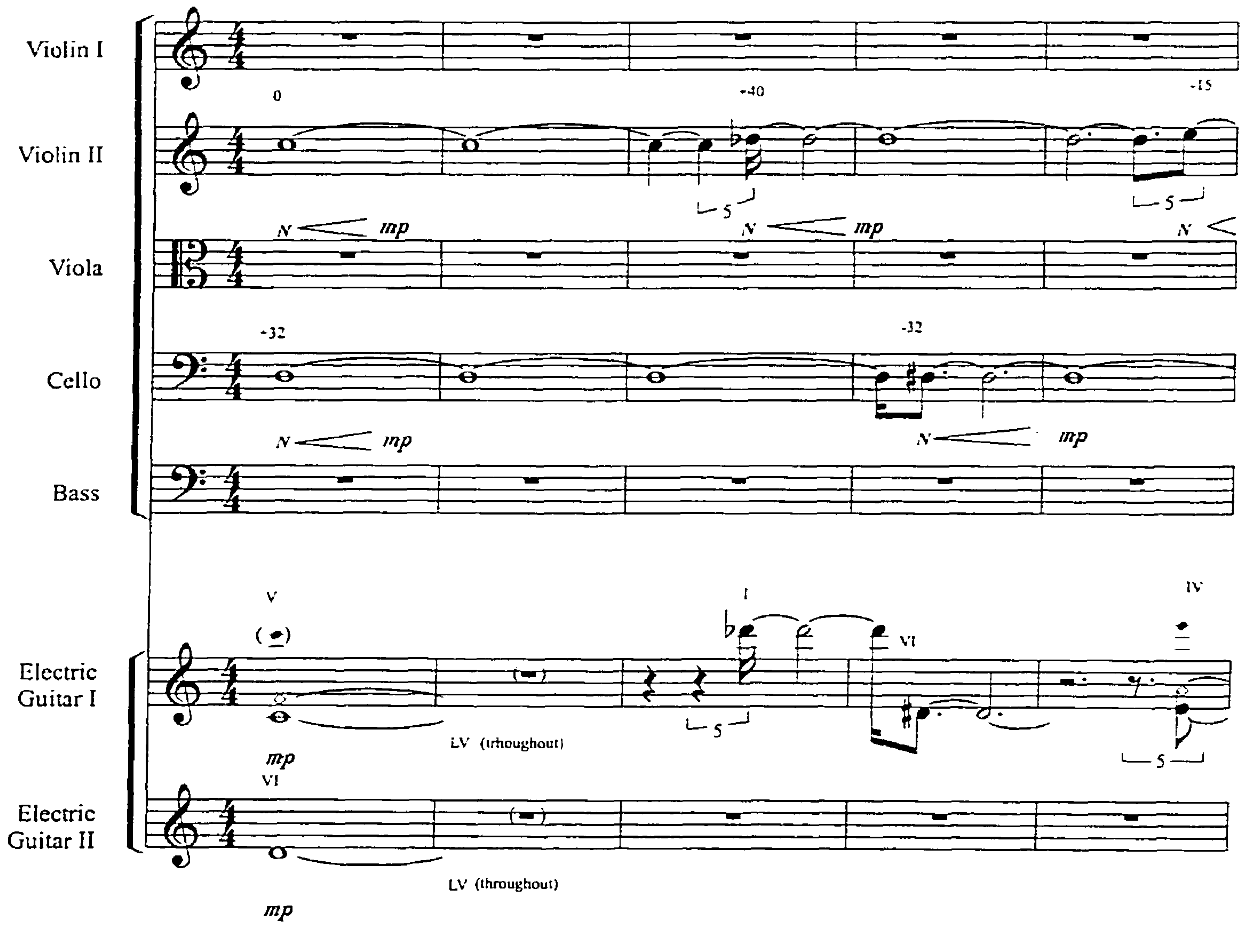
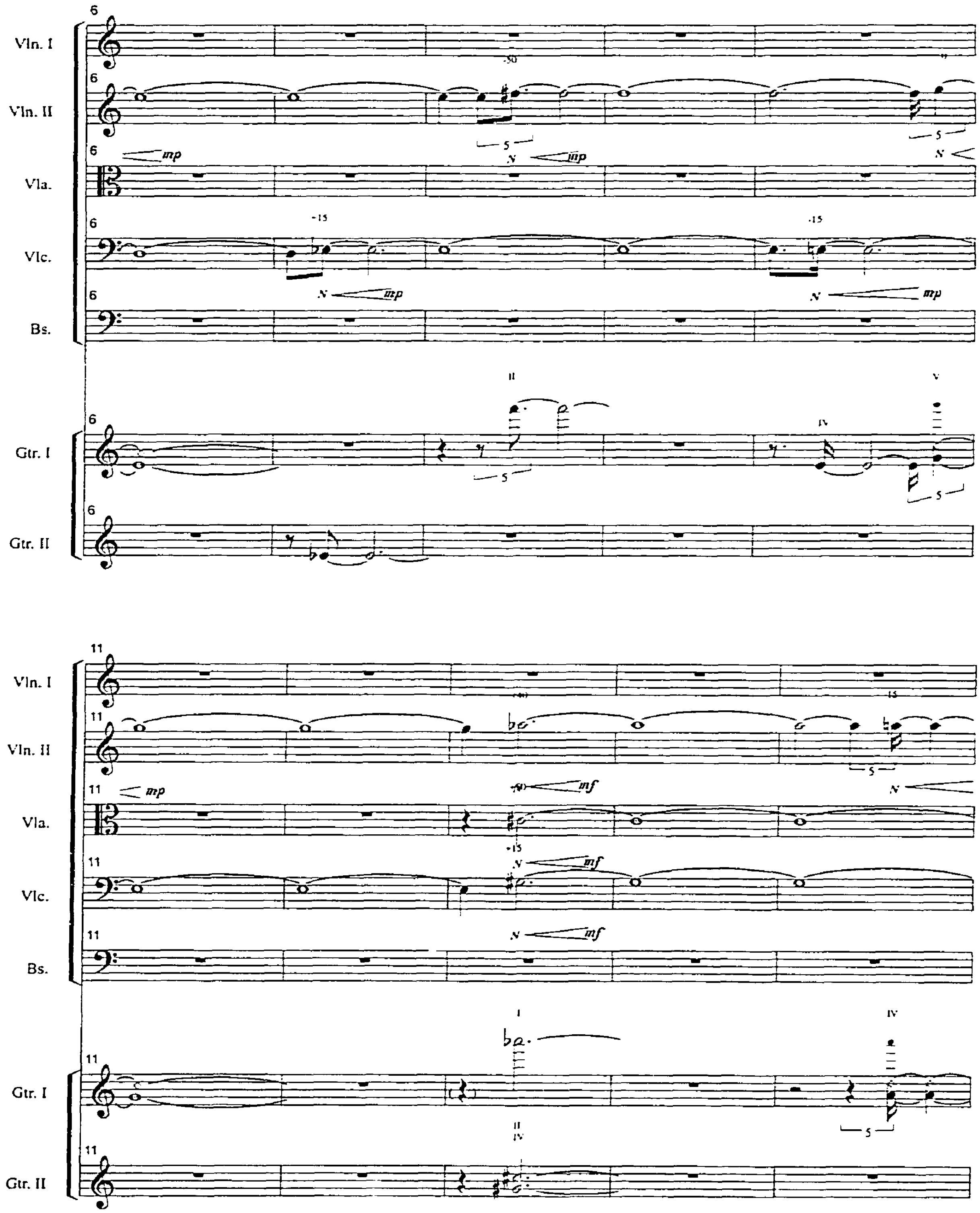
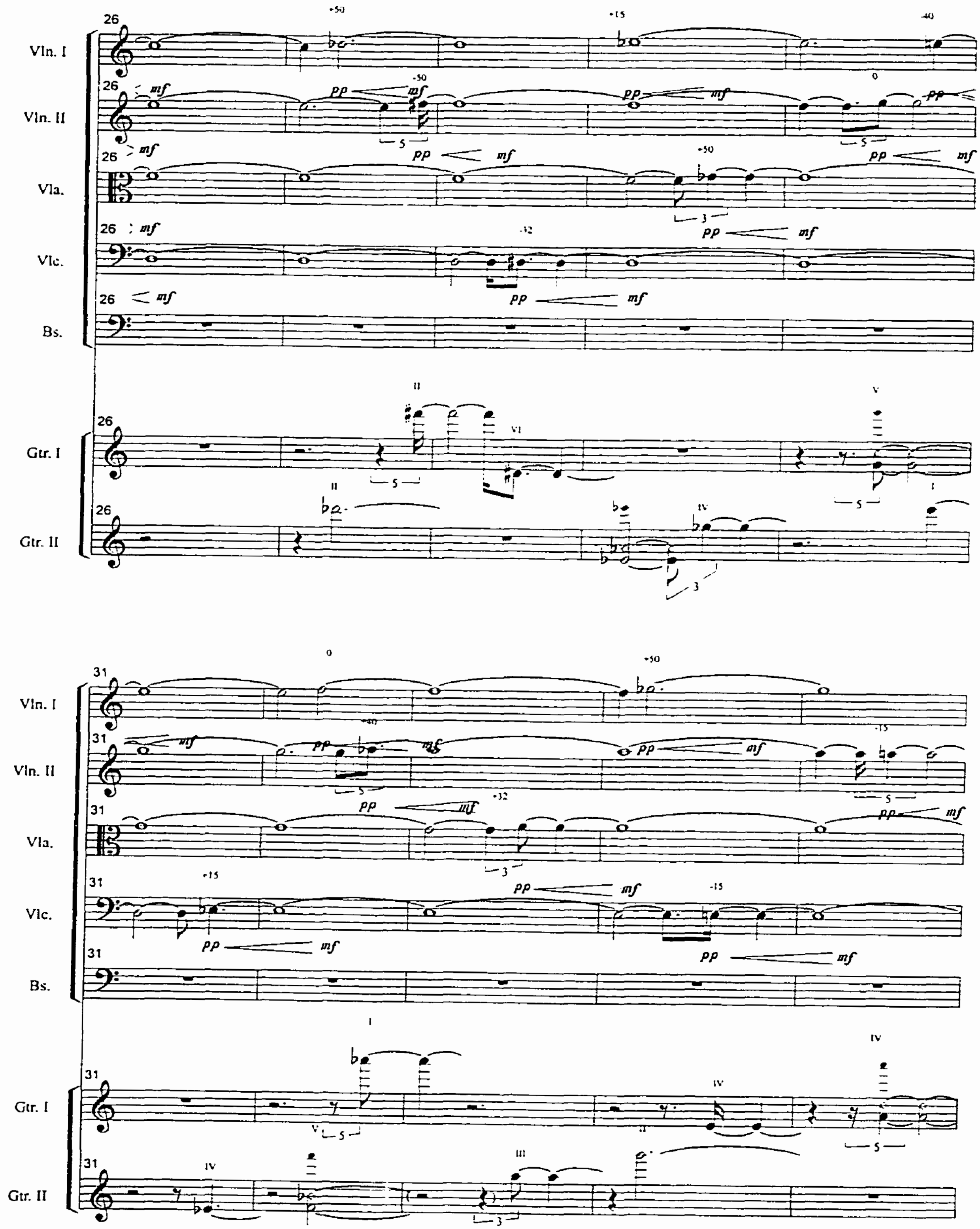
IV

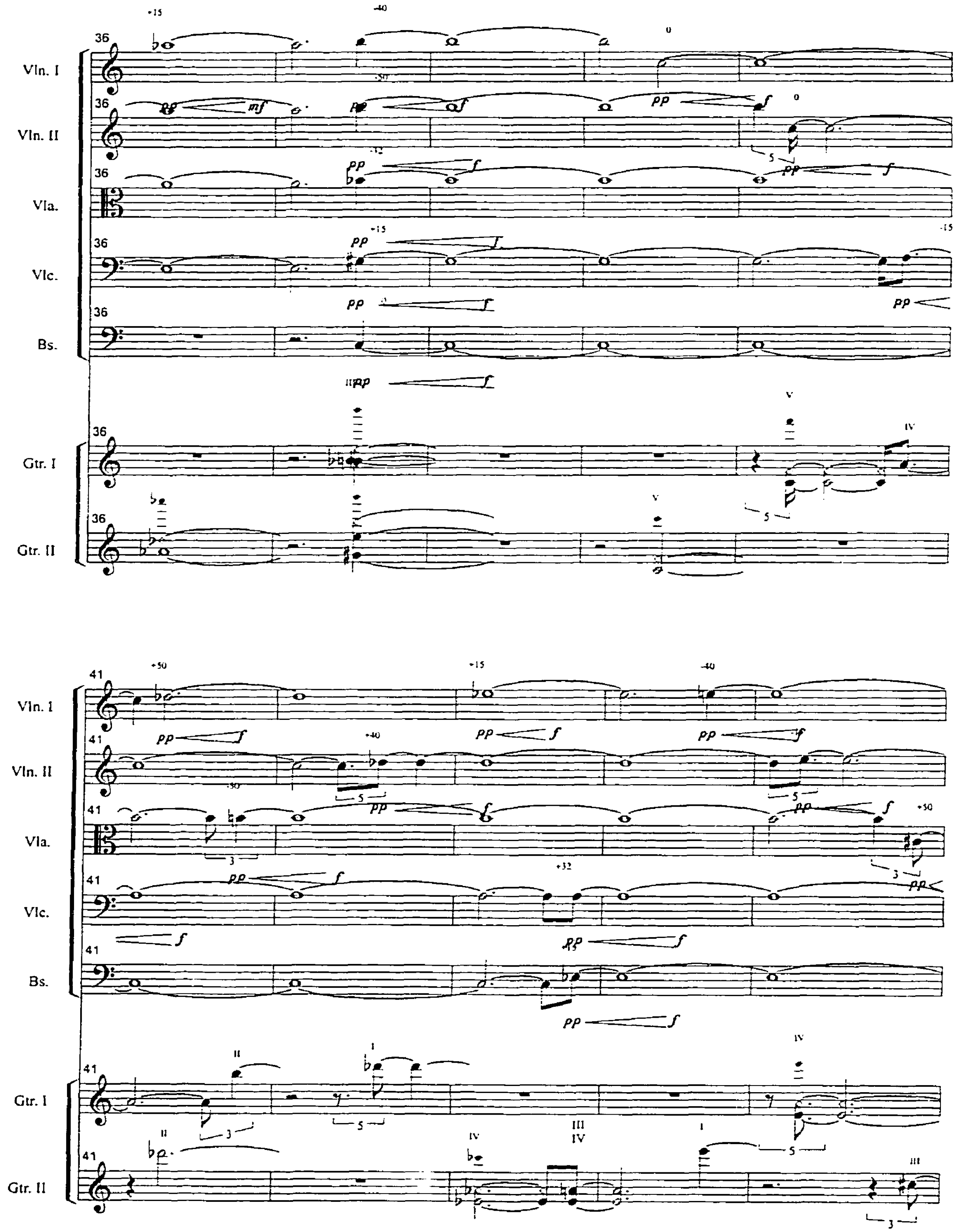
Music Box III

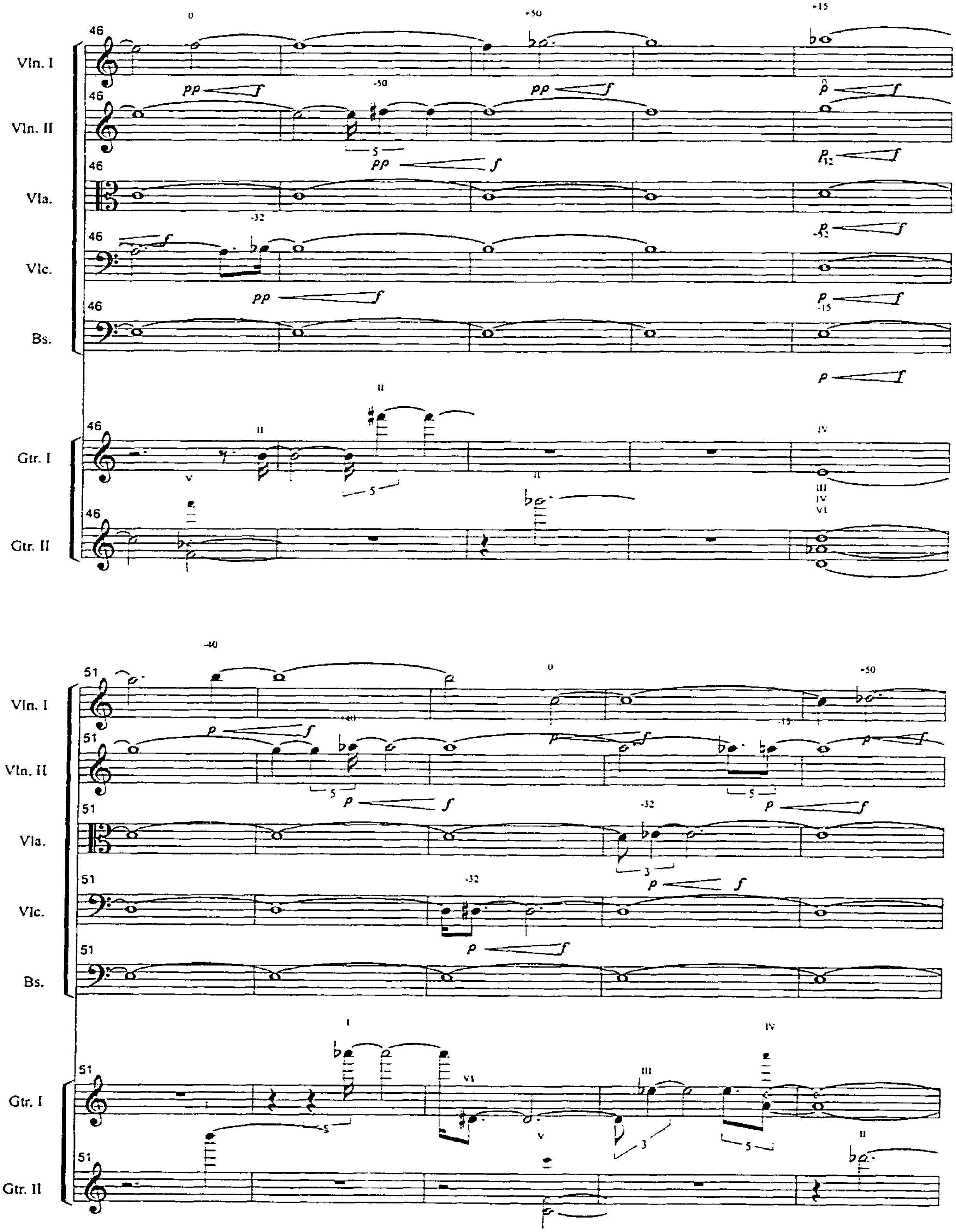



Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
mp
mf


16
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
mf
pp
21

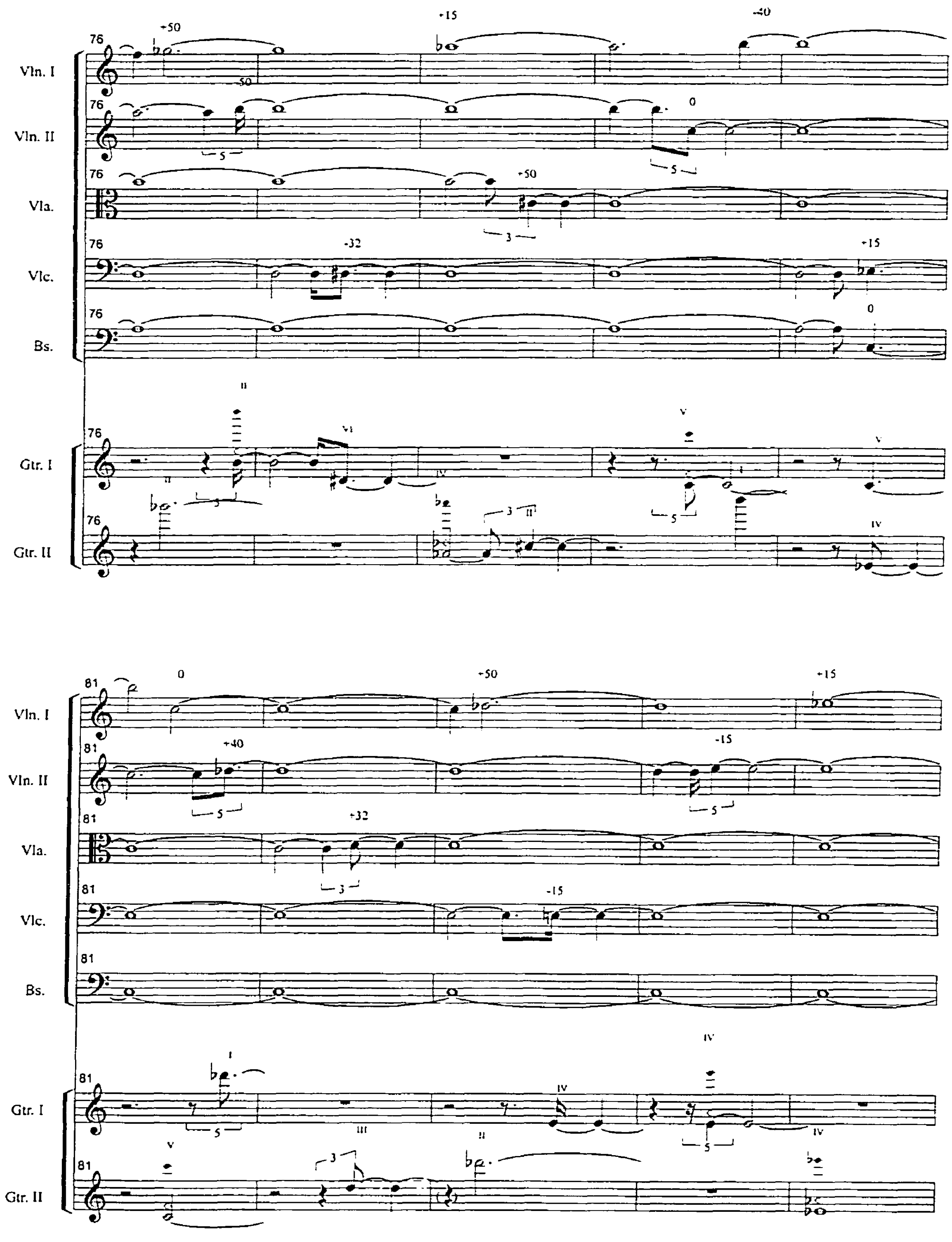

Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II


36
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
41
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II


Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II

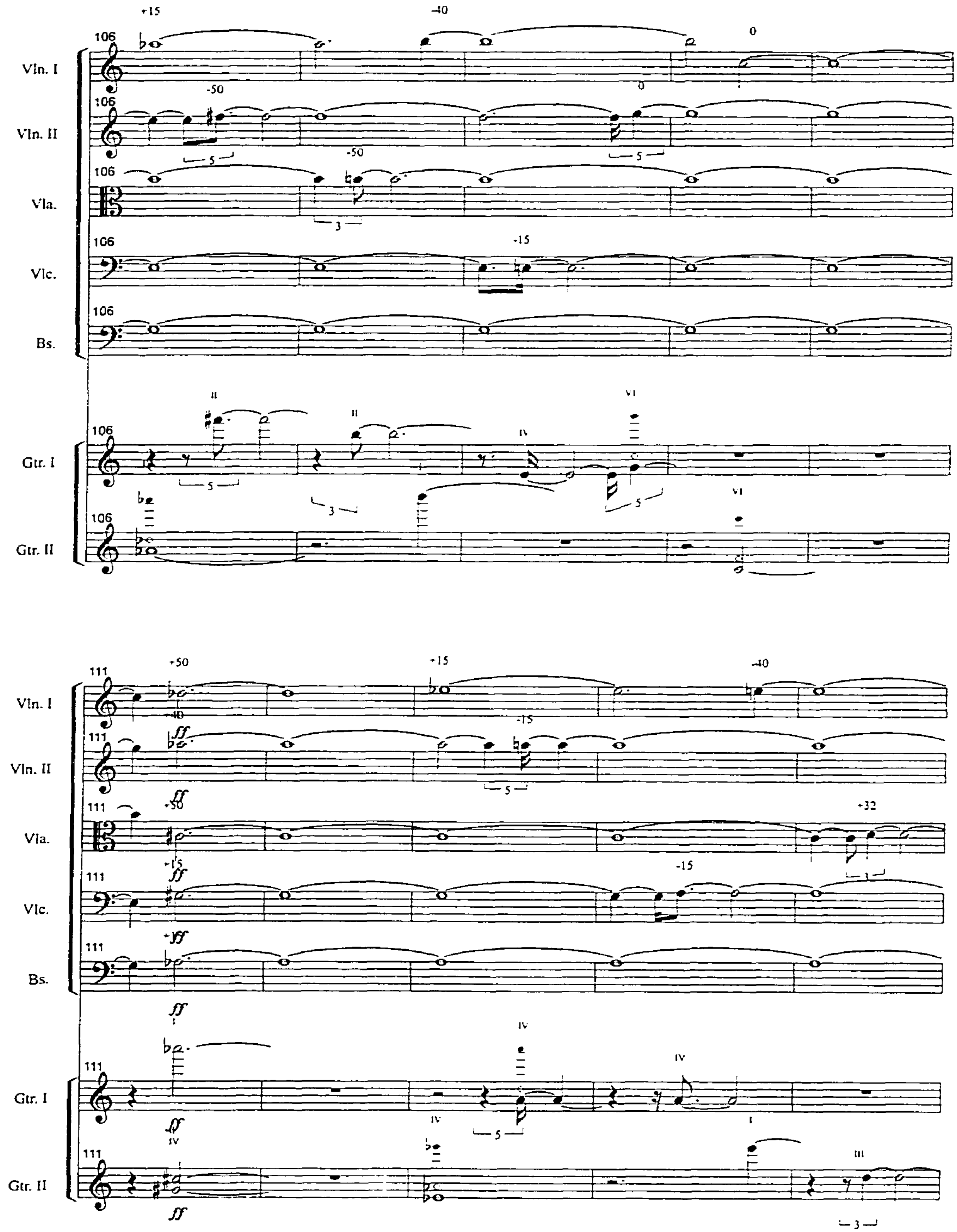

Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II

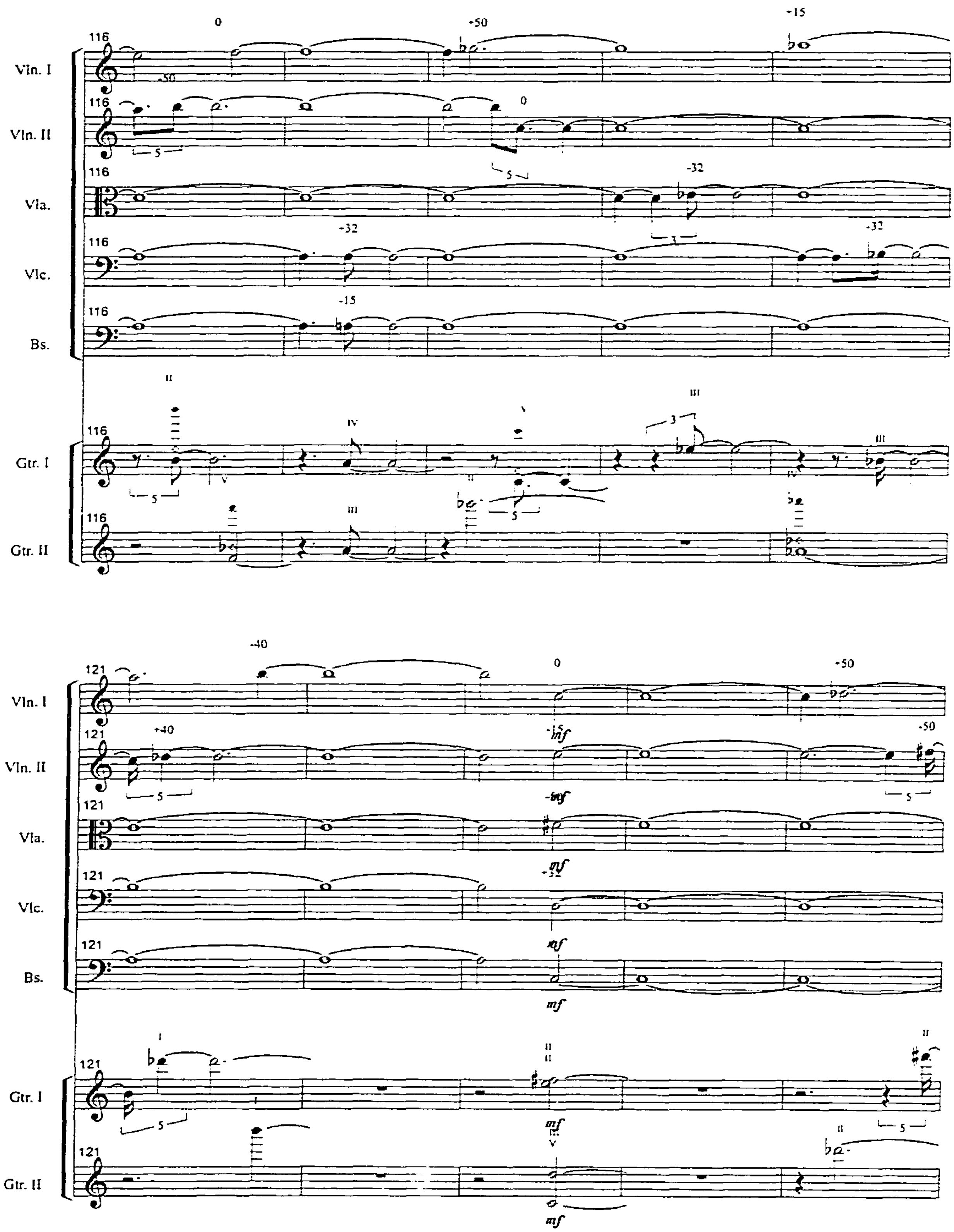

66
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
mf
f
0
+50
-50
+32
-32
II
III
V
IV - III
71
+15
-40
+40
-15
-50
I
IV

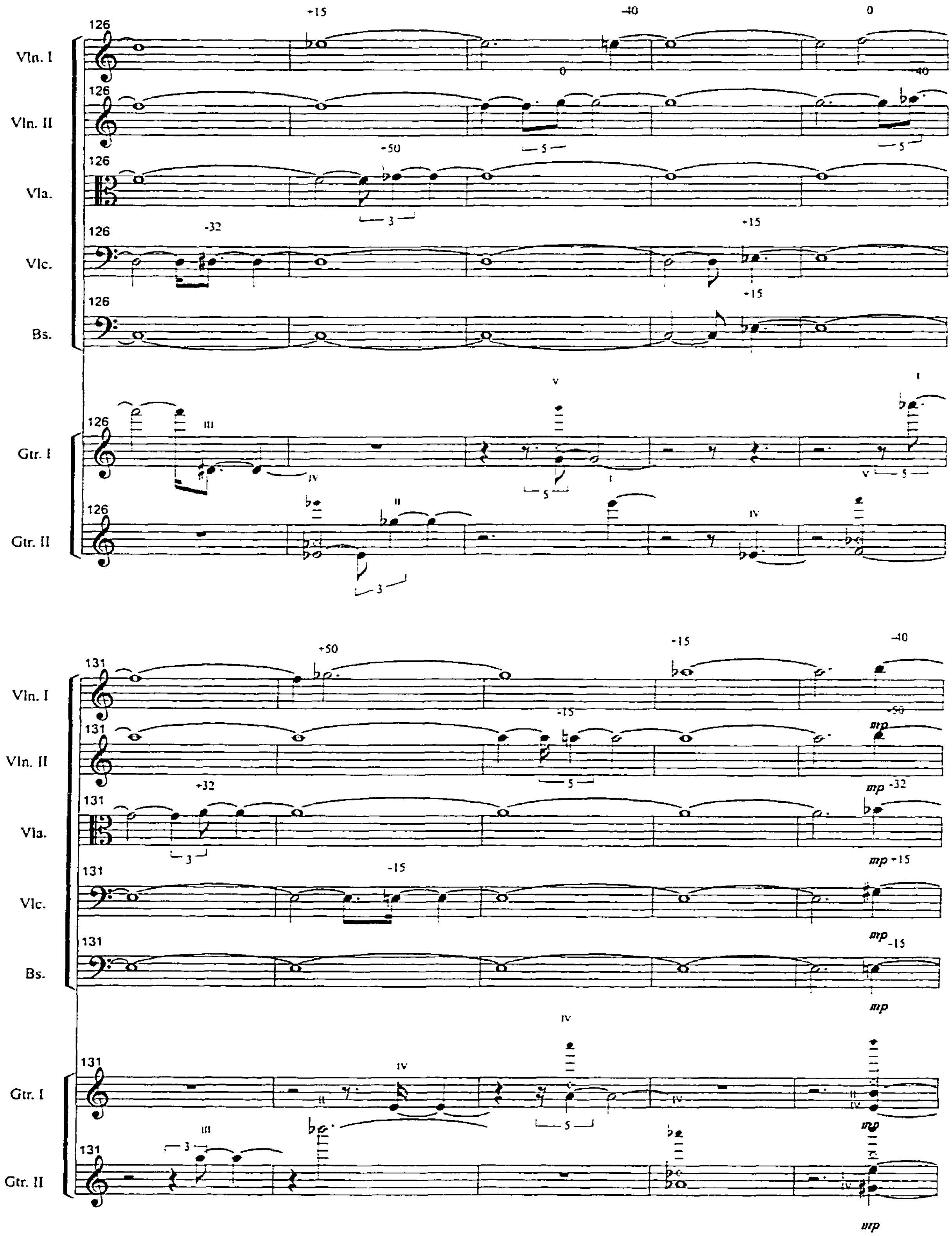

Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II


Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II

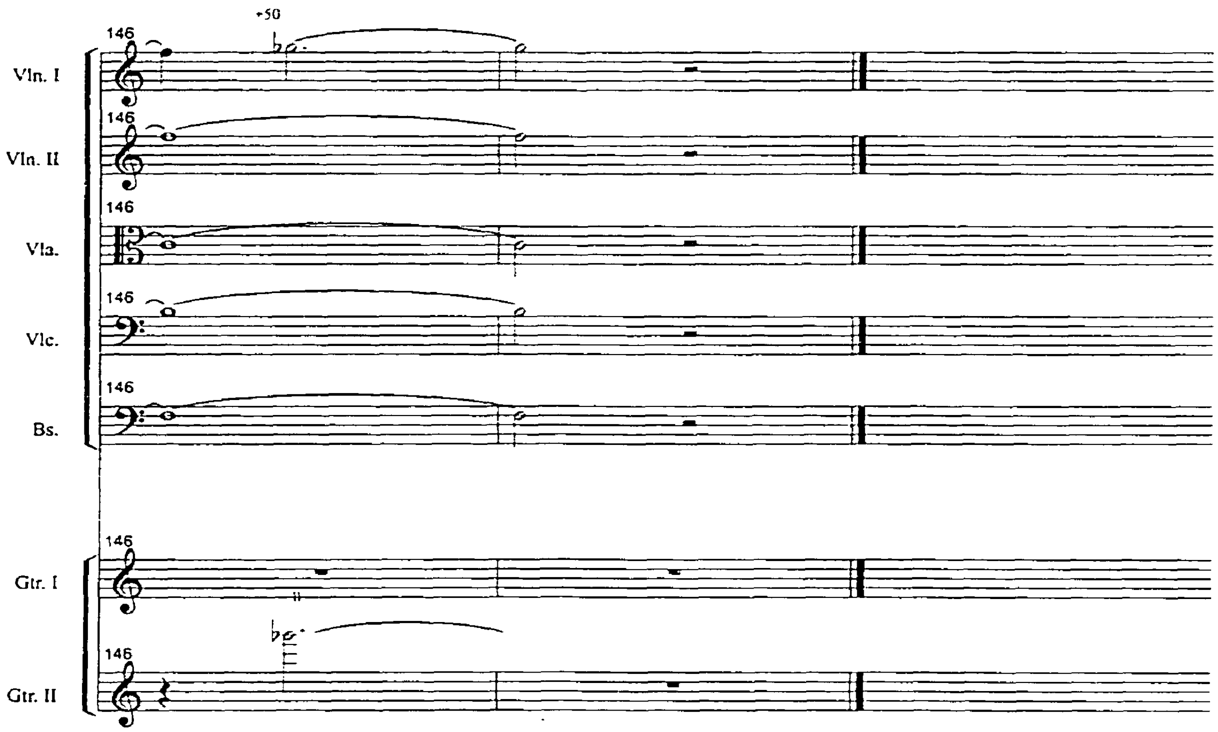

Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
96
101


106
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
111
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
ff

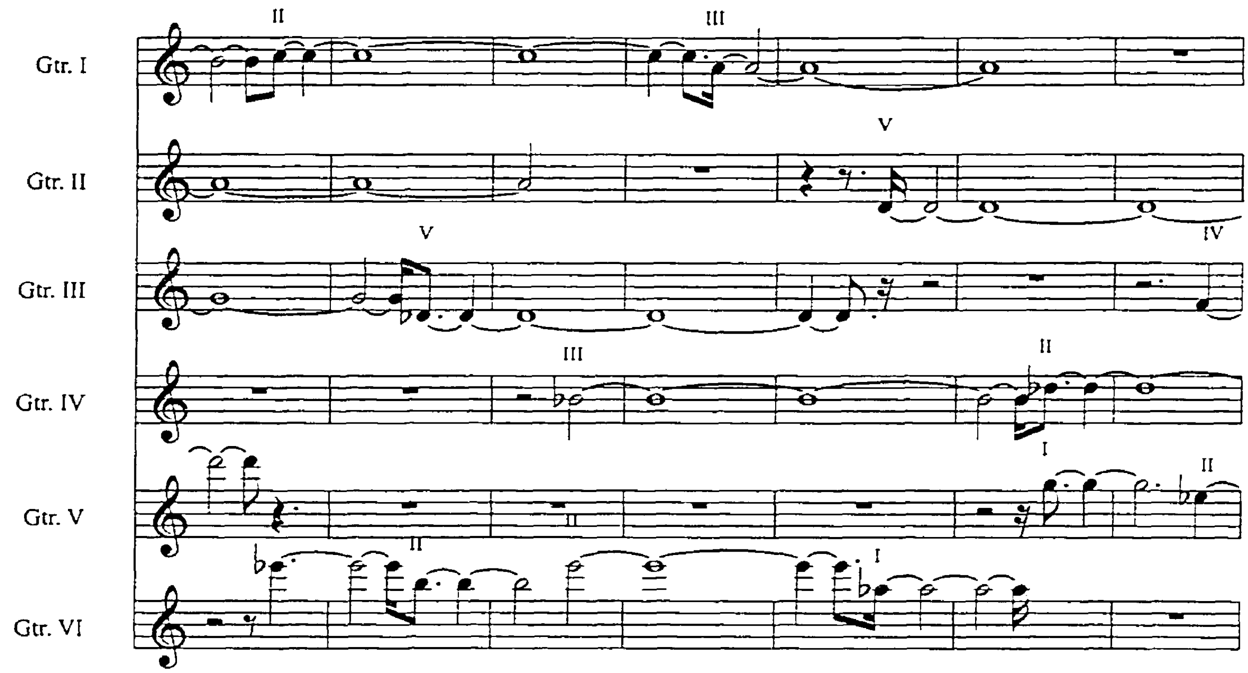

116
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
121
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
mf

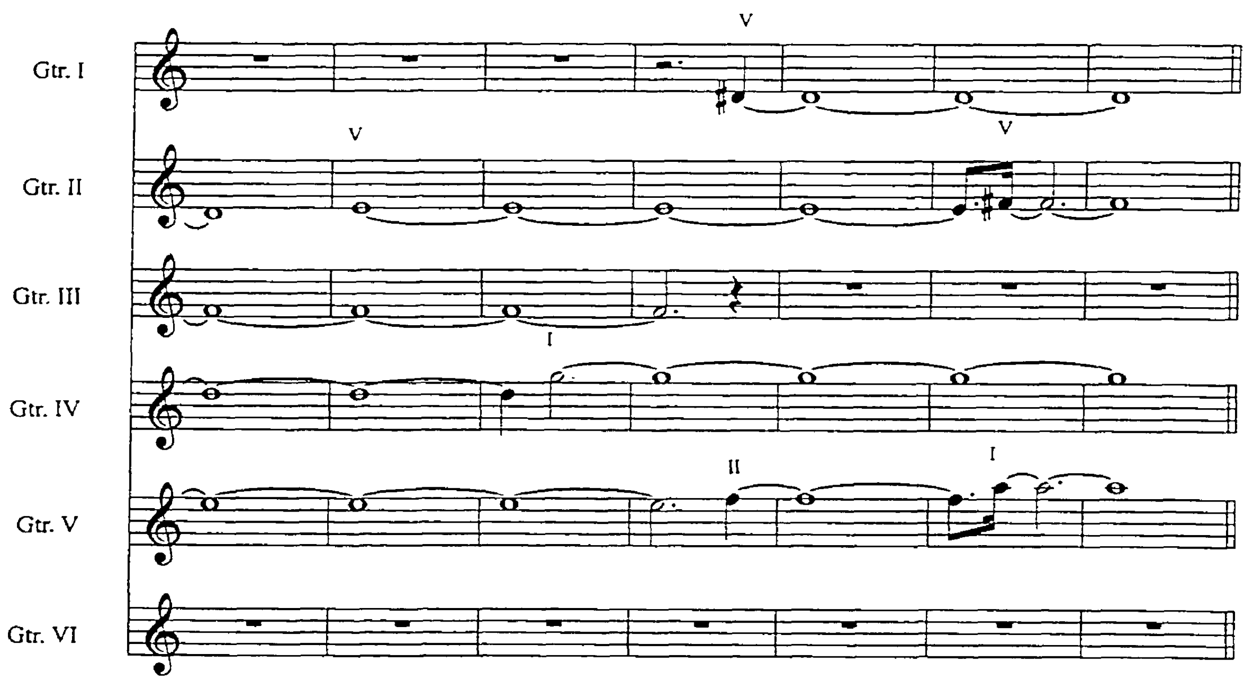

Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II


136
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II
141
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II

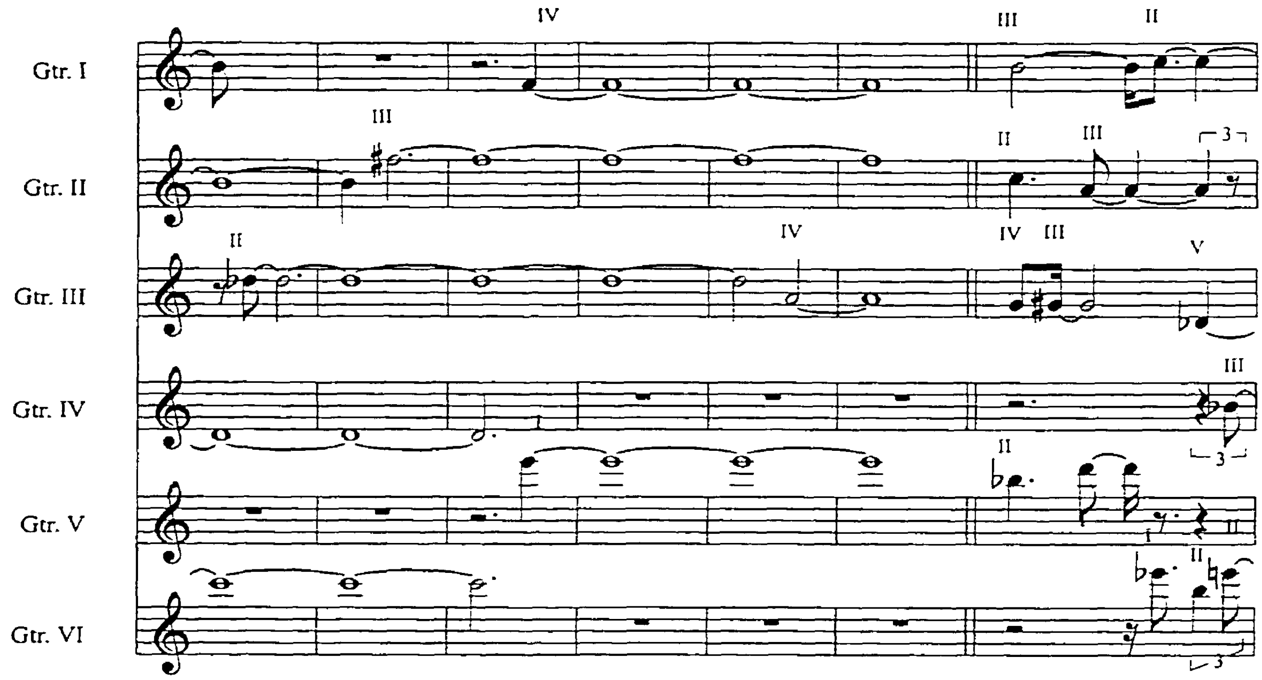

146
+50
Vln. I
Vln. II
Vla.
Vlc.
Bs.
Gtr. I
Gtr. II

Quartet for Six Guitars


Gtr. I
Gtr. II
Gtr. III
Gtr. IV
Gtr. V
Gtr. VI
II
III
V
V
IV
III
II
I
II
II
I


Gtr. I
Gtr. II
Gtr. III
Gtr. IV
Gtr. V
Gtr. VI
V
V
V
I
II
I


Gtr. I
Gtr. II
Gtr. III
Gtr. IV
Gtr. V
Gtr. VI
Gtr. I
Gtr. II
Gtr. III
Gtr. IV
Gtr. V
Gtr. VI


Gtr. I
Gtr. II
Gtr. III
Gtr. IV
Gtr. V
Gtr. VI


Gtr. I
Gtr. II
Gtr. III
Gtr. IV
Gtr. V
Gtr. VI


Gtr. I
Gtr. II
Gtr. III
Gtr. IV
Gtr. V
Gtr. VI
Gtr. I
Gtr. II
Gtr. III
Gtr. IV
Gtr. V
Gtr. VI


Gtr. I
Gtr. II
Gtr. III
Gtr. IV
Gtr. V
Gtr. VI
III
V
IV
II
I