Necessary and Sufficient Conditions so that a Commutative Ring Can be Embedded Into a strongly π-regular Ring.

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Abstract

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If R is commutative ring then R can be embedded into a strongly π-regular ring if and only if there exists a set of prime ideals $Y = \{P_\alpha\}\alpha \in \Lambda$ and for each P_α, a P_α-primary ideal Q_α such that:

I) Y is closed in the patch topology on Spec R.

II) $\bigcap_{\alpha \in \Lambda} Q_\alpha = \{0\}$.

III) for each $a \in R$ there is $n(a) \in N$ such that for $n > n(a)$, $\{P_\alpha \mid P_\alpha \in Y$ and $a^n \in Q_\alpha\}$ is patch open in Y.

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Dedication

I would like to dedicate my thesis to my nephew Alexander Bibic and to the memory of my mother Marianna Philippoussis.

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Introduction

In this thesis all rings will be assumed to have $1 \neq 0$ and that in all chapters, except chapter 2, all rings will be assumed to be commutative.

A ring S is said to be strongly π-regular if for every $a \in S$ there exists $n \geq 1$ such that $a^n \in a^{n+1}S$. It can equivalently be defined as S is strongly π-regular if for every $a \in S$ there exists $n \geq 1$ and $b \in S$ such that $a^n = a^{n+1}b$. G. Azumaya [2], W. D. Burgess, P. Menal [5], M. F. Dischinger [6] and Y. Hirano [7] have proven several properties of strongly π-regular rings using one or the other of the definitions above. The properties, necessary for the results of this thesis, are proven herein.

Two other ideas that are important in getting the desired result are that of patch R and the universal regular ring of R. The properties and the relations between patch R and Spec R were proven by M. Hochster [12]. Hochster showed that patch R must be the spectrum of some ring, which he described. Some of these properties will be used and proved in this thesis. The second idea - that of the universal regular ring of R -, its properties and its relation to patch R were proven by R. Wiegand [24] using the ring that Hochster found. Wiegand showed that Hochster's ring was a regular ring, that every homomorphism from R into a regular ring factors uniquely through φ, where φ is the map from R into the ring found by Hochster. So that ring must be the universal regular ring of R.

Chapter I

Definitions and Theorems

Definition 1.1: Let R be a ring. A right **R-module** is an Abelian group M together with a map $M \times R \to M$ written $(x,r) \to xr$, such that for all $x, y \in M$ and $r, s \in R$

a) $(x + y)\, r = xr + yr$,

b) $x\,(r + s) = xr + xs$,

c) $x\,(rs) = (xr)\, s$,

d) $x1 = x$.

Definition 1.2: Let A and B be two rings. M is an **A-B-bimodule** if M is a left A-module and a right B-module (with the same underlying additive group), such that $(ax)b = a(xb)$ for $a \in A$, $x \in M$ and $b \in B$.

Definition 1.3: Let M, N be R-modules. A map $\alpha: M \to N$ is an **R-linear map** if for $x, y \in M$, $a \in R$

a) $\alpha(x + y) = \alpha(x) + \alpha(y)$,

b) $\alpha(xa) = \alpha(x)a$.

Definition 1.4: Let φ be a homomorphism $\varphi: R \to R'$, the **dominion** $D = D(\varphi, R')$ is the largest subring of R' such that $\forall R''\ \forall \alpha,\beta: R' \to R''$ if $\alpha_{R} = \beta_{R}$ implies that $\alpha_{D} = \beta_{D}$.

Definition 1.5: Let φ be a homomorphism $\varphi: R \to R'$. φ is an **epimorphism** if $\forall R''$ $\forall \alpha,\beta: R' \to R''$ $\alpha\varphi = \beta\varphi$ implies $\alpha = \beta$.

Definition 1.6: A **maximal epimorphic extension** of φ, $E = E(\varphi,R')$ is the largest subring of R' such that $\varphi: R \to E$ is an epimorphism.

Definition 1.7: Let M be an ideal of R, $M \neq R$. M is called a **Maximal ideal** if $M \subseteq I \subseteq R$ then $I = R$ or $I = M$ for all I ideal of R.

Definition 1.8: Let R be a ring, the intersection of all maximal ideals of R is called the **Jacobson radical** of R and denoted **J(R)**.

Definition 1.9: P is said to be a **prime ideal** of R if

a) $P \neq R$,

b) $\forall x,y \in R$ $x,y \notin P \Rightarrow xy \notin P$.

Definition 1.10: Let R be a ring $a \in R$ is called **nilpotent** if $\exists n$ $a^n = 0$.

Definition 1.11: The **nilradical** of R is the set of all nilpotent elements of R and will be denoted by **nil(R)**. It is also the intersection of all the prime ideals of R.

Definition 1.12: Q is said to be a **primary ideal** of R if

a) $1 \notin Q$,

b) for $x,y \in R$, if $xy \in Q$ and $x \notin Q$ then $\exists n>0$ $y^n \in Q$.

Definition 1.13: A ring is **Noetherian** if every ideal is finitely generated.

Definition 1.14: A ring is **Local** if it is Noetherian and has exactly one maximal ideal.

Definition 1.15: Let R be a ring $S \subseteq R$ is a **multiplicative set** if

a) $x, y \in S \Rightarrow xy \in S$,

b) $0 \notin S$,

c) $1 \in S$.

Definition 1.16: Let R be a ring and S a multiplicative set. Let $F=\{(r,s) \mid r \in R \text{ and } s \in S\}$ $(r,s) \sim (r',s') \Leftrightarrow (rs' - r's)t = 0$ for some $t \in S$. The equivalence classes of (r,s) will be denoted by r/s and $S^{-1}R$ the set of equivalence classes. Note that $S^{-1}R$ is a commutative ring with operations $r/s + r'/s' = (rs' + r's) / ss'$ and $(r/s)(r'/s') = rr'/ss'$ and is called the **localization** of R and denoted by R_S

Definition 1.17: Let R be a ring, $a \in R$ is a **zero divisor** if there is a, $b \neq 0$ such that $ab = 0$.

Definition 1.18: Let R be a ring and A the set of all non zero divisors of R then the **classical ring of quotients** of R, $\mathbf{Q_{cl}(R)} = R_A$.

Definition 1.19: A ring R is called **regular** if $\forall r \in R\ \exists x \in R$ such that $r = rxr$.

Definition 1.20: Let R be a ring and $\varphi: R \to \hat{R}$, then $\hat{\mathbf{R}}$ is called the **universal regular ring of R** if for every homomorphism from R into a regular ring factors uniquely through φ.

Definition 1.21: A ring R is **strongly π-regular** if for every $a \in R$ there exists $n \geq 1$ such that $a^n \in a^{n-1}R$.

Definition 1.22: A commutative ring R is **π-regular** if every prime ideal of R is maximal.

Definition 1.23: Let R be a ring. The supremum of the lengths r taken over all strictly decreasing chains $P = P_0 \supset P_1 \supset P_2 \supset \ldots \supset P_r$ of prime ideals of R is called the **Krull dimension** and it is denoted by **dim(R)**.

Definition 1.24: The set of all prime ideals of a ring R is called the **Spectrum of R** and written **Spec R**.

Definition 1.25: Let R be a commutative ring, then $\mathbf{B(R)} = \{e \in R \mid e^2 = e\}$.

Definition 1.26: Let $a, b \in B(R)$ then $a \vee b = a + b - ab$.

Definition 1.27: $\emptyset \neq I \subseteq B(R)$ is called an ideal of $B(R)$ if for $a, b \in I$

a) $a \vee b \in I$,

b) $ac = c$ then $c \in I$.

Theorem 1.28: If I is an ideal of R then $I \cap B(R)$ is an ideal of $B(R)$.

Proof:

a) Let $a, b \in I \cap B(R)$ then $a + b - ab \in I$ so $a \vee b \in I$ and $(a \vee b)^2 = (a + b - ab)^2 = a^2 + ab - aab + ba + b^2 - bab - aba - abb + (ab)^2 = a + ab - ab + ab + b - ab - ab - ab + ab = a + b - ab = a \vee b$ so $a \vee b \in I \cap B(R)$.

b) Let $a \in I \cap B(R)$ and $c \in B(R)$ such that $ac = c$. $a \in I$ so $c = ac \in I$. $a, c \in B(R)$ so $(ac)^2 = acac = aacc = ac$ so $c = ac \in B(R)$. This shows that $c \in I \cap B(R)$.

Theorem 1.29: If I is an ideal of $B(R)$ then RI is an ideal of R.

Proof:

a) Let $a, b \in R$ and $e, f \in I$ then $ae, bf \in RI$ and $(ae + bf)(e \vee f) = (ae + bf)(e + f - ef) = aee + aef - aeef + bfe + bff - bfef = ae + aef - aef + bfe + bf - bfe = ae + bf$. But $(e \vee f) \in I$ and $(ae + bf) \in R$. So $(ae + bf) = (ae + bf)(e \vee f) \in RI$.

b) Let $a, c \in R$ and $e \in I$ so that $ae \in RI$ then $c(ae) = (ca)e$ but $ca \in R$ so $(ca)e \in RI$ and therefore $c(ae) \in RI$.

Definition 1.30: Let R be a ring and $M \in \text{Spec } B(R)$ then the **Pierce stalk at** M is R/RM.

Definition 1.31: Let A be a set and T a set of subsets of A. T is called a **topology** and the elements of T **open sets** if it satisfies the following conditions:

a) any union of open sets is open,

b) the intersection of two open sets is open,

c) the set A and the empty set are open.

Definition 1.32: $C \subseteq A$ is called **closed** if C\A is an open set.

Definition 1.33: $K \subseteq A$ is called **clopen** if K is both closed and open in A.

Definition 1.34: Let X be a topological space and $x \in X$ then a **neighbourhood** of x, **N(x)**, is a set containing an open set containing x.

Definition 1.35: A topological space is called **compact** if every open cover possesses a finite subcover.

Definition 1.36: A topological space is called **Hausdorff** if for any two different points a, b there exists N(a) and N(b) such that $N(a) \cap N(b) = \emptyset$.

Definition 1.37: Let X and Y be topological spaces. A map $f: X \to Y$ is called **continuous** if the inverse image of an open set in Y is always open in X.

Definition 1.38: A bijective map $f: X \to Y$ is called a **homeomorphism** f and f^{-1} are both continuous.

Definition 1.39: Let A be a subset of a commutative ring R. **V(A)** $= \{P \in \text{spec } R \mid A \subseteq P\}$ and **D(A)** $= (\text{Spec } R) \setminus V(A)$.

Theorem 1.40: $F = \{D(I) \mid I \text{ an ideal of } R\}$ forms a topology on spec(R) called the **Zariski topology**. To prove this we must show that F satisfies all three properties of a topology.

Proof:

a) Let $\{I_\alpha\}\alpha \in \Lambda$ be a set of ideals of R (not necessarily countable). $\bigcup D(I_\alpha) = \bigcup ((\text{Spec } R) \setminus V(I_\alpha)) = (\text{Spec } R) \setminus \bigcap V(I_\alpha) = (\text{Spec } R) \setminus V(\sum I_\alpha)$ since $\sum I_\alpha$ is an ideal of R there is I an ideal of R such that $\bigcup D(I_\alpha) = D(I)$.

b) Let I, J be ideals of R. $V(I) \cup V(J) = \{P \in \text{spec } R \mid I \subseteq P\} \cup \{P \in \text{spec } R \mid J \subseteq P\} = \{P \in \text{Spec } R \mid I \subseteq P \text{ or } J \subseteq P\} = V(I \cap J)$ but $I \cap J$ is an ideal of R so there is K an ideal of R

such that $D(I) \cap D(J) = (\text{spec } R) \setminus (V(I) \cup V(J)) = (\text{spec } R) \setminus V(K) = D(K)$.

c) $D(R) = \text{spec } R$ since $R \not\subseteq P$. Since $\{0\}$is an ideal of R and for any P $0 \in P$ $D(0) = \emptyset$.

Definition 1.41: Let X be a topological space. A set B of open sets is called an **open subbasis** for the topology if every open set is a union of finite intersections of sets in B.

Definition 1.42: Let R be a ring and X = Spec R then a new topology, called the **patch topology**, is created by taking all compact open sets of X and their complements as an open subbase.

Definition 1.43: Let R be a ring, then **patch R** is spec R with the patch topology.

Chapter II

If a commutative ring R can be embedded into a strongly π-regular ring S then it can be embedded into a commutative strongly π-regular ring

In this chapter (and only in this chapter) the term Ring will be applied to any ring (commutative and non-commutative), and commutative ring for any ring that must be commutative. If a commutative ring R can be embedded into a strongly π-regular ring S, it will be shown that the condition that S is strongly π-regular implies that there must be a strongly π-regular commutative ring T such that R can be embedded in T.

This must be proven since commutativity does not necessarily get transmitted by the function of embedding. One such example is the ring $\mathbb{R}$, which is a commutative ring and can be embedded into M_2 = {the ring of all 2x2 matrices} by $\varphi:\mathbb{R}\to M_2$ where $a \in \mathbb{R}$ and

$$\varphi(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

M_2 is not commutative but $\mathbb{R}$ is.

<u>Lemma 2.1:</u> Let R be a ring if $s, x \in R \ni s^n = s^{n-1} x$ then $s^n = s^{2n} x^n$.

<u>Proof:</u> $s^n = s^{n-1} x$

$= s s^n x$

$= s(s^{n-1}x)x$

$= s^2 s^n x^2$ Continuing in the same way

$\vdots$

$= s^{n-1}(s^{n-1}x)x^{n-1}$

$= s^{2n} x^n$

Lemma 2.2: Let R be a ring. If a, b, c $\in$ R and $a^r = a^{r-1}b$ and $b^m = b^{m-1}c$, then there exists n such that $a^n = a^{n-1}b$ and $b^n = b^{n-1}c$.

Proof: $a^{m+r} = a^m a^r = a^m a^{r-1}b = a^{m+r-1}b$ and $b^{r+m} = b^r b^m = b^r b^{m-1}c = b^{r+m-1}c$ so by letting $n = r + m = m + r$, $a^n = a^{n-1}b$ and $b^n = b^{n-1}c$.

Lemma 2.3: [most of this proof came from 6] Let S be strongly π-regular then $\forall s \in S\ \exists n \geq 1\ \exists x \in S \ni s^n = s^{n-1}x$ and $s^n x = xs^n$

Proof: Let s, y, z $\in$ R such that $sn^r = s^{r-1}y$ and $y^r = y^{r-1}z$. There is no problem for the two equations to have the same exponent because of lemma 2.2. Choosing $a = s^r$, $b = y^r$, $c = z^r$ and by lemma 2.1 $a = a^2b$ and $b = b^2c$. Let d be such that $(c - a)^r = (c - a)^{r-1}d$.

$ac = a^2bc = a(a^2b)bc = a^3(b^2c) = a(a^2b) = aa = a^2$ so $a^2 - ac = 0$ (1)

$abc = (a^2b)bc = a^2(b^2c) = a^2b = a$ (2)

Using (1) $(c - a)^2 = c^2 - ca - ac + a^2 = c^2 - ca = c(c - a)$ (3)

Using (1), (2) and (3) $ab(c - a)^2 = abc(c - a) = a(c - a) = ac - a^2 = 0$ (4)

Using $b^2c = b$ and (3) $b^2(c - a)^2 = b^2c(c - a) = b(c - a)$ (5)

Using (5) repeatedly $b^m(c - a)^m = b^{m-1}(c - a)^{m-1} = \ldots = b(c-a)$ for any m (6)

Using $(c - a)^{r-1}d = (c - a)^r$ and (6) $b^r(c - a)^{r-1}d = b^r(c - a)^r = b(c - a)$ (7)

Using (6) and (7) $b^2(c - a) = b^{r-1}(c - a)^{r-1}d = b(c - a)d$ (8)

Using (4), (5) and (7) $ab(c - a) = ab^r(c - a)^{r-1}d = ab(c - a)^2d = 0d = 0$ (9)

Using (8) and (9) $0 = ab(c - a) = ab(c - a)d = ab^2(c - a)$ (10)

Using $b = b^2c$ and (10) $ab^2a = ab^2c = ab$ (11)

Using (2) and (9) $aba = abc = a$ (12)

Using (11) and (12) $a = aba = (ab)a = (ab^2a)a = ab^2a^2$ (13)

By (13), and returning to our original notation $s^r = s^r a^{2r} s^{2r} = s^r a^{2r} s^{r-1} s^{r-1}$. Multiplying on the right by s^r, $s^{2r} = s^r a^{2r} s^{r-1} s^{2r-1}$. But $s^{2r-1} s^r a^{2r} s^{r-1} = s^{r-1} s^{2r} a^{2r} s^{r-1} = s^{r-1} s^r a^r s^{r-1} = ss^{2r} a^r s^{r-1} = ss^r s^{r-1} = s^{2r}$. So if $n = 2r$ and $x = s^r a^{2r} s^{r-1}$ then $s^n = xs^{n-1} = s^{n-1}x$.

for x and n as chosen above $s^n x = (xs^{n-1})x = x(s^{n-1}x) = xs^n$

<u>Lemma 2.4:</u> [2] Let S be strongly π-regular and $s \in S$. Then $\exists n \geq 1$ and a unique $x \in S$ such that $s^n = s^{2n} x$, $sx = xs$ and $x^2 s^n = x$

<u>Proof:</u> By lemma 2.3 $\exists y \in S$ $s^n = s^{n-1} y$ and $s^n y = ys^n$ Let $x = y^n s^n y^n$

$$
\begin{aligned}
s^{2n} x &= s^{2n} y^n s^n y^n \\
&= (s^{2n} y^n) s^n y^n \\
&= (s^n) s^n y^n && \text{By lemma 2.1} \\
&= s^{2n} y^n \\
&= s^n && \text{By lemma 2.1}
\end{aligned}
$$

$$sx = s\, y^n s^n y^n = s^{n-1} y^n y^n = s^n y^{2n-1} = y^{2n-1} s^n = y^{2n-1} y s^{n-1} = y^n s^n y^n s = xs$$

$$
\begin{aligned}
x^2 s^n &= xxs^n = (y^n s^n y^n)(y^n s^n y^n) s^n \\
&= y^n s^n y^n y^n (s^{2n} y^n) && \text{By lemma 2.3} \\
&= y^n s^n y^n y^n s^n && \text{By lemma 2.1} \\
&= y^n s^n y^n s^n y^n && \text{By lemma 2.3} \\
&= y^n (s^{2n} y^n) y^n && \text{By lemma 2.3} \\
&= y^n s^n y^n && \text{By lemma 2.1} \\
&= x.
\end{aligned}
$$

Assume that there is $x, y \in S$ such that x and y satisfy the equations.

Then $y = y^2 s^n = y^2 (s^{2n} x) = y (y s^{2n}) x = y (s^{2n} y) x = y s^n x = y (s^{2n} x) x = (y s^{2n}) x^2 = (s^{2n} y) x^2 = s^n x^2 = x$.

Therefore $x = y$ and the solution is unique.

Lemma 2.5: Let $\{S_\alpha\}_{\alpha\in\Lambda}$ be a set of strongly π-regular subrings of a ring R. Then $S = \bigcap S_\alpha$ is strongly π-regular.

Proof: Let $s \in S$ and fix $\alpha \in \Lambda$ then there is an n>0 and $x \in S_\alpha$ such that it satisfies lemma 2.4.

Now consider S_β, $s^n \in S_\beta$ since $s \in S_\beta$. By lemma 2.4 $\exists m>1$ $\exists y \in S_\beta$ $(s^n)^m = (s^n)^{2m}\,y$, $(s^n)y = y(s^n)$ and $y^2\,(s^n)^m = y$.

But

$s^{nm} = s^{n(m-1)}s^n = s^{n(m-1)}s^{2n}x = s^{n(m-2)}s^n s^{2n}x = s^{n(m-2)}s^{4n}x^2 = s^{n(m-3)}s^{6n}x^3 = \ldots = s^{2nm}x^m$

$s^n x^m = s^{n-1}xsx^{m-1} = s^{n-1}x^2sx^{m-2} = \ldots = s^{n-1}x^m s = \ldots = x^m s^n$

$x^{2m}s^{nm} = x^{2(m-1)}(x^2s^n)s^{n(m-1)} = xx^{2(m-2)}(x^2s^n)s^{n(m-2)} = x^2x^{2(m-3)}(x^2s^2)s^{n(m-3)} = \ldots = x^m$

if the property that $s \in S_\alpha$ is used.

But y must be unique, therefore $y = x^m \in S_\beta$. This holds for any β.

When m = 1 then $x \in S_\beta$ from the above line.

When m > 1

$x = x^2s^n = xxs^n = x(x^2s^n)s^n = x^2xs^{2n} = x^2(x^2s^n)s^{2n} = x^3xs^{3n} = \ldots = x^{m-2}xs^{(m-2)n} = x^m s^{(m-1)n}$

But $x^m \in S_\beta$ and $s \in S_\beta$ so $x^m s^{(m-1)n} \in S_\beta$ therefore $x \in S_\beta$ and $x \in S$.

Lemma 2.6: Let $\varphi: R \to T$ be a homomorphism with dominion D, then the following properties of $t \in T$ are equivalent:

a) $t \in D$.

b) If M is a T-T-bimodule and $x \in M$ has the property that $rx = xr$ for all $r \in R$ then $tx = xt$.

c) $t \otimes 1 = 1 \otimes t$ in $T \otimes_R T$.

d) If M and N are right T-modules and $\alpha: M \to N$ is R-linear, then $\alpha(xt) = \alpha(x)t \ \forall x \in M$.

Proof:

a) $\Rightarrow$ b) Let $\alpha, \beta: T \to T \times M$ as $\alpha(b) = (b,0)$ and $\beta(b) = (b, bx - xb)$. Both α and β are ring homomorphisms, and $\alpha\varphi = \beta\varphi$ by the hypothesis on x. So if $t \in D$, then $(t,0) = \alpha(t) = \beta(t) = (t, tx - xt)$ and hence $0 = tx - xt$ so $tx = xt$.

b) $\Rightarrow$ c) $T \otimes_R T$ is a T-T-bimodule and take $x = 1 \otimes 1$ which is the multiplicative identity in $T \otimes_R T$ so by b $t \otimes 1 = t(1 \otimes 1) = tx = xt = (1 \otimes 1)t = 1 \otimes t$.

c) $\Rightarrow$ d) Define $\beta: T \otimes_R T \to N$ as $\beta(b \otimes b') = \alpha(xb)b'$. This is well defined because α is R-linear. $\alpha(xt) = \alpha(xt)1 = \beta(t \otimes 1) = \beta(1 \otimes t) = \alpha(x1)t = \alpha(x)t$ since $t \otimes 1 = 1 \otimes t$ by c).

d) $\Rightarrow$ a) Let $\alpha, \beta: T \to T'$ be homomorphisms such that $\alpha\varphi = \beta\varphi$. If T' is considered as a right T-module by means of β, then α becomes an R-linear map. But then $\alpha(t) = \alpha(1t) = \alpha(1)\beta(t) = \beta(t)$. Therefore $t \in D$.

Lemma 2.7: [5] Suppose $R \to T$ is a homomorphism with dominion D. If for any $a \in D$ there is $b \in T$ satisfying $ab = ba$, $a = a^2b$ and $b^2a = b$ then $b \in D$.

Proof: Note that $x \in D$ if and only if $1 \otimes x = x \otimes 1$ in $T \otimes T$ (by lemma 2.5). From $a = a^2b$ and $a \neq 0$ we get $a(1 - ab) = a - a^2b = 0$ so $(a \otimes 1)(1 - ab) = a(1 - ab) \otimes 1 = 0 \otimes 1 = 0 = 1 \otimes a(1 - ab) = (1 \otimes a)(1 - ab)$. Since $a \neq 0$ $1 \otimes a \neq 0$ hence $1 - ab = 0$ and $ab = 1$. So $a \otimes 1 = a \otimes ab$ and $ba \otimes 1 = ba \otimes ab$. From $ab = ba$ and a similar argument we get $ba \otimes 1 = 1 \otimes ba$ so $ba \in D$. Let $\alpha, \beta : T \to T'$ such that $\alpha_R = \beta_R$.

$\alpha(b) = \alpha(b^2a)$

$= \alpha(b)\alpha(ba)$ because α is a homomorphism

$= \alpha(b)\beta(ba)$ since $ba \in D$

$= \alpha(b)\beta(ab)$ since $ba = ab$

$= \alpha(b)\beta(a)\beta(b)$ since β is a homomorphism

$= \alpha(b)\alpha(a)\beta(b)$ since $a \in D$

$= \alpha(ba)\beta(b)$

$= \beta(ba)\beta(b)$

$= \beta(bab)$

$= \beta(b^2a)$

$= \beta(b)$.

So $\alpha(b) = \beta(b)$ and therefore $b \in D$.

Lemma2.8: $D(\varphi,T) = T$ if and only if T is an epimorphic extension of R

Proof:

$\Rightarrow$ Let $\alpha,\beta: T \to T'$ and $\alpha\varphi = \beta\varphi$ then $\alpha(t) = \beta(t)$ for all $t \in T$. But T is the whole domain of α, β therefore $\alpha = \beta$ and T is an epimorphic extension.

$\Leftarrow$ Let T be an epimorphic extension of R and $\alpha\varphi = \beta\varphi$ then $\alpha = \beta$ but since $t \in T$ is in the domain of α, β, $\alpha(t) = \beta(t)$ so that $T = D(\varphi,T)$.

Theorem 2.9: Let R be a ring and S a strongly π-regular ring $\varphi: R \to S$ be a ring homomorphism. Then $D = D(\varphi,S)$ and $E = E(\varphi,S)$ are strongly π-regular.

Proof: Since $D \subseteq S$ for $d \in D$ $\exists n>0$ $\exists x \in S$ with $d^n = d^{2n}x$, $dx = xd$ and $x^2d^n = x$ by lemma 2.4. However, if $a = d^n$ and $b = x$ in lemma 2.7 it shows that $x \in D$, hence D is strongly π-regular.

Let $D_0 = D$ and for an ordinal α, $D_{\alpha+1} = D(\varphi, D_\alpha)$. This definition permits the creation of the chain $R \subseteq \ldots \subseteq D_3 \subseteq D_2 \subseteq D_1 \subseteq D_0 = D$ which shows that for some τ $D_\tau = D_{\tau+1}$ and $E = D_\tau = \bigcap D_\alpha$ and by lemma 2.7 E is strongly π-regular.

Lemma 2.10: Let R be a commutative ring and E an epimorphic extension of R, then E is commutative.

Proof: Let $D = D(\varphi,E)$. E is an E-E-bimodule and for each $x \in R \subseteq E$ $xr = rx$ for all $r \in R$ since R is commutative. But this means by lemma 2.6 that for all $d \in D$ and all $x \in R$ $dx = xd$. But E is an epimorphic extension so $D = E$ by lemma 2.8. Therefore for all $b \in E$ and all $x \in R$ $bx = xb$.

But this means that for each $x \in E$ $xr = rx$ for all $r \in R$ by above. Hence by lemma 2.6 for all $d \in D$ and all $x \in E$ $dx = xd$. Since $D = E$ by lemma 2.8 $\forall x,b \in E$ $bx = xb$. Therefore E is commutative.

Theorem 2.11: If R is a commutative ring and S' a strongly π-regular ring $\varphi:R \rightarrow S'$ then there is S a strongly π-regular ring that is commutative and $\varphi(R) \subseteq S$.

Proof: E is such a ring by theorem 2.9 and corollary 2.10.

Chapter III

If R is a commutative ring

then R is strongly π-regular if and only if R is π-regular

Lemma 3.1: Let R be a ring, if $\forall a \in R\ \exists n \geq 1\ \exists b \in R\ a^n = a^{n-1}b$, then $\forall I\ \forall \alpha \in R/I\ \alpha$ is a zero divisor or α is a unit.

Proof: If $\alpha \in R/I$ then $\exists a \in R\ \alpha = a + I$. Let n and b be such that $a^n = a^{n-1}b$

$(a + I)^n = a^n + I$ By multiplication in R/I

$= a^{n-1}b + I.$

This implies that $I = (a^n + I) - (a^{n-1}b + I) = (a^n - a^{n-1}b) + I = a^n(1 - ab) + I$. So $a^n(1 - ab) \in I$. This means that $a^n + I$ is a zero divisor of R/I or that $1 - ab + I = I$.

The first possibility is that $a^n + I$ is a zero divisor of R/I, then $(a + I)^n$ is a zero divisor of R/I. And consequently $a + I$ is a zero divisor of R/I.

The second possibility is that $I = (1 - ab) + I = (1 + I) - (ab + I)$. This results in $1 + I = I + (ab + I) = ab + I = (a + I)(b + I)$ which means that $a + I$ is a unit of R/I.

From these two possibilities it can be deduced that $\forall I\ \forall \alpha \in R/I\ \alpha$ is a zero divisor or α is a unit.

Lemma3.2: Let R be a ring. If $\forall I$ $\forall \alpha \in R/I$ α is a zero divisor or α is a unit, then every prime ideal of R is maximal.

Proof: Let P be any prime ideal of R, $\alpha \in R/P$ and a, b $\in$ R such that $a + P = \alpha$ and $(a + P)(b + P) \subseteq P$. This means that $ab + P \subseteq P$ and that $ab \in P$ but since P is a prime ideal $a \in P$ or $b \in P$. Therefore the only zero divisor of R/P is the 0 and every other element of R/P is a unit, demonstrating that R/P is a field and that P is a maximal ideal.

Lemma 3.3: Let R be a ring. If every prime ideal of R is maximal then:

a) $J(R) = nil(R)$.

b) $R/J(R)$ is regular.

Proof:

a) $J(R) = \bigcap M = \bigcap P = nil(R)$.

b) $R/J(R) \cong X_{\alpha\in\Lambda}\, R/M_\alpha$ but R/M_α is a field so every element is a unit. So R/M_α is a regular ring and $X_{\alpha\in\Lambda}\, R/M_\alpha$ is regular and consequently $R/J(R)$ is regular by isomorphism.

Lemma 3.4: Let R be a ring. If $J(R) = nil(R)$ and $R/J(R)$ is regular, then $\forall a \in R$ $\exists y \in R$ $a^n = a^{n-1} y$.

Proof: Let $a \in R$, then, by the regularity of $R/J(R)$, $a - awa \in J(R)$ for some $w \in R$.

But this implies that $(a - awa)^n = 0$ for some n. So $0 = (a - awa)^n = a^n - \sum y_i$ for i from 1 to n, but each $y_i = a^{n-1} x_i$ for some $x_i \in R$. So $0 = a^n - a^{n-1}\sum x_i$ and $a^n = a^{n-1} x$ where $x = \sum x_i$.

Theorem 3.5: For a commutative ring R, R is π-regular if and only if R is strongly π-regular.

Proof:

If R is strongly π-regular, then by lemma 3.1 and 3.2 R is π-regular.

If R is π-regular, then by lemma 3.3 and 3.4 R is strongly π-regular.

Chapter IV

Spec R, patch R and $\hat{R}$

This chapter will deal with theorems on Spec R, patch R and $\hat{R}$, It will also deal with the relationship between these three concepts. It should also be noted that in this chapter all rings will be assumed to be commutative.

Lemma 4.1: Let $a \in R$ then $D(a) = D(aR) = \{P \in \text{Spec } R \mid a \notin P\}$ is an open set of Spec R. Furthermore the set of all D(a) forms a basis for Spec R.

Proof: Since R is never a subset of P $aR \subseteq P \Leftrightarrow a \in P$, which means that $V(a) = V(aR)$ and, furthermore, $D(a) = D(aR)$. Since aR is an ideal, D(a) is an open set in spec R. But either $a \in P$ or $a \notin P$ and it cannot be both, so $D(a) = (\text{spec } R) \setminus V(a) = \{P \in \text{spec } R \mid a \notin P\}$.

$D(a) \cap D(b) = D(ab)$ because of the properties of prime ideals. $D(0) = \emptyset$ and $D(1) = \text{Spec } R$. $D(I) = D(\bigcap_{a \in I} \{aR\}) = \bigcup D(aR) = \bigcup D(a)$.

Lemma 4.2: Let R be a ring and M is a maximal ideal of R if and only if M/nil(R) is a maximal ideal of R/nil(R)

Proof:

$\Rightarrow$Since $\text{nil}(R) \subseteq M$, M/nil(R) is an ideal of R/nil(R). Let I be an ideal of R/nil(R)

such as $M/nil(R) \subsetneq I$. This implies that there exists J an ideal of R such that $J/nil(R) = I$. Since $M/nil(R) \subsetneq J/nil(R)$ and M is maximal $J = M$ or $J = R$ which implies that $I = M/nil(R)$ or $I = R/nil(R)$. So $M/nil(R)$ is maximal in $R/nil(R)$.

$\Leftarrow$ Let $M/nil(R)$ be a maximal ideal, then M is an ideal of R. Suppose $M \subsetneq I$ an ideal of R, then $I/nil(R)$ is an ideal of $R/nil(R)$. But $M/nil(R)$ is maximal, so $M/nil(R) = I/nil(R)$ or $R/nil(R) = I/nil(R)$. This implies that $I = M$ or $I = R$. Therefore M is maximal.

<u>Lemma 4.3:</u> Let S be a strongly π-regular then Spec S is homeomorphic to Spec $(S/J(S))$.

<u>Proof:</u> $S/J(S)$ is regular by lemma 3.3 so $S/J(S)$ is strongly π-regular. By lemma 4.2 we can define a bijective function φ that takes the prime ideals of Spec $(S/J(S))$ and maps them to the prime ideals of Spec S. Furthermore this function is a homeomorphism since for $a \in S$, $D(a)$ is mapped to $D(a + J(S))$ and vice versa.

<u>Lemma 4.4:</u> Let $\mathscr{T}$:Spec $R \to R$ such that if Y is a closed subset of Spec R $\mathscr{T}(Y)$ is the intersection of the prime ideals that belong to Y. If I, J are two ideals of R then $V(I) \subseteq V(J) \Leftrightarrow rad(I) \supseteq J \Leftrightarrow rad(I) \supseteq rad(J)$.

<u>Proof:</u> $V(I) \subseteq V(J) \Leftrightarrow \mathscr{T}(V(I)) \supseteq \mathscr{T}(V(J)) \Leftrightarrow rad(I) \supseteq rad(J) \Leftrightarrow rad(I) \supseteq J$.

Theorem 4.5: For $a \in R$, $D(a)$ is compact. In particular Spec R is compact.

Proof: Since these sets form a basis in Spec R it is sufficient to show that, if $\{b_i\} \subseteq R$ such that $D(a) \subseteq \bigcup D(b_i)$, then there is $\{b_n\} \subseteq \{b_i\}$ such that $\{b_n\}$ is finite and $D(a) \subseteq \bigcup(b_n)$. Let $D(a) \subseteq \bigcup D(b_i)$, then $\bigcap V(b_i) \subseteq V(a)$ and, furthermore, $V(\{b_i\}) \subseteq V(a)$. Using lemma 4.4, it can be seen that a must be in the radical of the ideal generated by the b_i's or that a^m for some $m \geq 1$ is in the ideal generated by the b_i's. This can be restated as $a^m = \sum g_i b_i$ for $g_i \in R$, but then there must be $\{b_n\} \subseteq \{b_i\}$ which is a finite set and $h_n \in R$ such that $a^m = \sum h_n b_n$. This implies that $V(a) = V(a^m) \supseteq V(\{b_n\}) = \bigcap V(b_n)$ for $\{b_n\}$ finite. Or, to conclude, $D(a) \subseteq \bigcup D(b_n)$.

Since $D(1) = $ Spec R, and by the first part of the proof, Spec R is compact.

Lemma 4.6: [12] Let R be a ring, then patch R is Hausdorff.

Proof: Let P and Q be two different prime ideals of R and $a \in P \backslash Q$ (if $P \subseteq Q$, then interchange P and Q). Then $P \in V(a)$ and $Q \in D(a)$, but $D(a)$ is a compact open set of Spec R (by lemma 4.5) and $V(a)$ is its complement so, by the definition of the patch topology, $V(a)$ and $D(a)$ are open sets in patch R. Therefore patch R is Hausdorff since $V(a) \cap D(a) = \emptyset$.

Lemma 4.7: If R is a regular ring then every principal ideal is generated by an idempotent.

Proof: R is regular $\Leftrightarrow$ for all $r \in R$ there exists $s \in R$ such that $(rs)^2 = (rsr)s = rs$. But $r = rsr \in rsR$ and $rs \in rR$ so $rsR = rR$.

Lemma 4.8: If R is a regular ring, then Spec R is a compact Hausdorff and totally disconnected space.

Proof: Spec R is compact by lemma 4.5. Since R is regular it is strongly π-regular (i.e. using $n = 1$) and furthermore by theorem 3.5 every prime ideal of R is maximal. Let $P \neq Q$ be two prime ideals of R, then there exists $e \in P\backslash Q$ such that e is an idempotent. This implies that D(e) and D(1-e) are disjoint open subsets of Spec R such that $Q \in D(e)$ and $P \in D(1\text{-}e)$ (or else e, $1\text{-}e \in P$ which would imply $1 = 1\text{-}e + e \in R$ which is impossible). Thus spec R is Hausdorff.

Let O be an open set in spec R and $P \in O$. There is $A \subseteq R$ such that $O = D(A)$, and, since $P \in D(A)$, there exists $a \in A\backslash P$. Then D(a) is an open subset of Spec R such that $P \in D(a) \subseteq D(A)$. Let e, an idempotent of R, be such that $eR = aR$, then $D(a) = D(e)$. Furthermore $D(e) = V(1\text{-}e)$, which is closed. Thus D(a) is a clopen subset of spec R such that $Q \in D(a) \subseteq O$

Lemma 4.9: A closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed.

Proof: Let X be compact, F a closed subset of X, F ' = X\F and U an open covering for F. Thus U $\cup$ {F '} is an open covering of X and so there must be a finite subcovering U' $\subseteq$ U such that U' $\cup$ {F '} is an open covering of X. Then U' is a finite cover of F, so F is compact.

Suppose now that X is a Hausdorff space and K a compact subset of X. Let $y \in X\backslash K$. Since X is Hausdorff, for each $x \in K$ there are disjoint open sets O_x and N_x such that $x \in O_x$ and $y \in N_x$. The sets $\{O_x \mid x \in K\}$ form an open covering of K, and so there is a finite subcovering $\{O_{x1}, ..., O_{xn}\}$ of K. Let $N_y = \bigcap N_{xi}$. Then N_y is an open set containing y and $N_y \subseteq X\backslash K$. But $X\backslash K = \bigcup N_y$ and so $X\backslash K$ is open, which proves that K is closed.

Lemma 4.10: The continuous image of a compact set is compact.

Proof: Let f be a continuous function that maps the compact set K onto a topological space Y. If U is an open covering for Y, then the collection of sets $f^{-1}[O]$ for all $O \in U$ is an open covering of K. By the compactness of K, there are a finite number $O_1, O_2, ..., O_n$ of sets of U such that $f^{-1}[0_i]$ cover K. Since f is onto, the sets $O_1, ..., O_n$ cover Y.

Theorem 4.11: Patch R is homeomorphic to the spectrum of its universal regular ring.

Proof: Let $\varphi: R \to S$ be the natural homomorphism from R into its universal regular ring S. In this proof subscripts will be used to identify in what space the sets belong, $_S$ will

indicate that the set belongs to **Spec S**, while on the other hand, $_R$ will be used to indicate that the set belong **patch R**.

Define f:Spec S→patch R such that for any $P \in S$, $f(P) = \varphi^{-1}(P)$. Let $a \in R$, then it is obvious that $f^{-1}(D_R(a))$ is open. But $f^{-1}(V_R(a)) = V_S(\varphi(a)) = V_S(e)$ for some e an idempotent of S, hence, $f^{-1}(V_R(a)) = D_S(1-e)$ so it is open. Since $V_R(a)$ and $D_R(a)$ form a subbasis of patch R f is a continuous function.

Let O_S be an open set of Spec S and $C_S = (\text{Spec } S)\backslash O_S$. By the first part of lemma 4.9 C_S must be compact but then $f(C_S)$ is a compact set in Patch R by lemma 4.10 and, furthermore, it must be closed by the second part of lemma 4.9. So $f(C_S) = C_R$ for a closed set in Patch R. Finally $f(O_S) = f((\text{Spec } S)\backslash C_S) = f(\text{Spec } S)\backslash f(C_S) =$ (patch R)$\backslash C_R$ which is an open set in patch R. So f^{-1} is continuous.

Since f is continuous and f^{-1} is continuous, f is a homeomorphism and Spec $\hat{R}$ is homeomorphic to Patch R.

Chapter V

When can a ring be embedded into a strongly π-regular ring?

Theorem 5.1: Let R be a ring with Krull dimension equal to 0, then R is π-regular.

Proof: If dim(R) = 0 then for any prime ideal P the only decreasing chain is $P = P_0$. This means that there is no P_1 (a prime ideal of R) such that $P_0 \supset P_1$. Therefore there is no ideal $I \neq R$ such that $I \supset P_1$. This implies that P_1 is a maximal ideal. But since this istrue for any prime ideal P, every prime ideal is maximal. Therefore R is π-regular.

Lemma 5.2: [5] Let $\alpha: R \to S$ be a monomorphism that is also an epimorphism in the category of rings, where S is a π-regular ring. Then the composition $R \xrightarrow{\alpha} S \to S/J(S)$ factors through φ_R via $\beta: \hat{R} \to S/J(S)$, which is a surjection.

Proof: Since S/J(S) is a regular ring (from lemma 3.3), β is given by the universal property of $\hat{R}$. It is an epimorphism, which must be a surjection since $\beta(\hat{R})$ is regular.

Lemma 5.3: If $\mathcal{M} \in$ Spec S and $M = \mathcal{M} \cap B(S)$ then $M \in$ Spec B(S).

Let $ab \in M$ and $a, b \in B(S)$. Since $ab \in M$ this implies that $ab \in \mathcal{M}$ and, furthermore, that $a \in \mathcal{M}$ or that $b \in \mathcal{M}$. But Since $a, b \in B(S)$, we get that $a \in M$ or that $b \in M$. Therefore $M \in$ Spec B(S).

Lemma 5.4: Let Q be an ideal of R, then Q is primary if in R/Q every zero divisor is nilpotent.

Proof: Let every zero divisor in R/Q be nilpotent. This is the same as saying that for all $r \in R$ if $\bar{r}$ is a zero divisor of R/Q, then $\exists n$ such that $\bar{r}^n = 0$. This however means that $\forall r \in R$ if $\exists a \in R \setminus Q$ such that $ra \in Q$, then $\exists n$ such that $r^n \in Q$. This can be written as: if $ra \in Q$ and $a \notin Q$ then $r^n \in Q$. This is the definition of primary and therefore Q is primary.

Theorem 5.5: [5] Let R be a commutative ring. Then R can be embedded into a commutative π-regular ring if and only if there exists a set of prime ideals $Y = \{P_\alpha\}\alpha \in \Lambda$ and for each P_α, a P_α-primary ideal Q_α such that:

I) Y is closed in the patch topology on Spec R.

II) $\bigcap_{\alpha \in \Lambda} Q_\alpha = \{0\}$.

III) for each $a \in R$ there is $n(a) \in N$ such that for $n \geq n(a)$, $\{P_\alpha \mid P_\alpha \in Y \text{ and } a^n \in Q_\alpha\}$ is patch open in Y.

Proof:

$\Rightarrow$ Suppose $\exists_\alpha$ α:R$\rightarrow$S. By Theorem 2.9 it may be assumed that the embedding is an epimorphism of rings. The Pierce stalks of S given by S/SM for $M \in$ Spec B(S) are local rings since M_1/SM and M_2/SM are maximal $\Leftrightarrow$ SM $\subseteq M_1$ and SM $\subseteq M_2$ are maximal in

$S \Rightarrow SM \subseteq \mathcal{M}_1 \cap B(S)$ and $SM \subseteq \mathcal{M}_2 \cap B(S) \Rightarrow \mathcal{M}_1 \cap B(S) = \mathcal{M}_2 \cap B(S) = M$ since they are all maximal $\Rightarrow \mathcal{M}_1 = \mathcal{M}_2 \Rightarrow S/SM$ has only one maximal ideal.

We can define γ:Spec S$\to$Spec R by $\gamma(\mathcal{M}) = \alpha^{-1}(\mathcal{M})$. Since $\dim(S) = 0$, γ is identical with the function defined by $\beta:\hat{R}\to S/N(S)$ given in lemma 5.2. In fact γ is a monomorphism and its imageY is a closed in patch R.

$$\begin{tikzcd}
 & S \arrow[dr] & \\
R \arrow[ur, "\alpha"] \arrow[dr] & & S/J(S) \\
 & \hat{R} \arrow[ur, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{Spec } S \arrow[dl, "\gamma"'] & \\
\text{patch } R & & \text{Spec } S/J(S) \arrow[ul, "\text{homeomorphic}"'] \arrow[dl] \\
 & \text{Spec } \hat{R} \arrow[ul, "\text{homeomorphic}"] &
\end{tikzcd}$$

Let Y be the set of prime ideals of R. For each $P \in Y$, $P = \gamma(\mathcal{M})$ for some $\mathcal{M} \in$ Spec S. Define $M = \mathcal{M} \cap B(S)$. By lemma 5.3 the Pierce stalk S/SM is local and π-regular. Set $q(P) = \alpha^{-1}(SM) \subseteq P$. Since α maps $P\to\mathcal{M}$, $q(P)\to SM$ and $R\to S$, there exists an embedding $\delta:R/q(P)\to S/SM$ which sends $P/q(P)$ into $\mathcal{M}/SM$ the Jacobson radical of S/SM. Since S/SM is π-regular $\mathcal{M}/SM$ is nil(S/SM) and, furthermore, $P/q(P)$ is nil(R/q(P)), therefore P is a unique minimal prime of q(P). If $r \in R\backslash P$, then $\delta(\bar{r}) \in (S/SM) \setminus (\mathcal{M}/SM)$ but S/SM is a local ring therefore $\delta(\bar{r})$ is a unit. This implies that $\delta(\bar{r})$ is a non-zero divisor, furthermore

that $\bar{r}$ is a non-zero divisor in $R/q(P)$ and, finally, that every zero divisor of $R/q(P)$ is nilpotent. Therefore, using lemma 5.4 $q(P)$ is P-primary.

Since $\bigcap q(p)$ is the kernel of α we get that $\bigcap q(p) - \{0\}$. But α is a monomorphism so $\bigcap q(p) = \{0\}$. So this shows that condition II) holds.

Let $a \in R$ define $U_a = \{M \in \text{Spec } B(S) \mid \alpha(a) \in SM\}$.

a) Let $A \subseteq R$. One possibility is that $\bigcup_{a \in A} U_a = \text{Spec } B(S) = U_0$. For the others, define $B = \{M \in \text{Spec } B(S) \mid \exists a \in A\ \alpha(a) \in SM\}$, then there is $e \in (\bigcap_{M \in B} M) \setminus (\bigcup_{M \notin B} M)$. But $e \in M$ for some M in Spec $B(S)$, which implies that $e \in SM$ for that M and that there is $r \in R$ such that $\alpha(r) = e$. If $M \in U_r$ then $e \in SM$ and that there is $a \in A$ such that $M \in U_a$ since all $M \notin B$ were eliminated. If $M \notin U_r$ then $e \notin SM$ and, furthermore, M does not contain any element of A since $M \notin B$. Therefore $U_r = \bigcup_A U_a$.

b) $U_a \cap U_b = \{M \in \text{Spec } B(S) \mid \alpha(a) \in SM\} \cap \{M \in \text{Spec } B(S) \mid \alpha(b) \in SM\} = \{M \in \text{Spec } B(S) \mid \alpha(a) \in SM \text{ and } \alpha(b) \in SM\} = \{M \in \text{Spec } B(S) \mid \alpha(ab) \in SM\} = U_{ab}$

c) $U_1 = \{M \in \text{Spec } B(S) \mid \alpha(1) \in SM\} =$ and $U_0 = \{M \in \text{Spec } B(S) \mid \alpha(0) \in SM\} =$ Spec $B(S)$

By a), b) and c) it has been proven that the set U_a is open. However since S is

π-regular, Spec B(S) is homeomorphic to Spec S via $M \to M \cap B(S)$. Hence $\gamma(U_a) = \{P \in \text{Spec } R \mid a \in q(p)\}$ is open in Spec R and in Patch R and also in Y. Which shows that III) holds.

$\Leftarrow$ To prove the converse the π-regular ring S will be constructed as a subring of $\prod_{\alpha \in \Lambda} S_\alpha$ where $S_\alpha = Q_{cl}(R/Q_\alpha)$.

Let $C(a) = \{P \in \text{patch } R \mid a \notin P\}$. Since D(a) is compact by theorem 4.5 C(a) is open, but (patch R)\C(a) is also open, therefore C(a) is clopen. Clearly $C(a) = C(a^2) = C(a^3) \ldots$ since $a \in P \Rightarrow a^n \in P$ because P is an ideal and $a^n \in P \Rightarrow a \in P$ because P is a prime ideal. If for some $\alpha \in \Lambda$, $a \in P_\alpha$, then $a^{n(a)} \in Q_\alpha$. By property III), $\{\{P_\alpha \mid a^n \in Q_\alpha\}\}_{n \geq n(a)}$ is an open covering of the compact set $Y \setminus C(a)$. Hence $\exists m \in \mathbb{N}\ \forall \alpha \in \Lambda\ a \in P_\alpha \Rightarrow a^m \in Q_\alpha$

For a, c $\in$ R, let $\bar{a}, \bar{c}$ denote the equivalence classes of a, c in R/Q_α and N a clopen subset of $C(c) \cap Y$.

$$[a,c] = \begin{cases} \bar{a}/\bar{c} \in S_\alpha & \text{if } P_\alpha \in N \\ 0 & \text{if } P_\alpha \notin N \end{cases}$$

Define S to be the subring of $\prod_{\alpha \in \Lambda} S_\alpha$ generated by the elements described above.

<u>Remark 5.6:</u> Let a, b, c, d $\in$R, N be a clopen subset of $C(c) \cap Y$ and L a clopen subset of $C(d) \cap Y$, then $[a,c]_N [b,d]_L = [ab,cd]_{N \cap L}$.

For $P \in N \cap L$, $[a,c]_N [b,d]_L = (\bar{a}/\bar{c})(\bar{b}/\bar{d}) = (\overline{ab})/(\overline{cd}) = [ab,cd]_{N \cap L}$. For $P \notin N$, $[a,c]_N [b,d]_L = 0[b,d]_L = 0 = [ab,cd]_{N \cap L}$. By the symmetry of commutativity of multiplication, it also holds for $P \notin L$ and thus holds for all S_α.

Remark 5.7: [5] Let a, b, c, d $\in R$, N be a clopen subset of $C(c) \cap Y$ and L a clopen subset of $C(d) \cap Y$. then $[a,c]_N + [b,d]_L = [a,c]_{N \setminus L} + [b,d]_{L \setminus N} + [ad+bc,cd]_{N \cap L}$.

Proof: For $P \in N \cap L$, $[a,c]_N + [b,d]_L = (\bar{a}/\bar{c}) + (\bar{b}/\bar{d}) = (\overline{ad} + \overline{bc}) / (\overline{cd}) = [ad+bc,cd]_{N \cap L}$, but $P \notin N \setminus L$ and $P \notin L \setminus N$ so $[a,c]_{N \setminus L} = [b,d]_{L \setminus N} = 0$. For $P \in N \setminus L$, $[a,c]_N + [b,d]_L = [a,c]_{N \setminus L} + 0 = [a,c]_{N \setminus L}$, but $P \notin L \setminus N$ and $P \notin L \cap N$ so $[b,d]_{L \setminus N} = [ad + bc,cd]_{N \cap L} = 0$. An identical proof can be used for $P \in L \setminus N$. For $P \notin L \cup N$, $[a,c]_N + [b,d]_L = 0 + 0 = 0$, but $P \notin N \setminus L$, $P \notin L \setminus N$ and $P \notin L \cap N$ so $[a,c]_{N \setminus L} = [b,d]_{L \setminus N} = [ad+bc,cd]_{N \cap L} = 0$.

Remark 5.8: [5] Every element of S can be expressed in the form $\sum_{i=1}^{n} [a_i,c_i]_{N_i}$ where the N_i 's are disjoint. (The expression is not unique but such a presentation is said to be of Standard form).

Proof: Let $[a,c]_N = \sum_{i=1}^{n} [a_i,c_i]_{N_i}$ where the $\bigcap N_i \neq \emptyset$ then a sequence can be made such that all the sets are disjoint. This is proven using mathematical induction:

For n = 2 it was proven in remark 5.7.

Assume that it is true for n = k

Then for n = k+1, $\sum_{i=1}^{k-1} [a_i,c_i]_{N_i} + [a_k,c_k]_{N_k} + [a_{k-1},c_{k-1}]_{N_{k-1}} = \sum_{i=1}^{k-1} [a_i,c_i]_{N_i} + [a_k,c_k]_{N_k \setminus N_{k-1}} +$

$[a_{k-1}, c_{k-1}]_{N_{k-2}N_k} + [a_{k-1}c_k + a_kc_{k-1}, c_kc_{k-1}]_{N_k \cap N_{k-1}}$. But each of the three new sets are disjoint, which implies that the longest chain such that the intersection is not empty has k elements. So by the assumption a chain can be found such that all the sets are disjoint.

Remark 5.9: [5] For N a clopen set of Y $[1,1]_N = e_N$ is an idempotent. If $e \in S$, $e = e^2$, then in each component S_α, e is 0 or 1. Hence $e = e_N$ for some N.

Proof: Let $1 \in R$ $[1,1]_N [1,1]_N = [1\bullet 1, 1\bullet 1]_{N \cap N} = [1,1]_N$. So $[1,1]_N$ is an idempotent.

Let $e = [a,c]_{N'}$ be idempotent then $[a,c]_{N'} = [a^2,c^2]_{N'}$ by remark 5.7. If $P \in N'$ then $\bar{c} = \bar{c}^2$ which means that $0_{RQ} = \bar{c}^2 - \bar{c} = \bar{c}(\bar{c} - 1_{RQ})$ but since $\bar{c}$ is not a zero divisor $\bar{c} - 1_{RQ} = 0_{RQ}$ and therefore $\bar{c} = 1_{RQ}$. For $\bar{a}$, $\bar{a} = \bar{a}^2$ and therefore $0_{RQ} = \bar{a}(\bar{a} - 1_{RQ})$. If $\bar{a} - 1_{RQ} = 0_{RQ}$ then $\bar{a} = 1_{RQ}$. If not then $\bar{a}^n = 0_{RQ}$, since Q is primary, but $\bar{a} = \bar{a}^2 = \overline{aa} = \overline{aa^2} = \bar{a}^3 = \ldots = \bar{a}^n = 0$. If $P \notin N'$ then it is 0 by definition. So for each component S_α e is 0 or 1.

Let $e = \{w_\alpha\}$ $\alpha \in \Lambda$ where $w_\alpha = 1$ or 0. If the number of 1's is finite let $A \subseteq \Lambda$ be the set of all α such that $w_\alpha = 1$. Then $e = \sum_{a \in A} [1,1]_{P_a}$, let $N = \{P_a\}_{a \in A}$ then $e = [1,1]_N$. If the number of 0's is finite let N' be the set of primes for which $w = 1$. $[1,1]_Y + [-1,1]_{N'} = [1,1]_{Y \setminus N'} + [-1,1]_{N' \setminus Y} + [1-1,1]_{N' \cap Y}$. But $[1,1]_{Y \setminus N'} = e$ since the 1's and 0's coincide, $[-1,1]_{N' \setminus Y} = 0$ since N'\Y is empty, and $[1-1,1]_{N' \cap Y} = [0,1]_{N' \cap Y} = 0$. So $e_{Y \setminus N'} = [1,1]_Y + [-1,1]_{N'}$. But S is a ring and $e_{Y \setminus N'} \in S$ so there exists a clopen set $N = Y \setminus N'$ hence $e = e_{Y \setminus N'} = e_N$.

Now it can be shown that S is a strongly π-regular ring. By remark 5.6 it can be shown that $[a,c]^k_N = [a^k,c^k]_N$. It should be noted that earlier on in this proof it was shown that $\exists m\in\mathbb{N}\ \forall\alpha\in\Lambda\ \ a \in P_\alpha \Rightarrow a^m \in Q_\alpha$. So $[a,c]^{m-1}_N\ [c,a]_{N\setminus C(a)} = [a^{m-1},c^{m-1}]_N\ [c,a]_{N\setminus C(a)} = [a^{m-1}c,c^{m-1}a]_{N\setminus C(a)}$ but either a coordinate is 0 or $\bar{a}$, $\bar{c}$ are not zero divisors so $[a^m(ac),c^m(ac)]_{N\setminus C(a)} = [a^m,c^m]_{N\setminus C(a)}$. However, for $N\setminus C(a)$, $P \in N\setminus C(a)$ then $P \notin C(a)$ and furthermore $a \in P$ but $a^m \in Q$, therefore $[a^m,c^m]_{N\setminus C(a)} = 0$. This results in $[a,c]^{m-1}_N\ [c,a]_{N\setminus C(a)} = [a^m,c^m]_{N\setminus C(a)} + 0 = [a^m,c^m]_{N\setminus C(a)} + [a^m,c^m]_{N\setminus C(a)} = [a^m,c^m]_N = [a,c]^m_N$. Thus S is a strongly π-regular ring.

To finish this proof it must be shown that R embeds into S. Let $a \in R$ then define $\varphi(a) = [a,1]_Y$. So $\varphi(a)\varphi(b) = [a,1]_Y\ [b,1]_Y = [ab,1]_Y = \varphi(ab)$. And $\varphi(a) + \varphi(b) = [a,1]_Y + [b,1]_Y = 0 + 0 + [a+b,1]_Y = \varphi(a+b)$. And φ is a monomorphism by II). Therefore R embeds into S.

Chapter VI

Examples

Three things will be shown in this chapter: 1) that even though it might appear from the previous chapter that R can be embedded into a π-regular ring iff it can be embedded into a regular ring it is not so; 2) that the set of rings that can be embedded into a π-regular ring is not empty; 3) that the set of rings that cannot be embedded into a π-regular ring is not empty:

Example 6.1: Let $R = \mathbf{Z}/4\mathbf{Z} = \{0, 1, 2, 3\}$. This is an example of a ring that can be embedded into a π-regular ring but not a regular ring.

Since $\dim(R) = 0$, by theorem 5.1 it can be embedded into a π-regular ring. Let's assume that φ maps R into R' a regular ring. Then for some $a \in R'$, $\varphi(2) = \varphi(2)^2 a = \varphi(0)a = 0a = 0$ therefore φ is not an embedding. So R is a ring that can be embedded into a π-regular ring but cannot be embedded into a regular ring.

Example 6.2: Since $\mathbf{Z}$ can be embedded into $\mathbb{R}$ and $\mathbb{Q}$ which are fields and so π-regular, it will be used to show how the theorem works. So let $R = \mathbf{Z}$, $Y = \operatorname{Spec} R$ and $Q_n = P_n$.

I) $Y = \text{Spec } R$ so Y is closed in the patch topology on Spec R

II) $\bigcap Q_n = \bigcap P_n = \{0\}$ since $P_0 = \{0\}$

III)Let $n(a) = 1$ $\{P_n \mid a^m \in Q_n\} = \{P_n \mid a^m \in P_n\} = \{P_n \mid a \in P_n\} = V(a)$ which is open in patch R.

Lemma 6.3: Let A, B be two rings then the prime ideals of A x B are of the form $A \times P_B$ or $P_A \times B$ where P stands for prime ideal.

Proof: Let P be a prime ideal of A x B, then $\{ab,cd\} \in P$ implies $\{a,c\}$ or $\{b,d\} \in P$ and that $\{b,c\}$ or $\{a,d\} \in P$, but this is true only if $ab \in I_A$ implies that a or $b \in I_A$ so $I_A = A$ or $I_A = P_A$. The same holds true for the component in B. $\{A,B\} = A \times B$ so it is not prime. Let $a \in P_A$, $c \in P_B$, $b \notin P_A$, $d \notin P_B$ and $P_AP_B = I$ then $\{ab,cd\} \in I$ but $\{a,d\}$, $\{b,c\} \notin I$ so I is not a prime ideal. Finally $\{ab,cd\} \in A \times P_B$ then a, $b \in A$ and c or $d \in P_B$ so $\{a,c\}$ or $\{b,d\} \in A \times P_B$ and that $\{b,c\}$ or $\{a,d\} \in A \times P_B$. By a similar argument, the same holds true for $P_A \times B$. So the only prime ideals are of the form $P_A \times B$ and $A \times P_B$.

Lemma 6.4: Let $R = \prod A_j$ be an infinite product of rings and $e_n \in R$ such that it has 1 in the n'th position and 0's everywhere else. Then for any prime ideal P_i, $e_n \in P_i$ for all $i \neq n$ where $P_i = (\prod_{j<i} A_j) \times P \times (\prod_{j>i} A_j)$ and P is a prime ideal of A_i.

Proof: By lemma 6.3 and mathematical induction, it is obvious that P_i is a prime ideal

of R. For P a prime of A_i and for any $j \neq i$, $0 \in A_j$, $1 \in A_j$, $0 \in P$ and $1 \notin P$. So en $\in P_i$.

Lemma 6.5: Let R be as above. Let Q_i be a primary ideal with radical P_i. Then $Q_i = (\Pi_{i\,1} A_i) \times Q \times (\Pi_{i\,1} A_j)$ where Q is a primary ideal with radical P and $e_n \in Q_i$ for $i \neq n$. Furthermore $\forall_{i \in N} \forall_{n \in N}\, q_n \in Q_i$ where $q_n = qe_n$ where $q \in Q$.

Proof: $e_n \in P_i$ if and only if $\exists_j\, e_n^j \in Q_i$. But e_n is an idempotent, so this is true if and only if $e_n \in Q_i$. So $e_n \in Q_i$ for all $i \neq n$ by lemma 6.3. The ideal $I_n = \{\{0\},...,\{0\},A_n,\{0\},...,\{0\}\}$ is generated by e_n. Therefore $\sum_{n \neq i} I_n \subseteq Q_i$ for all Q_i so $Q_i = (\Pi_{j\,1} A_j) \times Q \times (\Pi_{j\,1} A_j)$ is a primary ideal with radical P_i for Q a primary ideal with radical P. Since $e_n \in Q_i$ for $i \neq n$, $q_n = qe_n \in Q_i$ for $i \neq n$. But $q \in Q$ and $0 \in A_j$ for all j so $q_n \in Q_n$ which means that $q_n \in Q_i$ for all i and n.

Example 6.6: Let $V = \Pi_{j\,1} \mathbb{Z}/2^j\mathbb{Z}$ and $R = \mathbb{Z} \times V$ then R is an example of a ring that cannot be embedded into a strongly π-regular ring.

Spec R = $\{P_{0j}, P_n\}$ for n>1 where $P_{0j} = (p_j) \times V$. p_j is the j'th prime and $(p_0) = 0$. $\{0\} \times V \subseteq (2) \times V$ and both are prime ideals by the properties of integers and lemma 6.3. So Dim(R) $\neq 0$. So if R can be embedded into a strongly π-regular ring, then $\bigcap Q_n$ must be $\{0\}$. By lemma 6.5 $e_i \in Q_n$ for all $i \neq n$, so we must have $\{P_n\} \subseteq Y$ for all $n \in N$ and where

P_0 will be P_{00}.

Another problem that will limit the choice of Q's is that $q_i \in Q_n$ for all i and all n. If $P_0 = \{0\}V$ is chosen to be in Y, then $q_0 = 0$. But for n>0 $q_n = qe_n$ where $q \in Q$. So $0 = q_n$ if and only if $0 = q$ if and only if $\{0\} = Q$ if and only if $Q = P_n$. But, since this is an infinite sequence, this is impossible so $\bigcap Q_n \neq \{0\}$, so it cannot be embedded into a strongly π-regular ring.

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