

CONTRIBUTIONS TO THE STRUCTURAL ANALYSIS OF PARALLEL MANIPULATORS WITH REDUCED MOBILITY

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Abstract

At the embodiment stage of a robot design, fundamental tasks such as kinematic, static, dynamic and constraint-wrench analyses are essential. Moreover, by considering link elasticity, the elastostatic and elastodynamic analyses also become essential. However, the foregoing analyses are more challenging for parallel robots, when compared with their serial counterparts.

This thesis aims to develop comprehensive methodologies for the elastostatic, elastodynamic and constraint-wrench analyses of robotic mechanical systems, while focusing on parallel architectures with reduced mobility. To this end, first, a systematic method for the calculation of the constraint wrenches of a mechanical system, imposed by the kinematic pairs, is proposed. This method helps the designer readily handle the constraint-wrench analysis of robots with parallel architectures.

In the next step, considering link elasticity, a novel method for the elastostatic analysis of parallel robots is proposed. This method is based on the concept of generalized spring. With the essential ingredients, stiffness and mass matrices, the elastodynamic analysis of the robot under study is conducted.

Although the analyses are motivated by the special features of a novel two-limb parallel manipulator designed at McGill University, called the McGill Schönflies Motion Generator (SMG), they are applicable to any other two-limb parallel robots. The McGill SMG can produce the Schönflies displacement subgroup, which comprises translations in the three-dimensional Cartesian space and one rotation about

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a fixed-orientation axis, as needed for most common pick-and-place operations. Finally, as an application of the proposed analyses, the optimum structural design of the McGill SMG is obtained.

Résumé

A l'étape de réalisation durant la conception d'un robot, les analyses cinématique, statiques, dynamique et celles des torseurs statiques de contrainte sont des tâches fondamentales. En outre, si l'élasticité des maillons est considérée, les analyses élastostatique et élastodynamique deviennent nécessaires. Ces analyses sont plus difficiles pour les systèmes robotiques à architecture parallèle que pour ceux à architecture sérielle.

Cette thèse vise à développer une méthodologie complète des analyses élastostatique, élastodynamique et de torseurs statiques de contrainte de systèmes mécano robotiques, tout en mettant l'accent sur les architectures parallèles à mobilité réduite. D'abord, une méthode systématique pour le calcul des torseurs statiques de contrainte d'un système mécanique, imposée par les couples cinématiques, est proposée. Cette méthode permet au concepteur facilement gérer l'analyse de torseurs statiques de contraintes de robots à architecture parallèle.

À l'étape suivante, en considérant l'élasticité, une méthode originale pour l'analyse élastostatique des robots parallèles est proposée. Cette méthode est basée sur le concept du ressort généralisée. Avec les ingrédients essentiels, les matrices de raideur et de masse, l'analyse élastodynamique du robot est obtenue.

Bien que ses analyses soient motivés par les caractéristiques particulières d'un manipulateur à deux jambes en parallèle, appelé robot McGill générateur des mouvements de Schönflies (SMG), elles sont aussi applicables à tout autre robot parallèle à deux jambes. Le SMG de McGill est capable de produire le sous-groupe de

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déplacement dit de Schönflies, qui comprend des translations dans l'espace tridimensionnel cartésien et une rotation autour d'un axe à orientation fixe, nécessaire pour la plupart des opérations de transfert. Enfin, en utilisant les analyses proposées, la conception structurale optimale du SMG de McGill est obtenue.

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Claim of Originality

The analyses and algorithms introduced in this thesis are mainly directed toward the design of the McGill SMG, a two-limb Schönflies motion generator. In this vein, the author claims the originality of the ideas and results listed below:

- Kinematic constraint matrix of the helical, cylindrical, spherical, planar, universal and Π joints
- A novel procedure for the constraint-wrench analysis of parallel manipulators with passive kinematic pairs by resorting to the robot constraint matrix
- The actuator wrench-shaping matrix
- Use of isoconstrained versions of an overconstrained robot in the constraint-wrench analysis
- A new elastostatic modelling of a flexible component with passive lower kinematic pair in a multibody system
- Stiffness modeling of a Π -joint
- Two novel stiffness performance indices to measure the translational and rotational stiffness of a mechanical system
- Elastodynamic modelling of parallel manipulators with passive kinematic pairs
- A GA multiobjective optimization code for design purposes

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ABBREVIATIONS

BP:	base platform
C:	cylindrical joint
COM:	centre of mass
CPM:	cross-product matrix
EB:	elbow bracket
EE:	end-effector
F:	planar joint
FEA:	finite element analysis
GA:	genetic algorithms
H:	helical joint
HKP:	higher kinematic pair
LKP:	lower kinematic pair
MP:	moving platform
MSA:	matrix structural method
P:	prismatic joint
PPO:	pick-and-place operations
S:	spherical joint
SAD:	small-amplitude displacement

ABBREVIATIONS

SBX:	simulated binary crossover
SCARA:	Selective-Compliance Assembly Robot Arm
SII:	stiffness improvement index
SMG:	Schönflies Motion Generator
R:	revolute joint
U:	universal joint
VJM:	virtual-joint method
WB:	wrist bracket

CHAPTER 1

Introduction

1.1 Motivation

With the ever increasing demand on automation and robotics in industry, the related fields of study has attracted the attention of many researchers in different areas. In industry, robots have been extensively used in car-manufacturing operations such as spot-welding, arc-welding and spray painting as illustrated in Fig. 1.1. Although the extreme use of robots in the automotive industry has led to unemployment, robots have helped human beings in performing monotonous or dangerous tasks. Most robots found in this industry are composed of a serial chain of kinematic pairs and links, which constitute what is called *serial* robots. In addition to the automotive industry, robots have been successfully applied to assembly and pick-and-place operations (PPO). The high repeatability of robots has led to some new technologies in electronic assembly. PPO occurring in packaging and assembly of electronic printed-circuit boards, involve translations in three-dimensional space and one rotation about an axis of fixed orientation. This type of motions falls within the Schönflies displacement subgroup, a robot whose end-effector (EE) undergoes the foregoing motion being called a Schönflies motion generator (SMG). This motion is a subset of the *Cartesian* space whose dimension is known to be six, including three translations and three rotations. *Reduced mobility robots* are that aim to produce a subset of rigid-body motions. SCARA (Selective-Compliance Assembly Robot



FIGURE 1.1. Robots are painting

Arm) systems are reduced mobility robots that produce rigid body motion within the Schönflies displacement subgroup. A SCARA system with serial architecture is depicted in Fig. 1.2. As in serial robots, the heaviest motors move while in operation, the faster the task, the higher the load on the upstream motors. Demands on much



FIGURE 1.2. A serial SCARA robot

higher speeds than SCARA systems can deliver has motivated designers to seek superior alternatives. One of these alternative is robots with parallel architecture. Parallel robots are composed of two or more serial chains connecting the base platform (BP) to the moving platform (MP). In this type of robots, the motors are located on the BP; hence, they are stationary. As a result, parallel robots offer higher speeds than

their serial counterparts, due to their lighter and stiffer structures; accordingly, they have been used widely in fast PPO in industry. Two well known industrial parallel robots are Delta and Adept-Quattro, depicted in Figs. 1.3(a) and 1.3(b), respectively. The former produces translations in the Cartesian space while the latter moves within the Schönflies displacement subgroup. Recently, a novel parallel SMG was designed and prototyped at McGill University (Morozov and Angeles, 2007), called the McGill SMG, shown in Fig. 1.4. At the embodiment design stage of a robot, knowing the

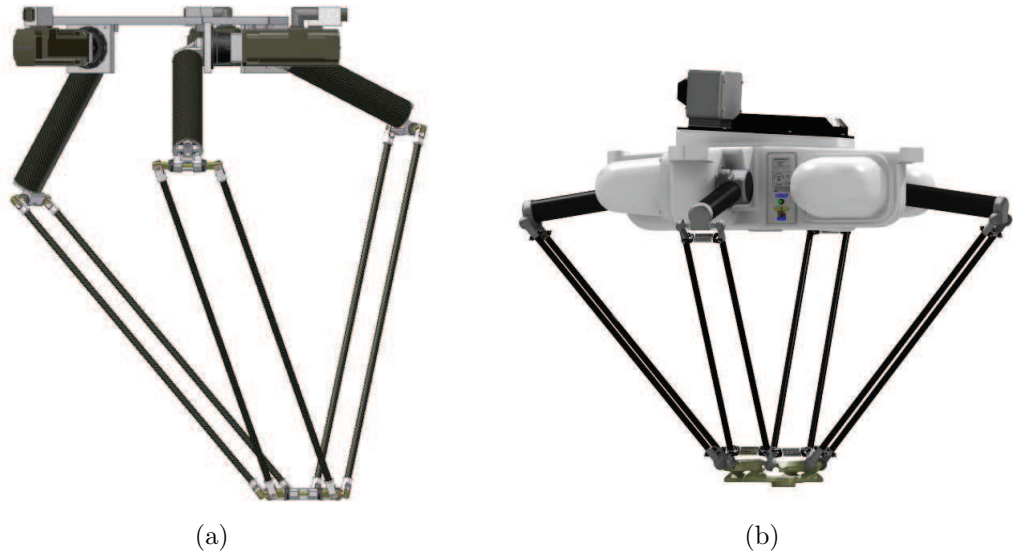


FIGURE 1.3. Two well-known parallel robots: (a) Nedquip's Delta and (b) Adept's Quattro

loads—forces and moments—applied on each structural element is the main issue. These forces and moments, which are posture-dependent, are imposed by the kinematic pairs; this calls for a comprehensive *constraint-wrench* analysis on the system. Moreover, in fast operations, the link flexibility comes into the picture. Since the pose of the MP is defined by six independent variables, in robotics, the Cartesian space is a six-dimensional space. Hence, in the *Cartesian* space, link flexibility is lumped in a 6×6 matrix, called the *Cartesian stiffness matrix*. As a robot stiffness is posture-dependent, it attains different values within the robot workspace. Elastostatic analysis helps the engineer identify the regions in which the robot is the

stiffest. In addition, for robots designed for highly repetitive tasks, the natural frequencies should be known. Because of *resonance*, a devastating phenomenon, the task frequency spectrum should lie far from the system natural frequencies. Hence, elastodynamic analysis is an essential task. This analysis completes its elastostatic counterpart, as it includes the effect of system mass distribution.



FIGURE 1.4. The McGill SMG

1.2 Thesis Objectives

The main objective of this thesis is to develop simple, yet comprehensive methods for the *constraint-wrench*, elastostatic, and elastodynamic analyses of parallel robots. By the presence of the passive kinematic pairs in the kinematic chain of parallel robots, the foregoing analyses are quite challenging.

To improve a robot structure, the maximum forces and moments imposed by the kinematic pairs at each link should be determined. For this matter, a reliable and fast constraint-wrench analysis is needed. In this study, such analysis, as pertaining to parallel robot structures with a single-loop kinematic chain is conducted, with the

aim of simultaneously calculating of both actuator torques and constraint wrenches. The analysis is based on the Newton-Euler formulation and the use of the *robot constraint matrix*. As a case study, the constraint wrench analysis of the McGill SMG is conducted along a test trajectory.

One of the main challenges in the design of current parallel robots is their *overconstrained* nature. As the MP of a parallel manipulator is connected to the BP by two or more serial chains, it may be redundantly constrained in one or more directions. This makes most parallel robots, such as Delta (Clavel, 1989), H4 (Pierrot et al., 2001) and, the McGill SMG, overconstrained. In reality, there is always a non-zero manufacturing tolerance; hence, without link flexibility, the assembly of parallel robots is not possible. For instance, deformable quadrilateral linkages are used to eliminate the overconstraints in the H4. However, link flexibility reduces the stiffness of the robot, which is a negative effect. Resolving the foregoing issue is imperative to enable the design of the links of a parallel robot with an acceptably high stiffness. To cope with this problem, *isoconstrained* versions of an overconstrained robot should be obtained. The isoconstrained versions of an overconstrained mechanism are obtained via plausible relaxations of the redundant constraints, that do not affect the motion of the original mechanism. This is elaborated through our case study, the McGill SMG. While an isoconstrained version is a design alternative, its dynamic model can be replaced with the original mechanism, and the results can be properly used for design purposes. Therefore, as in a isoconstrained version the Newton-Euler formulation brings enough equations, the constraint wrenches and actuator torques can be readily approximated without the links flexibility coming into the picture. It is noteworthy to clarify that the foregoing issue studied in this work is related to redundantly constrained parallel robots which are different from redundantly actuated counterparts; the latter is a highly specialized topic which lies outside the scope of the thesis.

In the second part, elastostatics, a novel stiffness-modeling methodology for parallel manipulators is developed, in which the effect of the passive kinematic pairs is

studied. The model is formulated for all six lower kinematic pairs (LKP), revolute (R), prismatic (P), helical (H), cylindrical (C), planar (F), and spherical (S) joints and two higher kinematic pairs (HKP), the universal (U) and Π joints. This representation is mainly inspired by the existing element assembly procedure in finite element analysis (FEA) while, here, it is defined with a notation that is more common in robotics; this results in a comprehensive method applicable to any type of parallel robots. As the outcome of the elastostatic analysis of a multibody system is a 6×6 Cartesian stiffness matrix, introducing a novel set of stiffness performance indices is quite important. These indices are useful for conducting a structural sensitivity analysis and optimization on a flexible mechanical system.

For a highly repetitive mechanical system, the calculation of the natural frequencies is an essential task. This helps the engineer place the system natural frequencies far from the excitation frequency spectrum, which calls for the elastodynamic analysis, with two ingredients, the stiffness and the mass matrices of the system. Here, for the computation of the mass matrix, a method, first proposed by Al Bassit et al. (2002) for serial robots, is extended to their parallel counterparts. Then, a novel elastodynamic analysis for parallel robots is proposed. While the method targets a especial case study, the McGill SMG, it is applicable to any robotic mechanical system.

At the end of this research work, the optimum design of the McGill SMG is conducted to improve its elastodynamic performance. By resorting to the novel elastostatic method introduced here and the two proposed stiffness indices, a multi-objective optimization problem is solved over the McGill SMG. In this vein, the structural parameters, such as link lengths and the link geometric parameters, are selected to enhance the stiffness performance indices of the system over a test trajectory.

1.3 Literature Survey

1.3.1 Inverse Dynamic Analysis of Robotic Mechanical Systems. Inverse dynamics is an essential analysis in robotics. In this analysis, the MP twist and

its rate are given, while the required actuator torques are calculated. Researchers have approached the inverse dynamics problem by using various methods. Staicu and Carp-Ciocardia (2003) used the virtual-power method to solve the inverse dynamics problem of Clavel’s DELTA robot. The inverse dynamics of the four-degree-of-freedom (dof) parallel robot H4 was reported by Choi et al. (2004), who used the Newton-Euler method for its model-based control. Based on Hamilton’s canonical equations, the direct and inverse dynamics problems of the NUWAR robot were solved by Miller (2001). Pang and Shahinpoor (1994) used the Lagrangian formulation for the inverse dynamics analysis of a three-dof parallel manipulator. Li and Qingsong (2005) employed the virtual work method to conduct the inverse dynamics of a modified Delta robot. While researchers mostly focused on the inverse dynamics of a special case study, Khalil and Ibrahim (2007) proposed a general method for the inverse and direct dynamics of parallel robots, which is based on the Newton-Euler formulation. Angeles and Lee (1989) introduced the *natural orthogonal complement* for the dynamic analysis of serial robots, which was later extended to parallel robots (Angeles, 2007). In all the foregoing works, the trajectory was known and the actuator torques were calculated. For torque control, the calculation of actuator torques suffices; however, robot structural design calls for the calculation of the constraint forces and moments, i.e., the *constraint wrenches*, as produced by the kinematic pairs, at every posture. This calls for the constraint-wrench analysis.

1.3.2 Structural Analysis of Robotic Mechanical Systems. The interaction of a parallel robot with the environment brings about wrenches applied to its MP. This is also the case of parallel robots conducting fast pick-and-place operations, whereby, the inertia forces of its links induce wrenches of the static type by virtue of D’Alembert’s Principle. Indeed, such operations are known to produce accelerations of the MP of the order of $15g$ (Nabat et al., 2006). The flexible elements of the robot, if properly designed, allow for “small”-amplitude displacement (SAD) *screws* of the MP in response to the applied wrenches. In the case of robots designed for pick-and-place operations, the calculation of these displacements is crucial, which calls for

an *elastostatic analysis*. A SAD screw is understood here in the sense of screw theory, wherein a line is represented by a six-dimensional array of Plücker coordinates, (Pottmann and Wallner, 2001), of which only four are independent; a screw is then defined as a line array with a pitch, its fifth independent parameter. A sixth parameter, the amplitude A , multiplying the screw array, then defines a twist—point velocity and angular velocity—or a wrench—force and moment—depending on whether A has units of angular velocity ω or of force F . If A is defined as $A = \omega\Delta t$, in which ω and Δt are angular velocity and time increment, respectively, and the product $\omega\Delta t \ll 1$, then a SAD screw is obtained. Elastostatic analysis hinges on the stiffness matrix of the robotic structure. This analysis has been mainly conducted by three methods: 1) finite element analysis (FEA); 2) the virtual joint method (VJM), and 3) matrix structural analysis (MSA). One of the most reliable methods for stiffness analysis is FEA, because by this means, each link, joint, and actuator can be modelled with its actual shape; however, FEA is highly demanding in computation time. Corradini et al. (2003) analyzed the stiffness of the H4 parallel robot by FEA, at one single posture, the results having been verified experimentally. The second method, VJM, is based on the work first proposed by Gosselin (1990). In the VJM, which is sometimes called *lumped-parameter method*, the compliance of the links is replaced by virtual compliant joints and rigid links. Then, by virtue of the elastic properties of the model, the equivalent virtual joint stiffnesses are derived. In this way, complex relations in the stiffness analysis are avoided and acceptable accuracy in short computation times is possible; however, because of the one-dimensional virtual springs in the model, the coupling effect between translational and rotational deflections is not taken into account. Later, the VJM was applied to conduct the stiffness analysis of spatial six-degree-of-freedom (dof) parallel robots with revolute joints (Gosselin and Zhang, 2002). Majou et al. (2007) applied the VJM to the Orthoglide robot in order to establish a parametric representation of the Cartesian stiffness matrix; however, coupling effects between translational and rotational deflections were neglected. Later, a modification was applied to the VJM in order to take into account

the coupling in question (Pashkevich and Chablat, 2008); this analysis was based on a multidimensional lumped-parameter model that replaces the link compliance with localized six-dof virtual springs. By this means the coupling between translational and rotational deflections was considered. To ensure high accuracy, the spring-stiffness values were calculated using FEA. The third method, MSA, is similar to FEA; however, instead of using a large number of elements, each part of the robot, link, joint, or actuator, is considered as a simple structural element, beam or rod, for example. By resorting to the theories of elasticity and kinematic chains, the stiffness matrix of the whole robot structure is obtained. The main feature that distinguishes MSA from other methods is that, within MSA, the stiffness matrix of the structure can be obtained in parametric form, which is important when the stiffness of the structure is to be optimized. Delbaise et al. (2006) found the Cartesian stiffness matrix of the Delta parallel structure by using this method, while Sales Gonalves and Mendes Carvalho (2008) employed the MSA for the stiffness analysis of the 6-RSS robot¹.

A novel method, based on the well-known concept of *generalized spring* (Lončarić, 1987), is proposed in this thesis; this method relies mostly on a form of the MSA, but enhanced here with a formulation for the modeling of the six lower kinematic pairs. The concept of generalized spring appears to have first been proposed by Lončarić in the foregoing reference as a suspension for a rigid body and composed of single linearly elastic translational springs. By means of the method proposed here, the 6×6 Cartesian stiffness matrix of the robot can be calculated *for any robot posture*, using stiffness matrices computed offline by means of parametric, whenever possible, or numerical methods. Moreover, although the method is limited to linearly elastic structural elements, it is general enough to accommodate elastic anisotropies, as appearing in composite materials. The method is especially attractive in the realm of parallel robots with what is known as Π -joints, which are *parallelogram four-bar linkages* (Hervé and Sparacino, 1992; Wohlhart, 1992). The specific property of this joint, under the assumption that its links are all rigid, is that any two opposite links

¹R denotes an actuated revolute joint, while S stands for spherical, or ball-and-socket, joint.

move under pure relative translation. Because of this property, II-joints are widely used in designing industrial parallel robots, e.g., ABB's Flexpicker and Adept's Quattro. By means of the generalized spring concept, the stiffness matrix of the II-joints can be readily calculated, as shown presently. The method is illustrated with the calculation of the stiffness matrix of a two-limb Schönflies Motion Generator. As an illustrative example, the robot stiffness matrix is computed along a MP trajectory. It is expected that the procedure will enable the real-time stiffness control of robotic structures, thereby allowing for faster, more accurate and safer operations.

As parallel robots, generally speaking, offer high speeds due to their lighter and stiffer structures, the dynamic response in high-speed applications is a practical design issue that calls for an elastodynamic analysis. Elastodynamics is the study of the dynamics of multibody systems with rigid and elastic bodies. This study pertains to the vibration analysis of flexible multibody systems. In this vein, the stiffness and the mass matrices of the whole mechanical system are first determined, from which the natural frequencies of the system are computed. These frequencies must lie above the frequency range of the task at hand. Imam and Sandor (1973) studied the elastodynamics of flexible manipulators, first, by resorting to the rigid kinematic analysis, and then, proportioning the areas of cross-section of the links optimally to account for the kineto-elastodynamic effects. The same researchers applied their method on a flexible four-bar linkage. By combining the rigid body harmonic analysis of a mechanism and FEA, Nath and Ghosh (1980*b*) reported the elastodynamic analysis of flexible mechanisms; these authors targeted the calculation of the steady-state deflections and stresses within the links. Ku and Chen (1990) developed a finite element model to conduct the vibration analysis of flexible robotic manipulators. In their method, the authors employed a generalized-Timoshenko-beam finite element. Many other studies have also addressed the elastodynamics of flexible mechanisms (Lowen and Jandrasits, 1979; Nath and Ghosh, 1980*a*; Cleghorn et al., 1984).

In the current study, the elastodynamics of the McGill SMG, depicted in Fig. 1.4, is the main motivation, with the purpose of designing the next generation of high-performance Schönflies Motion Generators.

1.3.3 Optimum Design. Optimum design includes minimization or maximization of one or more objective functions with or without constraints. A real-life engineering optimization problem usually consists of a number of conflicting objective functions and constraints with many design variables, which calls for multi-objective optimization. Generally speaking, optimization procedures are divided into two main categories, gradient and non-gradient methods. The former are based on the derivatives of the objective and the constraint functions, the latter only on function evaluation. In most engineering problems, objective and constraint functions are so complex that they cannot be described by formulas, the calculation of their derivatives then being impossible. Hence, *direct* methods, not requiring gradients, are commonly used for this type of problems. Many direct methods are available, e.g., *genetic algorithms* (GAs) (Goldberg, 1989), *simulated annealing* (SA) (Kirkpatrick et al., 1983), and *the complex method* (Box, 1965), among the best known. The optimization of robot manipulators has been studied for many years. Gosselin and Angeles (1988) reported the optimum kinematic design of a planar 3-DOF parallel manipulator. In this paper, to obtain the optimum kinematic design of a planar 3-DOF parallel manipulator, the authors established four different design criteria: (a) symmetry (b) the existence of a nonvanishing workspace for every orientation of the gripper (c) the maximization of the global workspace, and (d) the isotropy of the Jacobian of the manipulator. Liu et al. (2000) optimized the link lengths of 3-DOF spherical parallel manipulators (3-DOF SPMs) by resorting to the evaluation criteria, such as, the conditioning index and the stiffness. Performance improvement of robotic systems is one of the important issues being addressed in recent years. Stock and Miller (2003) conducted a kinematic optimization for parallel manipulators; they applied the method on the Delta robot.

Ceccarelli and Lanni (2004) solved a multi-objective optimization problem on general 3R manipulators, upon reducing the problem to single-variable optimization by means of weights on all objective functions. In the foregoing paper, the authors considered the workspace volume and the robot dimensions as objective functions. Based on the kinematic model and Jacobian matrix, Dou (2010) investigated the global conditioning index, global velocity index and global stiffness index of 3-RRR parallel manipulators; then, by resorting to a graphical representation of indices, the optimum design of 3-RRR parallel manipulators was obtained. The foregoing works mostly reported the kinematic optimization problem for robotic mechanical systems; however, in reality, to improve the overall performance of a robot the links elastic behaviour should also be taken into account. In a recent study, Gao et al. (2010) targeted the design optimization of a spatial six-degree-of-freedom parallel manipulator by taking into account the flexibility of the joints. By resorting to two indices, which are related to system stiffness and dexterity, a multi-objective optimization problem on a spatial six-degree-of-freedom parallel robot was established.

1.4 Thesis Outline

In Ch. 2, the constraint-wrench analysis of mechanisms, with focus on single-loop parallel manipulators, is sought. In this chapter, first, the kinematic constraint equations of all six LKPs, revolute (R), prismatic (P), helical (H), cylindrical (C), planar (F) and spherical (S), and of two HKPs, the Π -joint and the universal (U), are defined. Then, by introducing a systematic methodology, the constraint wrenches and applied torques of the McGill SMG, while tracking a trajectory, are obtained. In this analysis, all the links are assumed to be rigid.

In the next step, Ch. 3 is devoted to the elastostatic analysis of parallel robots with passive kinematic pairs. A novel formulation, inspired by FEA fundamentals, is introduced, which eases the calculation of the Cartesian stiffness matrix of parallel manipulators with passive kinematic pairs. Moreover, new stiffness indices, extracted from the Cartesian stiffness matrix, are introduced. The foregoing stiffness indices

can be readily used as objective functions in sensitivity analysis and optimization of robotic mechanical systems. As a case study, the time history of the stiffness indices of the McGill SMG while tracking a trajectory are obtained.

The subject of Ch. 4 is the elastodynamic analysis of parallel manipulators. The method is explained through a special case study, which is the McGill SMG. In this analysis, the posture-dependent stiffness and mass matrices of the robot, corresponding to the generalized coordinates chosen, are first calculated. With the essential ingredients at hand, the natural frequencies of the system are readily obtained.

In Ch. 5, by employing the stiffness indices introduced in Ch. 3, a multi-objective optimization problem over the McGill SMG is defined. The problem is solved by means of a genetic algorithm (GA). The possible optimum designs, which form a set of solutions, are obtained and evaluated.

Finally, in Ch. 6, a brief stiffness sensitivity analysis is conducted on the McGill SMG. Accordingly, the design recommendations for the McGill SMG are proposed. The thesis ends with closing remarks and recommendations for future research in Ch. 7.

CHAPTER 2

Constraint-wrench Analysis of Single-loop Robotic Manipulators

The constraint-wrench analysis of robotic mechanical systems with a single loop is the subject of this chapter. Although the method proposed here can be generalized to parallel robots with multiple-loop kinematic chains, this study is limited to single-loop systems. In this analysis, the constraint wrenches of a mechanism conducting a prescribed task are sought. To this end, a novel representation of the constraints imposed by the kinematic pairs is introduced. With this representation, the *constraint matrix* of a mechanism is readily derived. For the calculation of the constraint wrenches, by means of the constraint matrix and based on the Newton-Euler formulation, a new procedure is introduced. As a case study, the constraint wrench analysis of the McGill SMG, while tracking a test cycle adopted by the industry, is conducted.

2.1 Kinematic Constraints

In a mechanism, rigid links are connected by means of kinematic pairs. A kinematic pair imposes kinematic constraints on the links that they couple. Each constraint can be represented in terms of the twist arrays of the coupled bodies in linear homogeneous form (Angeles, 2007), namely,

$$\mathbf{K}_{i,i-1}\mathbf{t}_{i-1} + \mathbf{K}_{i,i}\mathbf{t}_i = \mathbf{0} \quad (2.1)$$

where $\mathbf{K}_{i,i-1}$ and $\mathbf{K}_{i,i}$ are 6×6 coefficient matrices that depend on the constraints imposed by the kinematic pair joining the $i - 1$ st and the (i) th links, while $\mathbf{t}_i$ is the six-dimensional twist array of i th link, containing its angular velocity $\boldsymbol{\omega}_i$ and the velocity $\dot{\mathbf{c}}_i$ of its center of mass, namely,

$$\mathbf{t}_i = \begin{bmatrix} \boldsymbol{\omega}_i^T & \dot{\mathbf{c}}_i^T \end{bmatrix}^T$$

This systematic formulation is extremely useful for mechanisms such as parallel robots, like the one depicted in Fig. 2.1. In this figure the BP and the MP are connected by two limbs, numbered I and II . The limbs are composed of several links coupled by kinematic pairs that constrain the motion of the links. For a robot with d degrees-of-freedom (dof) and n links, the constraint equations form a system of $6n$ scalar equations (Angeles, 2007), namely,

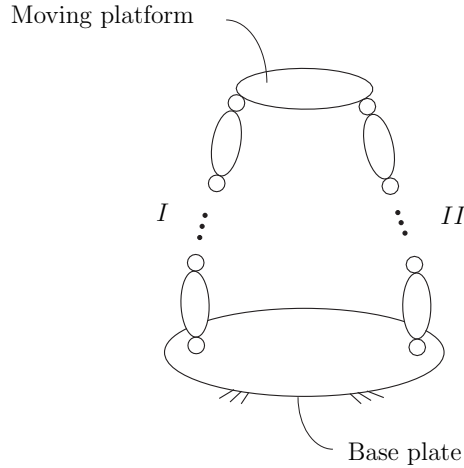


FIGURE 2.1. A parallel mechanism

$$\mathbf{K}\mathbf{t} = \mathbf{0} \tag{2.2}$$

in which the $6n \times 6(n - 1)$ matrix $\mathbf{K}$, containing the constraint coefficient matrices of each kinematic pair, as introduced in eq. (2.1), is called the *robot constraint matrix*. The $6(n - 1)$ -dimensional robot twist array $\mathbf{t}$ is composed of all rigid-body twist arrays. The dimension of the nullspace of $\mathbf{K}$, its *nullity*, is equal to d , the dof of robot. The constraint wrenches applied by the two kinematic pairs produces a resultant constraint

wrench $\mathbf{w}_i^C$ acting at the centre of mass (COM) of the i th link; the $6n$ -dimensional array $\mathbf{w}^C$ of these link wrenches is called the *robot constraint wrench* vector. Angeles (2007) showed that, for serial chains, $\mathbf{w}^C$ can be expressed as

$$\mathbf{w}^C = \mathbf{K}^T \boldsymbol{\lambda} \quad (2.3)$$

where $\boldsymbol{\lambda}$ is a $6n$ -dimensional vector of *joint-constraint wrenches*, whose components are constraint forces and moments transmitted by a kinematic pair to the two consecutive bodies connected by the pair. At each kinematic pair, the i th joint-constraint wrench does not develop any power on the rigid-body it couples, and that moves under a twist $\mathbf{t}_i$, as allowed by this pair, labelled i , namely,

$$\mathbf{t}_i^T \boldsymbol{\lambda}_i = 0 \quad (2.4)$$

From screw theory (Roth, 1984), a rigid body twist array is the product of an amplitude $A_i \neq 0$ with units of angular velocity by a unit screw $\mathbf{s}_i$, which results in

$$\mathbf{t}_i = A_i \mathbf{s}_i \quad (2.5)$$

A unit screw, in turn, is a line with a pitch p_i . If the Plücker coordinates (Roth, 1984) of the line $\mathcal{L}_i$ are $\begin{bmatrix} \mathbf{e}_i^T & (\mathbf{r}_i \times \mathbf{e}_i)^T \end{bmatrix}^T$, with $\mathbf{e}_i$ denoting the unit vector parallel to $\mathcal{L}_i$ and $\mathbf{r}_i$ the position vector of a point R_i on $\mathcal{L}_i$, then, $\mathbf{s}_i$ is given by

$$\mathbf{s}_i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{r}_i \times \mathbf{e}_i + p_i \mathbf{e}_i \end{bmatrix} \quad (2.6)$$

an array known to be represented in *ray coordinates* (Klein, 1871). Upon substitution of eq. (2.5) into eq. (2.4), with $\boldsymbol{\lambda}_i$ given in *axis coordinates*, i.e., with the moment as the top block and the force as its bottom counterpart¹,

$$A_i \mathbf{s}_i^T \boldsymbol{\lambda}_i = \mathbf{0} \quad (2.7)$$

¹This mode of representation of twist and wrench arrays is crucial here, for otherwise the inner product of eq. (2.7) would be meaningless.

As A_i is a nonzero scalar, eq. (2.7) yields

$$\mathbf{s}_i^T \boldsymbol{\lambda}_i = 0 \quad (2.8)$$

which reveals that the joint constraint vector is *reciprocal* to the joint screw². Upon assembling of all eqs. (2.7), a second relation is obtained for vector $\boldsymbol{\lambda}$:

$$\mathbf{S}^T \boldsymbol{\lambda} = \mathbf{0} \quad (2.9)$$

Matrix $\mathbf{S}$ is called henceforth the *joint-screw matrix*, which contains the screw vectors of all kinematic pairs of the mechanism. It is thus apparent that the vector of joint-constraint wrenches $\boldsymbol{\lambda}$ lies in the nullspace of $\mathbf{S}^T$. The matrix $\mathbf{S}$ is of $6n \times m$, where m is the sum of dof of all kinematic pairs.

Kinematic pairs are divided into two categories: lower and higher. Lower kinematic pairs (LKP) are characterized by the surface along which one of the coupled links “wraps” the other link. Contact on higher kinematic pairs (HKP), e.g., gears and cams, is either along a line or punctual (Hartenberg and Denavit, 1964). In the sequel, the kinematic constraint equations of the six LKPs, revolute (R), prismatic (P), screw (H), cylindrical (C), spherical (S), and planar (F) and of two higher kinematic pairs, the Π and the universal (U) joints, are derived.

2.1.1 Kinematic Constraint Equations for the Revolute Joint. A revolute joint, depicted in Fig. 2.2, is one of the most common kinematic pairs. Two rigid links, connected by means of a revolute joint can rotate with respect to each other about the axis of the joint. In the case at hand, this axis is the line passing through point P_i and parallel to the unit vector $\mathbf{e}_i$. Thus, the relative angular velocity of link i with respect to link $i - 1$ has a non-zero component only along the axis of the joint, namely,

$$\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i-1} = \dot{\theta}_i \mathbf{e}_i \quad (2.10)$$

²As screw arrays do not admit a norm, they neither admit the concept of orthogonality; what they admit is the concept of *reciprocity*, as per eq. (2.8), as long as one screw is in ray, the other is in axis coordinates.

The cross-product matrix (CPM) of $\mathbf{e}$ is defined for any three-dimensional vector $\mathbf{v}$ as (Angeles, 2007)

$$\mathbf{E} = \frac{\partial (\mathbf{e} \times \mathbf{v})}{\partial \mathbf{v}} \quad (2.11)$$

The constraint equation (2.10) can be expressed in linear homogeneous form in terms of the link angular velocities, upon multiplying from the left both sides of eq. (2.10) by $\mathbf{E}_i$, the CPM of the unit vector $\mathbf{e}_i$, and hence,

$$\mathbf{E}_i(\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i-1}) = \mathbf{0} \quad (2.12)$$

Furthermore, the velocities of the centers of mass of two successive links are related by:

$$\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \mathbf{R}_i \boldsymbol{\omega}_i + \mathbf{D}_{i-1} \boldsymbol{\omega}_{i-1} = \mathbf{0} \quad (2.13)$$

where $\mathbf{R}_i$ and $\mathbf{D}_{i-1}$ are the CPMs of vectors $\boldsymbol{\rho}_i$ and $\boldsymbol{\gamma}_{i-1}$, respectively, with $\boldsymbol{\rho}_i$ and $\boldsymbol{\gamma}_{i-1}$ shown in Fig. 2.2. Upon assembling eqs. (2.12) and (2.13), a six-dimensional homogeneous relation between the twists $\mathbf{t}_i$ and $\mathbf{t}_{i+1}$ of two consecutive links is obtained:

$$\mathbf{K}_{i,i-1} \mathbf{t}_{i-1} + \mathbf{K}_{i,i} \mathbf{t}_i = \mathbf{0} \quad (2.14)$$

with the 6×6 matrices $\mathbf{K}_{i,i-1}$ and $\mathbf{K}_{i,i}$, introduced in eq. (2.1), given by

$$\mathbf{K}_{i,i-1} = \begin{bmatrix} -\mathbf{E}_i & \mathbf{O} \\ \mathbf{D}_{i-1} & -\mathbf{1} \end{bmatrix}, \quad \mathbf{K}_{i,i} = \begin{bmatrix} \mathbf{E}_i & \mathbf{O} \\ \mathbf{R}_i & \mathbf{1} \end{bmatrix} \quad (2.15)$$

where $\mathbf{1}$ is the 3×3 identity matrix. The foregoing analysis is taken from (Angeles, 2007), reproduced here for quick reference. Because of its nature, the unit screw $\mathbf{s}_R^i$ of a revolute joint has a zero pitch, and hence,

$$\mathbf{s}_R^i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{p}_i \times \mathbf{e}_i \end{bmatrix} \quad (2.16)$$

Because eq. (2.8) always holds independent of the point to which the screw is referred, a reference point can be simply chosen on the joint axis, e.g., P_i of Fig. 2.2, which

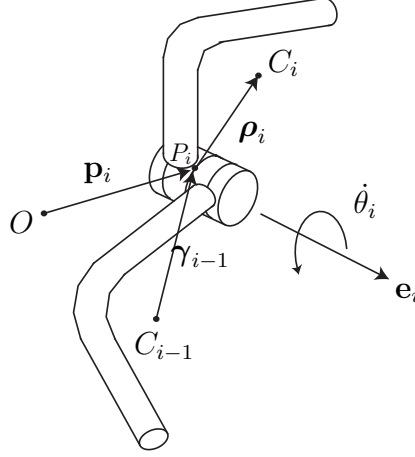


FIGURE 2.2. Two links connected by a revolute joint

then leads to

$$\mathbf{s}_R^i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{0} \end{bmatrix} \quad (2.17)$$

2.1.2 Kinematic Constraint Equations for the Prismatic Joint. Two successive links connected by means of a prismatic joint, as shown in Fig. 2.3, have the same angular velocity, which leads to

$$\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i-1} = \mathbf{0} \quad (2.18)$$

In this case, the relation between the velocities of the centers of mass of the two links becomes

$$\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \mathbf{R}'_i \boldsymbol{\omega}_i + \dot{b}_i \mathbf{e}_i = \mathbf{0} \quad (2.19)$$

where $\dot{b}_i \mathbf{e}_i$ accounts for the relative translational velocity of the links. This constraint equation then leads, upon premultiplication by $\mathbf{E}_i$, to

$$\mathbf{E}_i (\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \mathbf{R}'_i \boldsymbol{\omega}_i) = \mathbf{0} \quad (2.20)$$

where $\mathbf{E}_i$ is the CPM of the unit vector $\mathbf{e}_i$. Upon rearranging the six constraint equations (2.18) and (2.20), the coefficient matrices $\mathbf{K}_{i,i-1}$ and $\mathbf{K}_{i,i}$ of eq. (2.1) are

derived as

$$\mathbf{K}_{i,i-1} = \begin{bmatrix} -\mathbf{1} & \mathbf{O} \\ \mathbf{O} & -\mathbf{E}_i \end{bmatrix}, \quad \mathbf{K}_{i,i} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{E}_i \mathbf{R}'_i & \mathbf{E}_i \end{bmatrix} \quad (2.21)$$

where $\mathbf{R}'_i$ is the CPM of vector $\boldsymbol{\rho}_i + \boldsymbol{\gamma}_{i-1}$, the latter being depicted in Fig. 2.3. The foregoing analysis was also taken from (Angeles, 2007), reproduced here, again, for quick reference. A prismatic joint, allowing for pure translation along one direction,

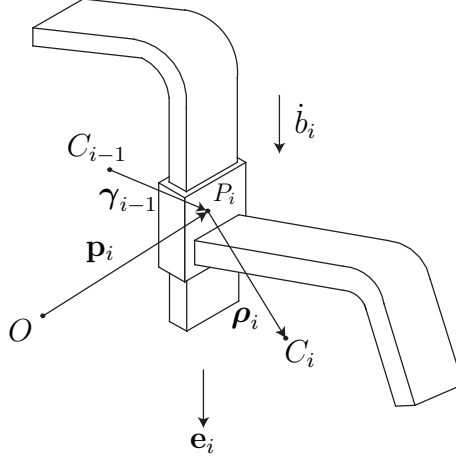


FIGURE 2.3. Two links connected by a prismatic joint

its unit screw has an infinite pitch, namely,

$$\mathbf{s}_P^i = \begin{bmatrix} 0 \\ \mathbf{e}_i \end{bmatrix} \quad (2.22)$$

2.1.3 Kinematic Constraint Equations for the Helical Joint. In the case of a helical—a.k.a screw—joint, with axis passing through P_i , parallel to the unit vector $\mathbf{e}_i$ and of pitch p_i , the link angular velocities are related by

$$\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i-1} = \dot{\theta}_i \mathbf{e}_i \quad (2.23)$$

in which b_i denotes the translation along the joint axis. Therefore, in a screw joint the constraint equation on the link angular velocities is the same as that applicable to the revolute and cylindrical joints, namely,

$$\mathbf{E}_i(\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i-1}) = \mathbf{0} \quad (2.24)$$

where $\mathbf{E}_i$ is, as before, the CPM of the unit vector $\mathbf{e}_i$. The relation between the velocities of the link COMs can be represented in turn as

$$\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \mathbf{R}_i \boldsymbol{\omega}_i + \mathbf{D}_{i-1} \boldsymbol{\omega}_{i-1} + \dot{b}_i \mathbf{e}_i = \mathbf{0}, \quad \dot{b}_i = p_i \dot{\theta}_i \quad (2.25)$$

Upon substitution of $\dot{b}_i$ in terms of $\dot{\theta}_i$ into eq. (2.25), one obtains a relation between the twists of the coupled links, i.e.,

$$\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + (\mathbf{R}_i + p_i \mathbf{1}) \boldsymbol{\omega}_i + (\mathbf{D}_{i-1} - p_i \mathbf{1}) \boldsymbol{\omega}_{i-1} = \mathbf{0} \quad (2.26)$$

Now, from eqs. (2.24) and (2.26) the coefficient matrices of eq. (2.1) can be readily obtained as

$$\mathbf{K}_{i,i-1} = \begin{bmatrix} -\mathbf{E}_i & \mathbf{O} \\ \mathbf{D}_{i-1} - p_i \mathbf{1} & -\mathbf{1} \end{bmatrix}, \quad \mathbf{K}_{i,i} = \begin{bmatrix} \mathbf{E}_i & \mathbf{O} \\ \mathbf{R}_i + p_i \mathbf{1} & \mathbf{1} \end{bmatrix} \quad (2.27)$$

The screw of the H joint of Fig. 2.4 with respect to a point O is, thus,

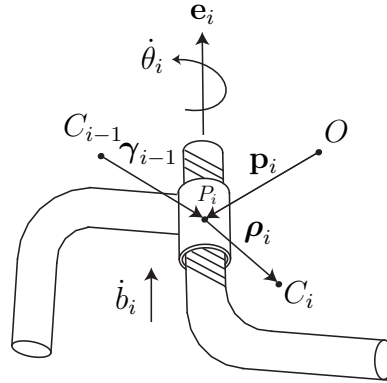


FIGURE 2.4. Two links are connected by a helical joint

$$\mathbf{s}_H^i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{e}_i \times \mathbf{p}_i + p \mathbf{e}_i \end{bmatrix} \quad (2.28)$$

By choosing the reference point P_i on the axis of rotation, $\mathbf{s}_H^i$ becomes

$$\mathbf{s}_H^i = \begin{bmatrix} \mathbf{e}_i \\ p_i \mathbf{e}_i \end{bmatrix} \quad (2.29)$$

2.1.4 Kinematic Constraint Equations for the Cylindrical Joint. Kinetically, a cylindrical joint can be replaced by the series array of a revolute joint and a prismatic joint, the latter allowing for relative translation in the direction of the axis of the former. Therefore, if two successive links are connected by a cylindrical joint the relation between their angular velocities is the same as that obtained for a revolute joint, eq. (2.12), and the relation between the velocities of the link centers of mass is identical with the one derived for a prismatic joint. Therefore, in the case of a cylindrical joint, matrices $\mathbf{K}_{i,i-1}$ and $\mathbf{K}_{i,i}$ of eq. (2.1) are

$$\mathbf{K}_{i,i-1} = \begin{bmatrix} -\mathbf{E}_i & \mathbf{O} \\ \mathbf{E}_i \mathbf{D}_{i-1} & -\mathbf{E}_i \end{bmatrix}, \quad \mathbf{K}_{i,i} = \begin{bmatrix} \mathbf{E}_i & \mathbf{O} \\ \mathbf{E}_i \mathbf{R}_i & \mathbf{E}_i \end{bmatrix} \quad (2.30)$$

in which, $\mathbf{D}_{i-1}$, $\mathbf{R}_i$, and $\mathbf{E}_i$ are the CPMs of vectors $\boldsymbol{\gamma}_{i-1}$, $\boldsymbol{\rho}_i$, and $\mathbf{e}_i$, respectively,

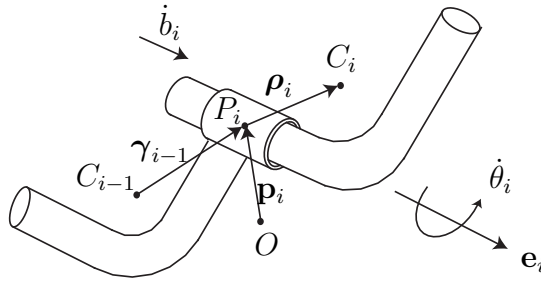


FIGURE 2.5. Two links connected by a cylindrical joint

depicted in Fig. 2.5 As a C joint can be realized by the concatenation of a revolute and a prismatic joints, rather than a screw vector array, this joint has an array of screw vectors, i.e., a screw matrix $\mathbf{S}_C^i$. This matrix is composed of the two screw arrays $\mathbf{s}_R^i$ and $\mathbf{s}_P^i$, defined in eqs. (2.17) and (2.22), respectively:

$$\mathbf{S}_C^i = \begin{bmatrix} \mathbf{s}_R^i & \mathbf{s}_P^i \end{bmatrix} \quad (2.31)$$

2.1.5 Kinematic Constraint Equations for the Spherical Joint. In this case, the angular velocities of the two coupled links are unconstrained, while the constraint equation on the twists of the links becomes the same as that obtained for

the R joint:

$$\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \mathbf{R}_i \boldsymbol{\omega}_i + \mathbf{D}_{i-1} \boldsymbol{\omega}_{i-1} = \mathbf{0} \quad (2.32)$$

where $\mathbf{D}_{i-1}$ and $\mathbf{R}_i$ are the CPM of vectors $\boldsymbol{\gamma}_{i-1}$ and $\boldsymbol{\rho}_i$, shown in Fig. 2.6. Thus, the coefficient matrices $\mathbf{K}_{i,i-1}$ and $\mathbf{K}_{i,i}$ of eq. (2.1) for a S joint are

$$\mathbf{K}_{i,i-1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{D}_{i-1} & -\mathbf{1} \end{bmatrix}, \quad \mathbf{K}_{i,i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{R}_i & \mathbf{1} \end{bmatrix} \quad (2.33)$$

Because of its multi-dof nature, this joint also admits a screw matrix, rather than

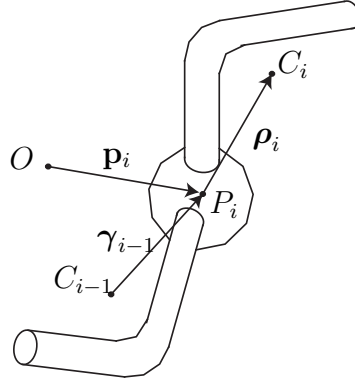


FIGURE 2.6. Two links are connected by a spherical joint

a screw vector. The joint screw matrix of this joint contains three revolute-joint screws about three concurrent axes, which can be conveniently chosen as the mutually orthogonal axes passing through P_i , of directions parallel to the unit vectors $\mathbf{e}_i$, $\mathbf{f}_i$, and $\mathbf{g}_i$, namely,

$$\mathbf{S}_S^i = \begin{bmatrix} \mathbf{s}_e^i & \mathbf{s}_f^i & \mathbf{s}_g^i \end{bmatrix} \quad (2.34)$$

where $\mathbf{s}_e^i$, $\mathbf{s}_f^i$ and $\mathbf{s}_g^i$ are, respectively, unit screws of R joints with axes of rotation along $\mathbf{e}_i$, $\mathbf{f}_i$ and $\mathbf{g}_i$, as introduced in eq. (2.17).

2.1.6 Kinematic Constraint Equations for the Planar Joint. A planar joint restricts the relative motion of the two coupled links in a plane, as depicted in Fig. 2.7, to two translations in the plane along the directions of the unit vectors $\mathbf{f}_i$ and $\mathbf{g}_i$, and one rotation about an axis passing through point P_i , in the direction of

$\mathbf{e}_i$, the unit vector in the direction of the normal to the plane. The relative angular velocity of the two links coupled by a planar joint is

$$\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i-1} = \dot{\theta}_i \mathbf{e}_i \quad (2.35)$$

whence,

$$\mathbf{E}_i(\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i-1}) = \mathbf{0} \quad (2.36)$$

where $\mathbf{E}_i$ is the CPM of vector $\mathbf{e}_i$. The relation between the velocities of the centers of mass of the two links becomes

$$\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \mathbf{R}_i \boldsymbol{\omega}_i + \dot{f}_i \mathbf{f}_i + \dot{g}_i \mathbf{g}_i = \mathbf{0} \quad (2.37)$$

where $\dot{f}_i$ and $\dot{g}_i$ account for the relative translational velocities of the links in the direction of the distinct unit vectors $\mathbf{f}_i$ and $\mathbf{g}_i$. This constraint equation then leads, upon premultiplication sequentially by $\mathbf{F}_i$ and $\mathbf{E}_i$, to

$$\mathbf{E}_i \mathbf{F}_i (\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \mathbf{R}_i \boldsymbol{\omega}_i) = \mathbf{0} \quad (2.38)$$

where $\mathbf{F}_i$ and $\mathbf{E}_i$ are defined as usual. Upon rearranging the six constraint equations (2.36) and (2.38), the coefficient matrices $\mathbf{K}_{i,i-1}$ and $\mathbf{K}_{i,i}$ of eq. (2.1) are derived as

$$\mathbf{K}_{i,i-1} = \begin{bmatrix} -\mathbf{E}_i & \mathbf{O} \\ \mathbf{O} & -\mathbf{E}_i \mathbf{F}_i \end{bmatrix}, \quad \mathbf{K}_{i,i} = \begin{bmatrix} \mathbf{E}_i & \mathbf{O} \\ \mathbf{E}_i \mathbf{F}_i \mathbf{R}_i & \mathbf{E}_i \mathbf{F}_i \end{bmatrix} \quad (2.39)$$

where $\mathbf{R}_i$ is the CPM of vector $\boldsymbol{\rho}_i + \boldsymbol{\gamma}_{i-1}$, the latter depicted in Fig. 2.7. As a planar joint can be replaced kinematically with two prismatic and one revolute joints, its joint screw matrix bears three screw arrays, namely,

$$\mathbf{S}_F = \begin{bmatrix} \mathbf{s}_f^i & \mathbf{s}_g^i & \mathbf{s}_e^i \end{bmatrix} \quad (2.40)$$

where $\mathbf{s}_f^i$ and $\mathbf{s}_g^i$ are, respectively, the unit screws of the prismatic joints, with translation directions $\mathbf{f}_i$ and $\mathbf{g}_i$. Moreover, $\mathbf{s}_e^i$ is the unit screw of the revolute of axis passing through P_i and parallel to $\mathbf{e}_i$.

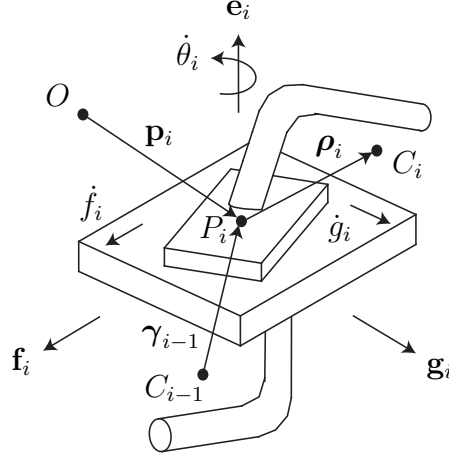


FIGURE 2.7. Two links are connected by a planar joint

2.1.7 Kinematic Constraint Equations for the Universal Joint. A universal joint can be kinematically replaced by two revolute joints of axes intersecting at right angles. The revolute joint axes are parallel to the unit vectors $\mathbf{e}_i$ and $\mathbf{f}_i$, normal to each other, as shown in Fig. 2.8. Therefore, the relative angular velocity of the two successive links connected by a universal joint is

$$\boldsymbol{\omega}_{i-1} - \boldsymbol{\omega}_i = \dot{\epsilon}_i \mathbf{e}_i + \dot{\phi}_i \mathbf{f}_i \quad (2.41)$$

in which $\dot{\epsilon}_i$ and $\dot{\phi}_i$ account for the relative angular velocities of the links about the unit vectors $\mathbf{e}_i$ and $\mathbf{f}_i$, respectively, whence,

$$\mathbf{F}_i \mathbf{E}_i (\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i-1}) = \mathbf{0} \quad (2.42)$$

where $\mathbf{F}_i$ and $\mathbf{E}_i$ are the CPMs of vectors $\mathbf{e}_i \times \mathbf{f}_i$ and $\mathbf{e}_i$, respectively. The relation between the velocities of the link COMs is the same as that derived for a revolute joint and shown in eq. (2.13). As a result, the coefficient matrices $\mathbf{K}_{i,i-1}$ and $\mathbf{K}_{i,i}$, as pertaining to U joint, are obtained as

$$\mathbf{K}_{i,i-1} = \begin{bmatrix} -(\mathbf{F}_i \mathbf{E}_i) & 1 \\ \mathbf{D}_{i-1} & -1 \end{bmatrix}, \quad \mathbf{K}_{i,i} = \begin{bmatrix} (\mathbf{F}_i \mathbf{E}_i) & 0 \\ \mathbf{R}_i & 1 \end{bmatrix} \quad (2.43)$$

where $\mathbf{R}_i$ and $\mathbf{D}_{i-1}$ are, respectively, the CPM of vectors $\boldsymbol{\rho}_i$ and $\boldsymbol{\gamma}_{i-1}$, shown in Fig. 2.8. As a universal joint can be replaced kinematically with two revolutes, its

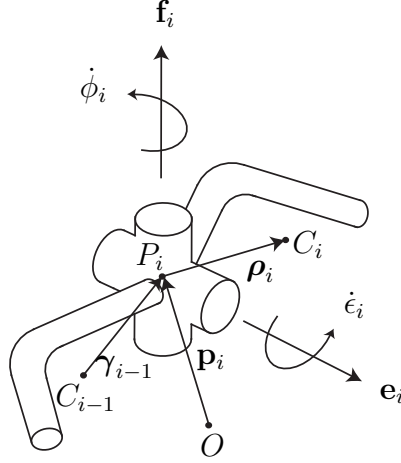


FIGURE 2.8. Two links are connected by a universal joint

joint screw matrix has two screw arrays, each corresponding to one revolute, namely,

$$\mathbf{S}_U = \begin{bmatrix} \mathbf{s}_e^i & \mathbf{s}_f^i \end{bmatrix} \quad (2.44)$$

where $\mathbf{s}_e^i$ and $\mathbf{s}_f^i$ are, respectively, the unit screws of the revolutes with axes of rotation intersecting at point P_i and parallel to the unit vectors $\mathbf{e}_i$ and $\mathbf{f}_i$. Apparently, contact between the two links coupled by this HKP is punctual, namely, at P_i .

2.1.8 Kinematic Constraint Equations for the Π Joint. A novel kinematic pair, known as the Π -joint, which is commonly used in a new generation of parallel robots, is also included. The Π -joint is a *parallelogram four-bar linkage* that couples two links. The specific property of this joint, under the assumption that its links are all rigid, is that any two opposite links move under pure relative translation, its points tracing circles, on the other link, with variable centre and radius identical to the length of the two other parallelogram links (Hervé and Sparacino, 1992; Wohlhart, 1992). Hence, the angular velocities of two successive rigid links connected by a Π -joint are equal, i.e.,

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} \quad (2.45)$$

The velocities of the link COMs are derived from the geometry of Fig. 2.9, which leads to

$$\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \boldsymbol{\omega}_{i-1} \times (\boldsymbol{\gamma}_{i-1} + \mathbf{u}_i) + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_i + \dot{\theta}_i \mathbf{e}_i \times \mathbf{u}_i = \mathbf{0} \quad (2.46)$$

A Π -joint lets the coupled links have one relative translational motion, which results in three constraint equations, as derived from eq. (2.46), of which only two are independent, namely,

$$\mathbf{L}_i \left[\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} - \boldsymbol{\omega}_{i-1} \times (\boldsymbol{\gamma}_{i-1} + \mathbf{u}_i) - \boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \right] = \mathbf{0} \quad (2.47)$$

in which $\mathbf{L}_i$ is the CPM of vector $\mathbf{e}_i \times \mathbf{u}_i$. Thus, $\mathbf{K}_{i,i-1}$ and $\mathbf{K}_{i,i}$ of eq. (2.1) are obtained as

$$\mathbf{K}_{i,i-1} = \begin{bmatrix} -\mathbf{1} & \mathbf{O} \\ \mathbf{L}_i \mathbf{D}_{i-1} & -\mathbf{L}_i \end{bmatrix}, \quad \mathbf{K}_{i,i} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{L}_i \mathbf{R}_i & \mathbf{L}_i \end{bmatrix} \quad (2.48)$$

As a Π -joint produces pure translation, its screw has an infinite pitch. Therefore,

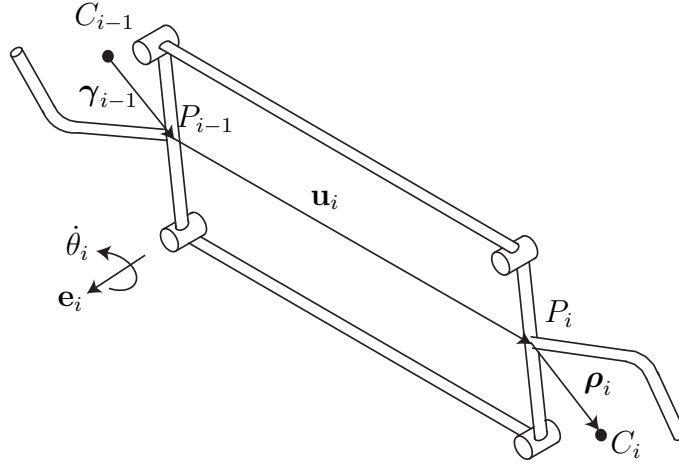


FIGURE 2.9. Two links are connected by means of a Π joint

the unit screw of a Π -joint is that of a prismatic joint whose direction is $\mathbf{e}_i \times \mathbf{u}_i / \|\mathbf{u}_i\|$, namely,

$$\mathbf{s}_\Pi = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \times \mathbf{u}_i / \|\mathbf{u}_i\| \end{bmatrix} \quad (2.49)$$

Similar to a prismatic joint, the joint variable associated with a Π joint is a translational speed, namely, $\|\mathbf{u}_i\| \dot{\theta}_i$. Different from the prismatic joint, the unit screw of

the Π -joint is of *variable direction*, as “seen” by the two coupled links. In order to identify the set of contact points of the links coupled by this joint, the joint has to be regarded as a *planar linkage*. In this light, the contact point of the two links coupled by a Π -joint lies at infinity, in the plane of motion of the parallelogram linkage. However, its line of sight³ changes continuously as the linkage moves, always parallel to line $P_{i-1}P_i$ of Fig. 2.9. Coming back to three-dimensional space, this point turns out to be the intersection of a line normal to the plane of motion with this plane itself. The two links coupled by a Π -joint thus maintain contact along a moving line at infinity.

2.2 Constraint-Wrench Analysis

For the structural design of a robot, knowing the constraint wrenches at any posture during a task plays an important role. This requires *constraint-wrench analysis*. To this end, the time-histories of position, velocity, and acceleration of the MP are given, actuated torques and constraint wrenches at the kinematic pairs being required. To compute these quantities, the equations of motion of all rigid links should be derived. The Newton-Euler equations of motion for the i th link, before coupling, are (Angeles, 2007)

$$\mathbf{M}_i \dot{\mathbf{t}}_i = -\mathbf{W}_i \mathbf{M}_i \mathbf{t}_i + \mathbf{w}_i^A + \mathbf{w}_i^S + \mathbf{w}_i^C, \quad i = 1, \dots, n \quad (2.50)$$

where $\mathbf{M}_i$ and $\mathbf{W}_i$ are the 6×6 inertia and angular-velocity dyads, namely,

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i \mathbf{1} \end{bmatrix}, \quad \mathbf{W}_i = \begin{bmatrix} \boldsymbol{\Omega}_i & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \quad (2.51)$$

in which $\mathbf{1}$ and $\mathbf{O}$ denote the 3×3 identity and zero matrices, respectively, while $\boldsymbol{\Omega}_i$ and $\mathbf{I}_i$ are the angular-velocity and the inertia matrices of the i th rigid link, which are defined with respect to the COM C_i of this link (Angeles, 2007); the mass of this body is denoted by m_i . Moreover, $\mathbf{w}_i^A$, $\mathbf{w}_i^S$, and $\mathbf{w}_i^C$ are, respectively, the 6-dimensional

³Line of sight is defined as a line between an observer and a point at infinity.

vectors of *actuator*, *static-load*, and *nonworking constraint wrenches* exerted on the i th rigid link with forces acting at its COM. For a robot with $n - 1$ moving links the equations of motion are assembled in the form

$$\mathbf{M}\dot{\mathbf{t}} = -\mathbf{W}\mathbf{M}\mathbf{t} + \mathbf{w}^A + \mathbf{w}^S + \mathbf{w}^C \quad (2.52)$$

where $\mathbf{t}$ was already defined as the *robot twist vector*, which contains the twist vectors of all the $n - 1$ moving bodies, namely,

$$\mathbf{t} = \begin{bmatrix} \mathbf{t}_1^T & \mathbf{t}_2^T & \dots & \mathbf{t}_{n-1}^T \end{bmatrix}^T$$

while $\mathbf{w}^A$, $\mathbf{w}^S$ and $\mathbf{w}^C$ are the robot actuator, static-load and constraint wrenches, while

$$\mathbf{M} = \text{diag}(\mathbf{M}_1, \dots, \mathbf{M}_{n-1}), \quad \mathbf{W} = \text{diag}(\mathbf{W}_1, \dots, \mathbf{W}_{n-1})$$

are the *uncoupled* robot mass, and angular-velocity matrices. The robot twist vector $\mathbf{t}$ can be represented as a linear transformation of the vector of joint rates, i.e.,

$$\mathbf{t} = \mathbf{T}\dot{\boldsymbol{\theta}} \quad (2.53)$$

where $\boldsymbol{\theta}$ contains the joint variables, both active and passive. In this analysis, parallel robots with single-loop kinematic chains are targeted; in this type, the number of kinematic pairs equals that of rigid links, i.e., $n - 1$ moving links plus a base. Vector $\dot{\boldsymbol{\theta}}$ is composed of d active and p passive joint rates, where d is the degree-of-freedom of the robot. Moreover, $\mathbf{T}$ is the $6(n - 1) \times (d + p)$ *twist-shaping matrix*, defined as

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 \\ \vdots \\ \mathbf{T}_{n-1} \end{bmatrix} \quad (2.54)$$

in which $\mathbf{T}_i$ is the $6 \times (d + p)$ twist-shaping matrix of the i th link (Angeles, 2007). It is worth mentioning that if each kinematic pair brings only one *kinematically effective dof*—to be defined presently—then, the number of passive joint rates p equals $n - d$. In robots with reduced mobility, it may happen that not all the dof of a kinematic

pair affect the robot kinematics; the *kinematically effective dof* accounts for those dof that contribute to the motion of the MP. Given a trajectory, the joint-variable vector $\boldsymbol{\theta}$ is first derived from inverse kinematics. The robot-actuator wrench vector $\mathbf{w}^A$ can be represented as a linear transformation of actuator forces or torques, namely,

$$\mathbf{w}^A = \mathbf{W}^A \boldsymbol{\tau} \quad (2.55)$$

in which $\mathbf{W}^A$ is termed here the *robot actuator wrench-shaping matrix*, and $\boldsymbol{\tau}$ is the d -dimensional vector of actuator forces or torques. Hence, substitution of eqs. (2.3), (2.53), and (2.55) into eq. (2.52) yields

$$\mathbf{K}^T \boldsymbol{\lambda} + \mathbf{W}^A \boldsymbol{\tau} = \boldsymbol{\eta} \quad (2.56)$$

in which

$$\boldsymbol{\eta} = \mathbf{M}\mathbf{T}\ddot{\boldsymbol{\theta}} + (\mathbf{W}\mathbf{M}\mathbf{T} + \mathbf{M}\dot{\mathbf{T}})\dot{\boldsymbol{\theta}} - \mathbf{w}^S \quad (2.57)$$

while vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ are unknown. Moreover, the reciprocity between each joint screw and its constraint wrench is recalled, which for all joints can be assembled in the form of eq. (2.8). Equations (2.8) and (2.56) can be cast in a compact system of $6n + d$ linear equations, namely,

$$\begin{bmatrix} \mathbf{K}^T & \mathbf{W}^A \\ \mathbf{S}^T & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\tau} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{0} \end{bmatrix} \quad (2.58)$$

In the static case, when the system is stationary, eq. (2.58) is still applicable, if with $\boldsymbol{\eta}$ reduced to $-\mathbf{w}^S$.

2.3 A Case Study: the McGill SMG

The McGill SMG, as depicted in Fig. 2.10, is a parallel robot whose MP can produce motions within the Schönflies displacement subgroup (Hervé, 1994). This subgroup, denoted $\chi(\mathbf{e})$, comprises three translations and one rotation about an axis of fixed orientation parallel to the unit vector $\mathbf{e}$. The McGill SMG is composed of eight single degree-of-freedom kinematic pairs, arranged in a RIIIRRIIIR kinematic chain,

and eight rigid links, one of them being the stationary base (Morozov and Angeles, 2007). As a result, in the corresponding constraint wrench analysis, 42 equations are

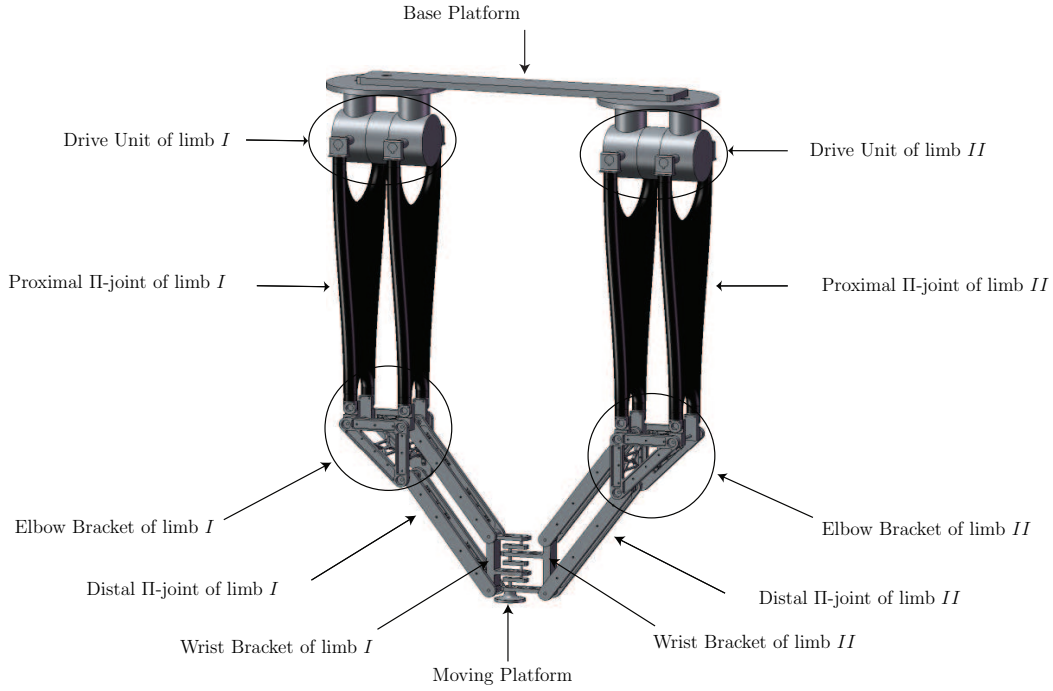


FIGURE 2.10. The McGill SMG

derived from seven moving rigid bodies, with 44 unknowns, 40 constraint forces and moments imposed by the kinematic pairs, and four actuated torques. As a result, the McGill SMG is an overconstrained linkage. In such a linkage the special geometry of the system renders some of the constraints idle; however, their full wrenches still appear in vector λ . In reality, links are flexible, and thus, the shortage of equations can be accommodated by taking link elasticity into account. Blajer et al. (1994), Bayo and Ledesma (1996), and Zahariev and Cuadrado (2007) studied the dynamics of overconstrained flexible multibody systems. While link flexibility makes a robot model more accurate, it makes the problem much more complicated. Apart from this, at the embodiment design stage (Pahl et al., 2007), the rigid link model is preferred. However, due to the lack of equations, the unknown wrench components cannot be computed. One way to cope with this problem is to analyze, instead of the actual case, an *isoconstrained* version of the same mechanism. As a matter of fact, an

isoconstrained version of the linkage at hand also allows for assembly, which, by virtue of the unavoidable manufacturing errors, is impossible in an overconstrained linkage. This version is a kinematic chain with the same mobility, but with non-redundant constraints. Isoconstrained versions of the McGill SMG are introduced below.

2.3.1 The Isoconstrained Versions of the McGill SMG. The McGill SMG is made of two limbs, each carrying a RIIIR kinematic chain that produces Schönflies displacements $\chi(\mathbf{e}_k)$, i.e., three translations and one rotation about an axis parallel to the unit vector $\mathbf{e}_k$. This implies that no limb can undergo rotations about axes normal to $\mathbf{e}_k$. As a result, the rotations of the MP about horizontal axes are prevented by the two limbs. Hence, by eliminating the redundant constraints imposed on the MP by the two limbs, isoconstrained versions of the McGill SMG are obtained. Constraint-elimination is possible via two approaches: 1) relaxing the two redundant constraints from one limb; and 2) relaxing one of the redundant constraint from each limb. For the first approach, a possibility is to replace one of the two R joints attached to the MP with a S joint, as shown in Fig. 2.11(a). The second solution can be implemented by adding two R joints, one to each limb, with a horizontal axis of rotation intersecting its counterpart of vertical axis of rotation. One of the possibilities is to replace the R joints of the MP with U joints, with one axis parallel to $\mathbf{e}_k$, the other horizontal, as depicted in Fig. 2.11(b). After conducting simulation runs

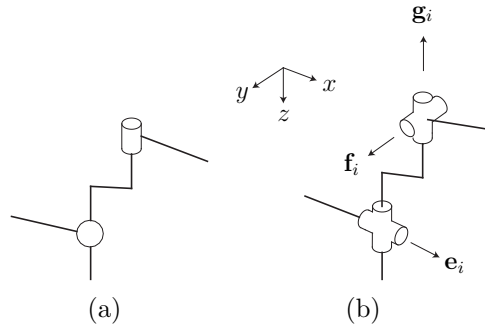


FIGURE 2.11. Two alternative versions for coupling of the MP to each limb: (a) S-R; and (b) U-U

for both versions, it was revealed that the latter brought about singularities inside

the usable workspace, which is undesirable. Hence, the constraint-wrench analysis is conducted on the former, i.e, the one with one S joint replacing one R joint attached to the MP. In the sequel, the constraint and the screw matrices of the chosen type and the actuator wrench-shaping matrix of the McGill SMG will be derived. For the sake of brevity, the angular velocity dyad, the twist-shaping matrix of the foregoing isoconstrained version are included in Appendices A and B.

2.3.2 Derivation of the Constraint Matrix. The isoconstrained McGill SMG, described in Subsection 2.3.1, with the version depicted in Fig. 2.11(a), is made of eight kinematic pairs with a kinematic chain of the RIIISRRIIR type. These eight joints constrain the motion of the seven moving rigid links. Thus, eight systems of kinematic-constraint equations are imposed on the robot, namely,

$$\begin{aligned} \mathbf{K}_{11}\mathbf{t}_1 &= \mathbf{0} \\ \mathbf{K}_{i,i-1}\mathbf{t}_{i-1} + \mathbf{K}_{ii}\mathbf{t}_i &= \mathbf{0}, \quad i = 2, \dots, 7 \\ \mathbf{K}_{87}\mathbf{t}_7 &= \mathbf{0} \end{aligned} \tag{2.59}$$

where

$$\begin{aligned} \mathbf{K}_{11} &= \begin{bmatrix} \mathbf{E}_{I1} & \mathbf{O} \\ \mathbf{R}_{I1} & \mathbf{1} \end{bmatrix}, \quad \mathbf{K}_{21} = \begin{bmatrix} -\mathbf{1} & \mathbf{O} \\ \mathbf{L}_{IP}\mathbf{D}_{I1} & -\mathbf{L}_{IP} \end{bmatrix}, \quad \mathbf{K}_{22} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{L}_{IP}\mathbf{R}_{I2} & \mathbf{L}_{IP} \end{bmatrix} \\ \mathbf{K}_{32} &= \begin{bmatrix} -\mathbf{1} & \mathbf{O} \\ \mathbf{L}_{ID}\mathbf{D}_{I2} & -\mathbf{L}_{ID} \end{bmatrix}, \quad \mathbf{K}_{33} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{L}_{I3}\mathbf{R}_{I3} & \mathbf{L}_{I3} \end{bmatrix}, \quad \mathbf{K}_{43} = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{D}_{I3} & -\mathbf{1} \end{bmatrix} \\ \mathbf{K}_{44} &= \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{R}_M & \mathbf{1} \end{bmatrix}, \quad \mathbf{K}_{54} = \begin{bmatrix} -\mathbf{E}_{II4} & \mathbf{O} \\ \mathbf{D}_M & -\mathbf{1} \end{bmatrix}, \quad \mathbf{K}_{55} = \begin{bmatrix} \mathbf{E}_{II4} & \mathbf{O} \\ \mathbf{R}_{II3} & \mathbf{1} \end{bmatrix} \\ \mathbf{K}_{65} &= \begin{bmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{L}_{IID}\mathbf{D}_{II3} & -\mathbf{L}_{IID} \end{bmatrix}, \quad \mathbf{K}_{66} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{L}_{IID}\mathbf{R}_{II2} & \mathbf{L}_{IID} \end{bmatrix}, \quad \mathbf{K}_{76} = \begin{bmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{L}_{IIP}\mathbf{D}_{II2} & -\mathbf{L}_{IIP} \end{bmatrix} \\ \mathbf{K}_{77} &= \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{L}_{IIP}\mathbf{R}_{II1} & \mathbf{L}_{IIP} \end{bmatrix}, \quad \mathbf{K}_{87} = \begin{bmatrix} -\mathbf{E}_{II1} & \mathbf{0} \\ \mathbf{D}_{II1} & -\mathbf{1} \end{bmatrix} \end{aligned} \tag{2.60a}$$

$$\begin{aligned}
\mathbf{E}_{I1} &= \text{CPM}(\mathbf{e}_{I1}), \quad \mathbf{R}_{I1} = \text{CPM}(\boldsymbol{\rho}_{I1}), \quad \mathbf{L}_{PI} = \text{CPM}(\mathbf{e}_{I2} \times \mathbf{u}_{IP}) \\
\mathbf{D}_{I1} &= \text{CPM}(\boldsymbol{\gamma}_{I1} + \mathbf{u}_{IP}), \quad \mathbf{R}_{I2} = \text{CPM}(\boldsymbol{\rho}_{I2}) \\
\mathbf{L}_{DI} &= \text{CPM}(\mathbf{e}_{I3} \times \mathbf{u}_{ID}), \quad \mathbf{D}_{I3} = \text{CPM}(\boldsymbol{\gamma}_{I3} + \mathbf{u}_{ID}), \quad \mathbf{R}_{I3} = \text{CPM}(\boldsymbol{\rho}_{I3}) \\
\mathbf{E}_{I4} &= \text{CPM}(\mathbf{e}_{I4}), \quad \mathbf{D}_M = \text{CPM}(\boldsymbol{\gamma}_M), \quad \mathbf{R}_M = \text{CPM}(\boldsymbol{\rho}_M) \\
\mathbf{E}_{II4} &= \text{CPM}(\mathbf{e}_{II4}), \quad \mathbf{D}_{I3} = \text{CPM}(\boldsymbol{\gamma}_{I3}) \\
\mathbf{L}_{DII} &= \text{CPM}(\mathbf{e}_{II3} \times \mathbf{u}_{IID}), \quad \mathbf{D}_{II3} = \text{CPM}(\boldsymbol{\gamma}_{II3} + \mathbf{u}_{IID}), \quad \mathbf{R}_{II3} = \text{CPM}(\boldsymbol{\rho}_{II3}) \\
\mathbf{L}_{IIP} &= \text{CPM}(\mathbf{e}_{II2} \times \mathbf{u}_{IIP}), \quad \mathbf{R}_{II1} = \text{CPM}(\boldsymbol{\rho}_{II1}), \quad \mathbf{D}_{II2} = \text{CPM}(\boldsymbol{\gamma}_{II2} + \mathbf{u}_{IIP}) \\
\mathbf{E}_{II1} &= \text{CPM}(\mathbf{e}_{II1}), \quad \mathbf{D}_{II1} = \text{CPM}(\boldsymbol{\gamma}_{II1}), \quad \mathbf{R}_{II2} = \text{CPM}(\boldsymbol{\rho}_{II2})
\end{aligned} \tag{2.60b}$$

The kinematic-constraint equations (2.59) can be cast in the form of eq. (2.2), if

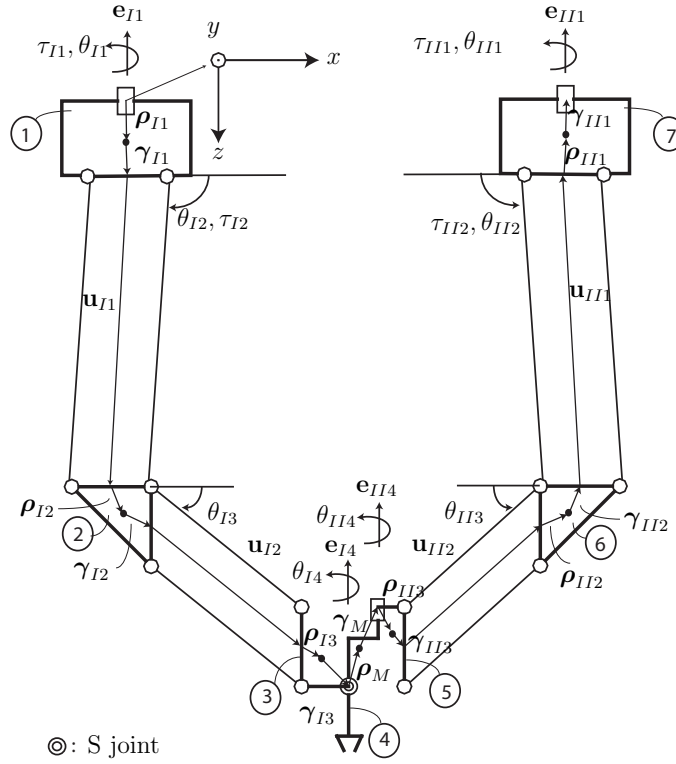


FIGURE 2.12. The dynamic model of the McGill SMG

with the *constraint matrix* $\mathbf{K}$ of 48×42 and $\mathbf{t}$ becoming the 42-dimensional vector

displayed below:

$$\mathbf{t} = \begin{bmatrix} \mathbf{t}_1^T & \mathbf{t}_2^T & \mathbf{t}_3^T & \mathbf{t}_4^T & \mathbf{t}_5^T & \mathbf{t}_6^T & \mathbf{t}_7^T \end{bmatrix}^T \quad (2.61)$$

2.3.3 Derivation of the Screw Matrix. The isoconstrained version chosen bears eight joints, of which seven are single-dof and one is of three-dof, thereby adding 10 constraints, which can be expressed in compact form as

$$\mathbf{S}^T \boldsymbol{\lambda} = \mathbf{0}_{10} \quad (2.62)$$

where $\mathbf{S}$ is the block-diagonal screw matrix of the isoconstrained robot containing the screw arrays of all kinematic pairs, namely,

$$\mathbf{S} = \text{diag}(\mathbf{s}_R^I, \mathbf{s}_{IIP}^I, \mathbf{s}_{IID}^I, \mathbf{S}_S^I, \mathbf{s}_{RM}^{II}, \mathbf{s}_{IID}^{II}, \mathbf{s}_{IIP}^{II}, \mathbf{s}_R^{II}) \quad (2.63)$$

where

$$\begin{aligned} \mathbf{s}_R^J &= \begin{bmatrix} \mathbf{e}_{J1} \\ \mathbf{0}_3 \end{bmatrix}, \quad \mathbf{s}_{IIP}^J = \frac{1}{\|\mathbf{u}_{JP}\|} \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{e}_{J2} \times \mathbf{u}_{JP} \end{bmatrix}, \quad \mathbf{s}_{IID}^J = \frac{1}{\|\mathbf{u}_{JD}\|} \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{e}_{J3} \times \mathbf{u}_{JD} \end{bmatrix}, \quad J = I, II \\ \mathbf{S}_S^I &= \begin{bmatrix} \mathbf{e}_{I3} & \mathbf{e}_{I2} & \mathbf{e}_{I1} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}, \quad \mathbf{s}_{RM}^{II} = \begin{bmatrix} \mathbf{e}_{J1} \\ \mathbf{0}_3 \end{bmatrix} \end{aligned} \quad (2.64)$$

with I and II standing for the limb labels.

2.3.4 Derivation of the Actuator Wrench-Shaping Matrix. Each actuator produces an actuated wrench on the rigid links to which it is attached. From eq. (2.55) the actuator wrench-shaping matrix maps actuator-applied forces or moments into wrenches applied on the rigid links of the system. The McGill SMG has four active joints, two R and two II. The active R joints of angles θ_{I1} and θ_{II1} apply moments τ_{I1} and τ_{II1} , respectively. These moments induce applied wrenches on links 1 and 7, with the moments depicted in Fig. 2.12, namely:

$$\mathbf{w}_1^A = \mathbf{s}_R^I \tau_{I1}, \quad \mathbf{w}_7^A = \mathbf{s}_R^{II} \tau_{II1} \quad (2.65)$$

The derivation of the wrenches induced by the active Π -joints is not as straightforward as those induced by the R active joints. In an active Π joint of the robot under study, two of the four R joints are active. A torque produced by an active R joint, driving the first rigid link, is transmitted to the next rigid link by means of parallel links, thereby inducing an active wrench on it. In the McGill SMG, each limb has an active proximal Π -joint. That of the J th limb is actuated by the two top R joints by means of bevel gears. Therefore, the active wrenches induced on the centre of mass of each elbow bracket (EB), the second and the sixth rigid links in the closed loop, can be obtained as

$$\begin{aligned} \mathbf{w}_2^A &= \frac{1}{2l_2^2} \begin{bmatrix} \mathbf{1} & -(\mathbf{A}_I + \mathbf{B}_I) \\ \mathbf{O} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_{I2} \times \mathbf{u}_{IP} \end{bmatrix} \tau_{I2} \equiv \mathbf{n}_I \mathbf{s}_{\Pi P}^I \tau_{I2} \\ \mathbf{w}_6^A &= \frac{1}{2l_2^2} \begin{bmatrix} \mathbf{1} & -(\mathbf{A}_{II} + \mathbf{B}_{II}) \\ \mathbf{O} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_{II2} \times \mathbf{u}_{IP} \end{bmatrix} \tau_{II2} \equiv \mathbf{n}_{II} \mathbf{s}_{\Pi P}^{II} \tau_{II2} \end{aligned} \quad (2.66)$$

Moreover, $\mathbf{A}_J$ and $\mathbf{B}_J$ are the CPMs of vectors $\mathbf{a}_J$ and $\mathbf{b}_J$, respectively, as depicted in Fig. 2.13. As no actuator wrenches are applied on bodies number 3, 4, and 5 one

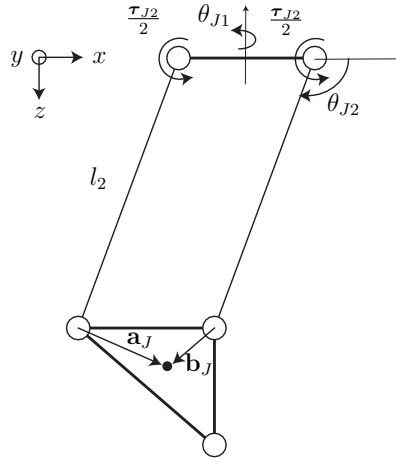


FIGURE 2.13. An active proximal Π joint

has

$$\mathbf{w}_3^A = \mathbf{w}_4^A = \mathbf{w}_5^A = \mathbf{0}_6 \quad (2.67)$$

The actuator torques and the applied wrench arrays are defined below:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{I1} & \tau_{I2} & \tau_{II1} & \tau_{II2} \end{bmatrix}^T$$

$$\mathbf{w}^A = \begin{bmatrix} (\mathbf{w}_1^A)^T & (\mathbf{w}_2^A)^T & (\mathbf{w}_3^A)^T & (\mathbf{w}_4^A)^T & (\mathbf{w}_5^A)^T & (\mathbf{w}_6^A)^T & (\mathbf{w}_7^A)^T \end{bmatrix}^T$$

Hence, by resorting to eq. (2.55) and the actuator wrenches derived above, the actuator wrench-shaping matrix of the McGill SMG is obtained as

$$\mathbf{T}^A = \begin{bmatrix} \mathbf{s}_R^I & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{n}_I \mathbf{s}_{IIP}^I & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_{18} & \mathbf{0}_{18} & \mathbf{0}_{18} & \mathbf{0}_{18} \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{n}_{II} \mathbf{s}_{IIP}^{II} \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{s}_R^{II} & \mathbf{0}_6 \end{bmatrix} \quad (2.68)$$

in which $\mathbf{0}_6$ and $\mathbf{0}_{18}$ are the six- and the 18-dimensional zero vectors, respectively.

2.4 The Test Trajectory

The SMGs are typically in use for pick-and-place operations in industry. For this type of motions, a trajectory has been adopted by industry in order to compare the efficiency of different systems on the market (Berthod, 1989). The standard test trajectory involves a 25 mm vertical upward translation, a 300 mm horizontal translation while rotating 180°, and a 25 mm downward vertical translation of its MP, after which the MP returns to its original pose by following the same trajectory backwards. Gauthier et al. (2008) proposed the smoothed test trajectory depicted in Fig. 2.14. As a pick-and-place operation also involves time, the whole task should be

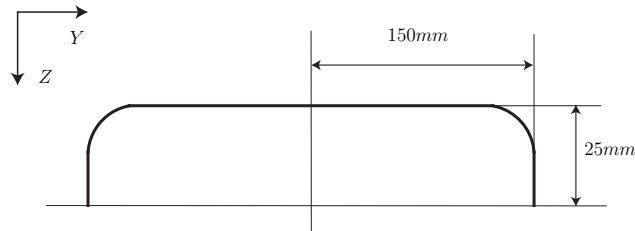


FIGURE 2.14. The smoothed test trajectory obtained in Gauthier et al. (2008)

defined as three time-dependent functions that describe the translations along the Y - and Z -axes and the rotation about the Z -axis, namely,

$$y = y(t), \quad z = z(t), \quad \phi = \phi(t) \quad (2.69)$$

These three functions, as defined by Gauthier et al. (2008) are depicted in Fig 2.15. According to this motion, each pick-and-place operation cycle takes 500 ms, which

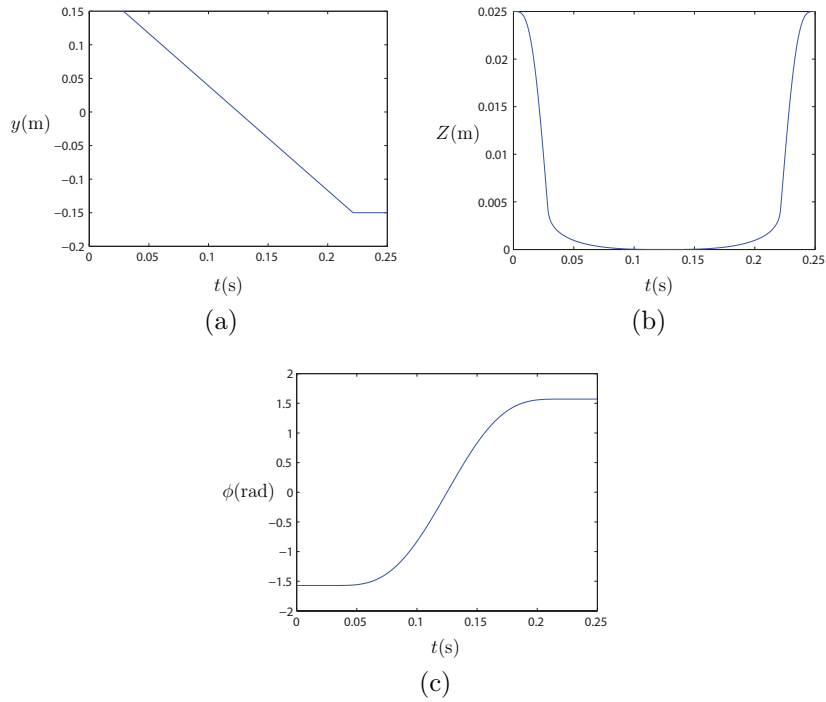


FIGURE 2.15. The trajectory plots: (a) the translation along the Y -axis, (b) the translation along the Z -axis, and (c) the rotation about the Z -axis

implies that the task frequency is 2 Hz. It is noteworthy that all three plots of Fig. 2.15 are C^2 -continuous. The slope discontinuity apparent in Figs. 2.15(a), (b) and (c) is a result of the low resolution of the plot. The corresponding velocities and acceleration profiles of the MP are also shown in Fig. 2.16.

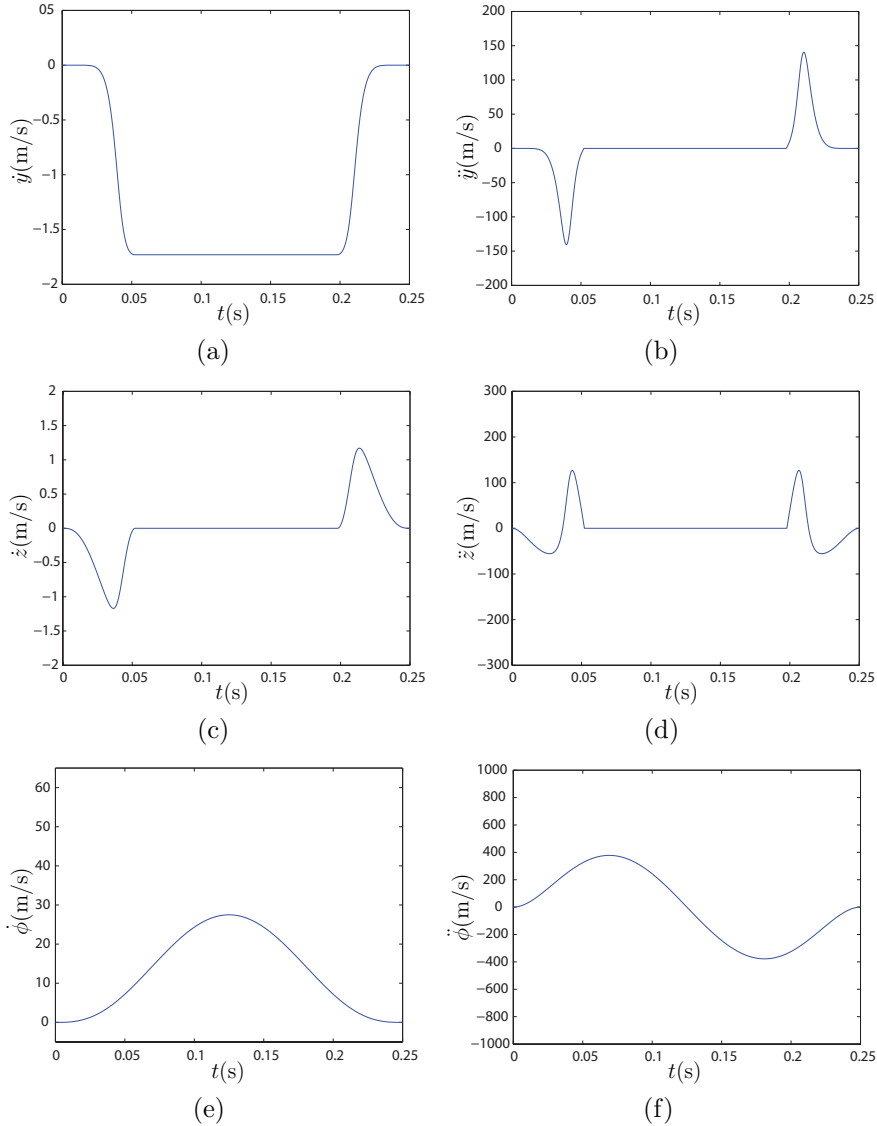


FIGURE 2.16. The velocity and acceleration time-histories of the MP: (a) $\dot{y}$; (b) $\ddot{y}$; (c) $\dot{z}$; (d) $\ddot{z}$; (e) $\dot{\phi}$; and (f) $\ddot{\phi}$

2.5 Numerical Results

Now we let the McGill SMG perform a pick-and-place operation over the smoothed test trajectory. The constraint-wrench analysis of the isoconstrained McGill SMG can be conducted by using the procedure described in Section 2.2. As a result, the constraint wrenches imposed on the kinematic pairs of the robot are obtained along with the required actuator torques. The latter are plotted in Fig. 2.17, while

the evolution of the constraint forces and moments through the test trajectory are plotted in Figs. 2.18–2.25. As depicted in Figs. 2.16, over the smoothed test trajectory and between 0.05 s and 0.2 s, the MP undergoes zero acceleration along the Y - and Z -axes, while undergoing an angular acceleration about the Z -axis; hence, most of the constraint forces and moments vanish during this time interval. From the Figs. 2.18–2.25, it is also apparent that the results are not symmetric through the test trajectory. There are three main reasons that can explain this asymmetric behavior: 1) the isoconstrained kinematic chain RIIISRIIIIR is not symmetric; 2) the velocity and acceleration profiles of the smoothed test cycle in the Y direction, depicted in Figs. 2.16(a) and (c), and the angular acceleration profile $\ddot{\phi}$, shown in Fig. 2.16(f), are quasi-symmetric; and 3) the geometry of the current MP is asymmetric.

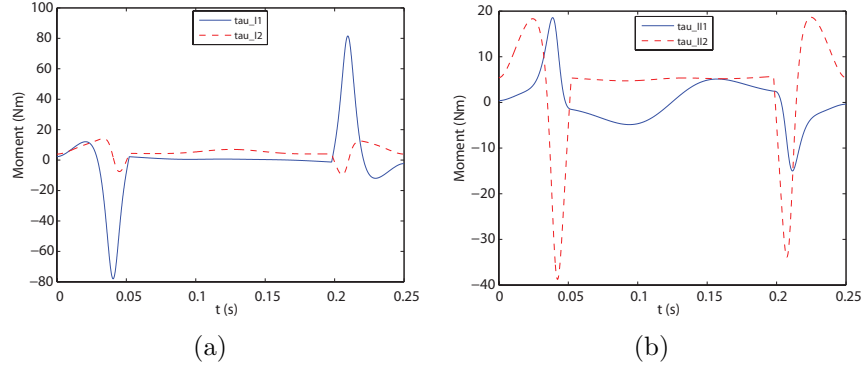


FIGURE 2.17. The time-histories of the actuator torques: (a) τ_{I1} and τ_{I2} (N.m) and (b) τ_{II1} and τ_{II2} (N.m)

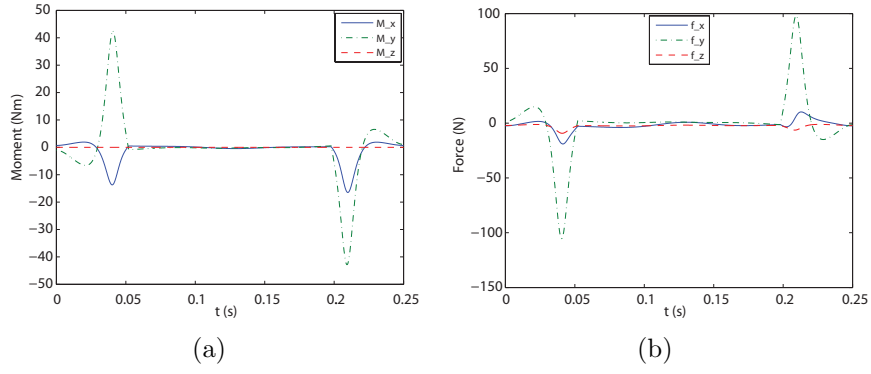


FIGURE 2.18. The evolution of the constraint forces and moments at the first revolute joints of limb *I* along the test trajectory: (a) M_x , M_y , and M_z (Nm) and (b) f_x , f_y , and f_z (N)

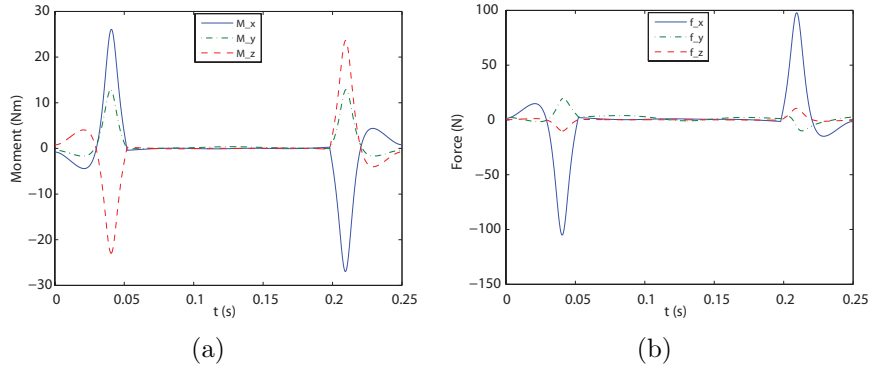


FIGURE 2.19. The evolution of the constraint forces and moments at the proximal II joint of limb *I* along the test trajectory: (a) M_x , M_y , and M_z (Nm) and (b) f_x , f_y , and f_z (N)

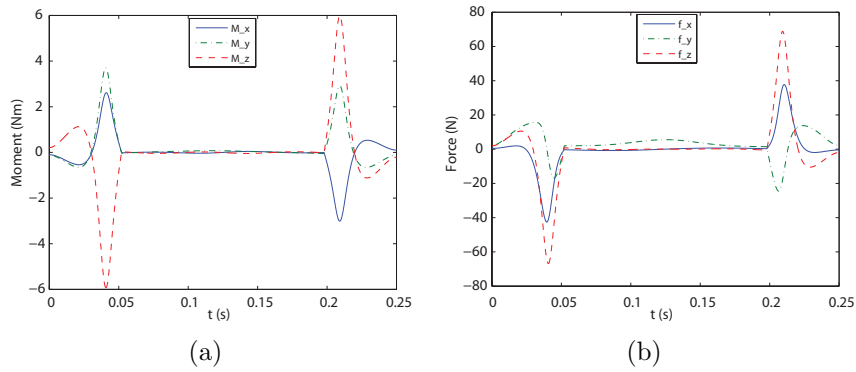


FIGURE 2.20. The evolution of the constraint forces and moments at the distal II joint of limb *I* along the test trajectory: (a) M_x , M_y , and M_z (Nm) and (b) f_x , f_y , and f_z (N)

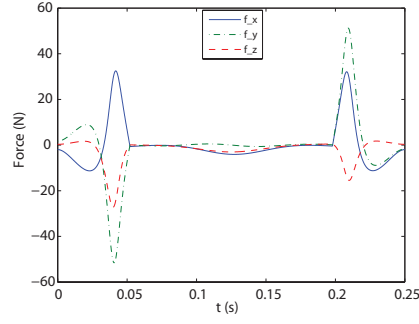


FIGURE 2.21. The evolution of the constraint forces at the spherical joint of limb *I* over the test trajectory

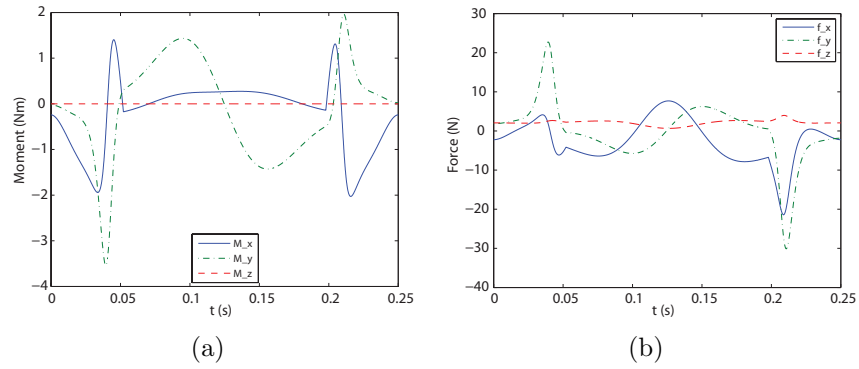


FIGURE 2.22. The evolution of the constraint forces and moments at the first revolute joints of the limb *II* along the test trajectory (a) M_x , M_y , and M_z (Nm) and (b) f_x , f_y , and f_z (N)

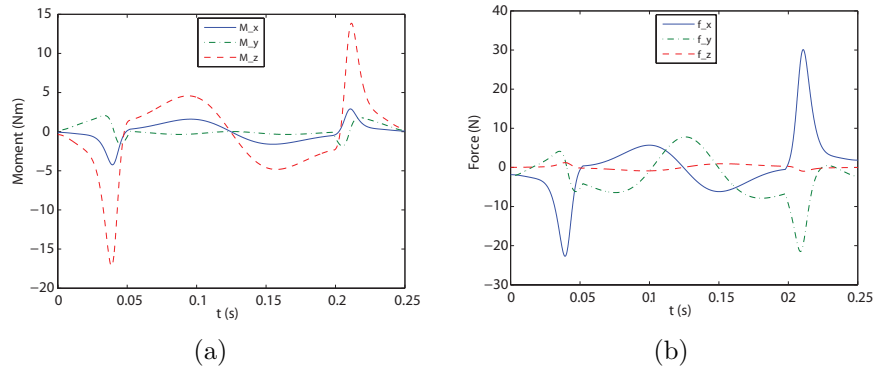


FIGURE 2.23. The evolution of the constraint forces and moments at the proximal *II* joint of limb *II* along the test trajectory: (a) M_x , M_y , and M_z (N.m) and (b) f_x , f_y , and f_z (N)

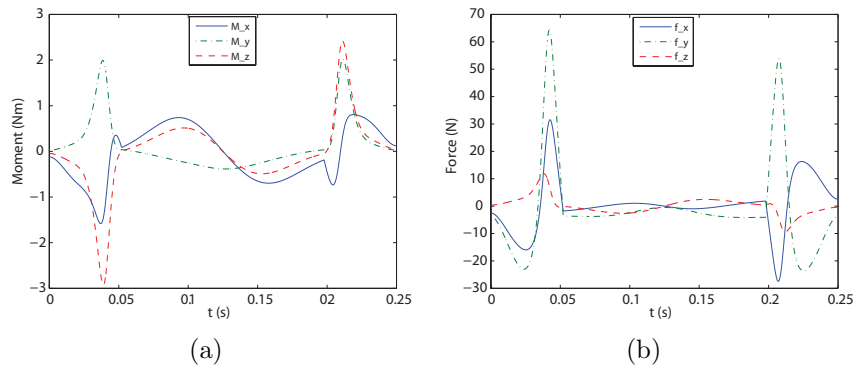


FIGURE 2.24. The evolution of the constraint forces and moments at the distal II joint of limb II along the test trajectory: (a) M_x , M_y , and M_z (Nm) and (b) f_x , f_y , and f_z (N)

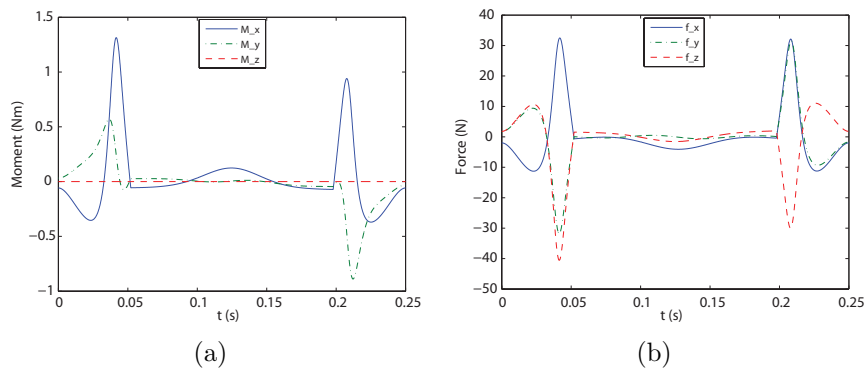


FIGURE 2.25. The evolution of the constraint forces and moments at the second revolute joints of limb II along the test trajectory: (a) M_x , M_y , and M_z (Nm) and (b) f_x , f_y , and f_z (N)

CHAPTER 3

Elastostatic Analysis of Multibody Systems

The subject of this chapter is the elastostatic analysis of robot structures. The method of analysis is based on the concept of *generalized spring*, in which each flexible link is replaced with a six-dimensional generalized spring. The stiffness matrix of this spring, when the link complexity so requires, should be obtained numerically, via FEA. In this study, a novel formulation for the modelling of all six LKPs, when connecting two flexible links, is introduced. By resorting to this formulation, a compact formulation for the stiffness matrix of a parallelogram with flexible linkages, what is called a Π joint, is obtained. As an illustrative example, the procedure is applied to the McGill SMG that features Π joints. In order to illustrate the online feasibility of the computations involved, the minimum eigenvalue of a dimensionless factor of the stiffness matrix, used as stiffness performance index, is plotted along the test trajectory.

3.1 The Small-rotation Matrix

The rotation of a rigid body displaced from an initial attitude to a new one can be represented in a variety of forms. In this study the *natural invariants* of the rotation are used; these are the angle of rotation ϕ and the unit vector $\mathbf{e}$ parallel to the axis

of rotation (Angeles, 2007). The rotation matrix is then represented as

$$\mathbf{R} = \mathbf{1} + (\sin \phi) \mathbf{E} + (1 - \cos \phi) \mathbf{E}^2 \quad (3.1)$$

with $\mathbf{E}$ denoting the CPM of $\mathbf{e}$. The angular velocity of the rigid body in terms of $\mathbf{e}$, ϕ and their time-derivatives takes the form (Angeles, 2007)

$$\boldsymbol{\omega} = \begin{bmatrix} (\sin \phi) \mathbf{1} + (1 - \cos \phi) \mathbf{E} & \mathbf{e} \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\phi} \end{bmatrix} \quad (3.2)$$

If ϕ is small, then $\sin \phi \rightarrow \phi$ and $\cos \phi \rightarrow 1$, the rotation matrix $\mathbf{R}$ thus becoming

$$\mathbf{R} = \mathbf{1} + \phi \mathbf{E} \quad (3.3)$$

Under the small-angle assumption, a rigid-body rotation admits a vector represen-

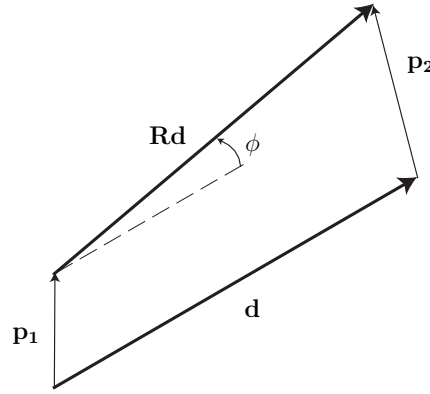


FIGURE 3.1. A vector attached to a rigid body after and before translation and small rotation

tation, namely,

$$\boldsymbol{\phi} \equiv \phi \mathbf{e} \quad (3.4)$$

the angular velocity $\boldsymbol{\omega}$ for small ϕ then reducing to

$$\boldsymbol{\omega} = \begin{bmatrix} \phi \mathbf{1} & \mathbf{e} \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\phi} \end{bmatrix}$$

i.e.,

$$\boldsymbol{\omega} = \dot{\boldsymbol{\phi}} = \phi \dot{\mathbf{e}} + \dot{\phi} \mathbf{e}$$

In this particular case, the angular-velocity vector becomes a time-derivative. In Fig. 3.1 a vector $\mathbf{d}$ attached to a rigid body is shown before and after the body undergoes a SAD screw. The algebraic relations among the vectors of Fig. 3.1 lead to

$$\mathbf{p}_2 = \mathbf{p}_1 + (\mathbf{R} - \mathbf{1}) \mathbf{d} \quad (3.5)$$

From eq. (3.3), after some algebraic manipulations, the above expression for $\mathbf{p}_2$ leads to

$$\mathbf{p}_2 = \mathbf{p}_1 - \mathbf{D}\boldsymbol{\phi} \quad (3.6)$$

in which $\mathbf{D}$ is the CPM of $\mathbf{d}$. A unit screw is a line $\mathcal{L}$ with a pitch p . If the Plücker coordinates (Roth, 1984) of the line are $\begin{bmatrix} \mathbf{e}^T & (\mathbf{r} \times \mathbf{e})^T \end{bmatrix}^T$, with $\mathbf{e}$ denoting the unit vector parallel to $\mathcal{L}$, and $\mathbf{r}$ the position vector of a point R on $\mathcal{L}$, then, the *unit screw* $\mathbf{s}$ is given by

$$\mathbf{s} = \begin{bmatrix} \mathbf{e} \\ \mathbf{r} \times \mathbf{e} + p\mathbf{e} \end{bmatrix} \quad (3.7)$$

The six-dimensional SAD screw $\mathbf{u}_i$, for $i = 1, 2$, is defined as a unit screw multiplied by the “small”-amplitude angle of rotation ϕ , namely,

$$\mathbf{u}_i = \begin{bmatrix} \phi \mathbf{e}_i \\ \mathbf{r}_i \times \phi \mathbf{e}_i + p_i \phi \mathbf{e}_i \end{bmatrix} = \begin{bmatrix} \phi \\ \mathbf{p}_i \end{bmatrix} \quad (3.8)$$

Finally, by resorting to eq. (3.6), the six-dimensional SAD screws of a rigid body, referred to two distinct points P_1 and P_2 , of the body, of position vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, take the form

$$\begin{bmatrix} \phi \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{D} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \phi \\ \mathbf{p}_1 \end{bmatrix} \quad (3.9)$$

where $\mathbf{O}$ is the 3×3 zero matrix. Hence, the relation between the two screws can be expressed as

$$\mathbf{u}_2 = \mathbf{W}\mathbf{u}_1, \quad \mathbf{W} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{D} & \mathbf{1} \end{bmatrix} \quad (3.10)$$

which is the SAD version of the twist-transfer formula introduced in (Angeles, 2007).

3.2 Generalized Spring

In multibody systems some flexible links are rigidly connected to each other or to other much stiffer bodies that can be safely assumed rigid. The main source of potential energy is the elastic energy stored in the former. Each of these can be considered a generalized massless spring, as shown in Fig. 3.2. Such a spring has two

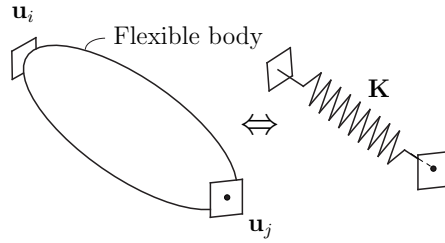


FIGURE 3.2. A generalized spring

free ends, considered as rigid plates, which undergo SAD screws $\mathbf{u}_i$ and $\mathbf{u}_j$. A force $\mathbf{f}_i$ and a moment $\mathbf{n}_i$, grouped in the six-dimensional wrench array $\mathbf{w}_i$, can be applied on the i th rigid plate. The wrench arrays $\mathbf{w}_i$ and $\mathbf{w}_j$, acting on the rigid plates are now arrayed in a 12-dimensional vector $\bar{\mathbf{w}} = \begin{bmatrix} \mathbf{w}_i^T & \mathbf{w}_j^T \end{bmatrix}^T$; by the same token the SAD screws $\mathbf{u}_i$ and $\mathbf{u}_j$ can be cast in a 12-dimensional array, namely, $\bar{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_i^T & \mathbf{u}_j^T \end{bmatrix}^T$. The 12-dimensional wrench array $\bar{\mathbf{w}}$ is the image of the 12-dimensional SAD screw array $\bar{\mathbf{u}}$ under a transformation represented by the 12×12 stiffness matrix $\mathbf{K}$, as partitioned below in 6×6 blocks:

$$\bar{\mathbf{w}} = \mathbf{K}\bar{\mathbf{u}}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ \mathbf{K}_{ij}^T & \mathbf{K}_{jj} \end{bmatrix} \quad (3.11)$$

For example, for the flexible beam depicted in Fig. 3.3, of length L , cross-section area A , cross-section moments of inertia I_y and I_z , cross-section torsional constant J ,

isotropic and homogeneous, with Young modulus E and shear modulus G , the blocks of the stiffness matrix are:

$$\mathbf{K}_{ii} = \frac{E}{L} \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & 12I_z/L^2 & 0 & 0 & 0 & -6I_z/L^2 \\ 0 & 0 & 12I_y/L^2 & 0 & 6I_y/L^2 & 0 \\ 0 & 0 & 0 & JG/E & 0 & 0 \\ 0 & 0 & 6I_y/L^2 & 0 & 4I_y & 0 \\ 0 & -6I_z/L^2 & 0 & 0 & 0 & 4I_z \end{bmatrix} \quad (3.12)$$

$$\mathbf{K}_{ij} = \frac{E}{L} \begin{bmatrix} -A & 0 & 0 & 0 & 0 & 0 \\ 0 & -12I_z/L^2 & 0 & 0 & 0 & -6I_z/L \\ 0 & 0 & -12I_y/L^2 & 0 & 6I_y/L & 0 \\ 0 & 0 & 0 & -JG/E & 0 & 0 \\ 0 & 0 & -6I_y/L & 0 & 4I_y & 0 \\ 0 & 6I_z/L & 0 & 0 & 0 & 4I_z \end{bmatrix} \quad (3.13)$$

$$\mathbf{K}_{jj} = \frac{E}{L} \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & 12I_z/L^2 & 0 & 0 & 0 & 6I_z/L \\ 0 & 0 & 12I_y/L^2 & 0 & -6I_y/L & 0 \\ 0 & 0 & 0 & JG/E & 0 & 0 \\ 0 & 0 & -6I_y/L & 0 & 4I_y & 0 \\ 0 & 6I_z/L & 0 & 0 & 0 & 4I_z \end{bmatrix} \quad (3.14)$$

The potential energy of a generalized spring, i.e., the elastic energy stored in the spring, is computed as

$$V = \frac{1}{2} \begin{bmatrix} \mathbf{u}_i^T & \mathbf{u}_j^T \end{bmatrix} \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ \mathbf{K}_{ij}^T & \mathbf{K}_{jj} \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} \quad (3.15)$$

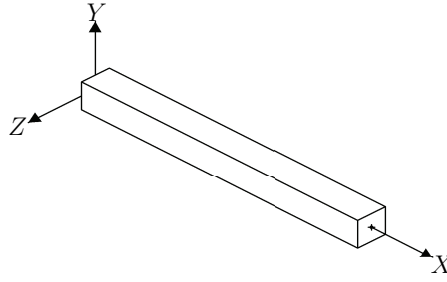


FIGURE 3.3. An elastic beam as a generalized spring

3.3 The Stiffness Matrix of an Articulated Flexible Body

When a passive kinematic pair couples two links, it applies constraint wrenches on the coupled links. Each end-plate of a flexible link comprises six dof, which results in 12 elastic dof brought about by the two links at the connecting point. There are thus 12 elastic displacements, at least six of them independent, which produce potential energy. If a kinematic pair applies m constraints, then the number of independent elastic dof at the connecting point is $12 - m$. For example, as shown in Fig. 3.4, when a revolute joint couples the two rigid end-plates of the flexible links, it constrains five elastic dof of the coupled plates, two rotations and three translations; therefore, seven independent elastic dof contribute to the potential energy of the mechanical system at hand. These dof are the six independent displacements of the two bodies as one single rigid body and the relative rotation allowed by the R joint. Hence, If the SAD

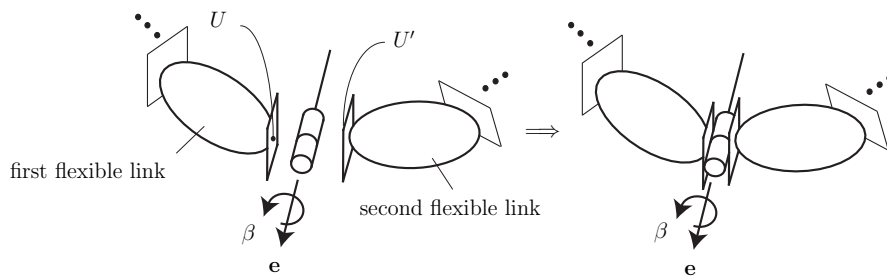


FIGURE 3.4. An articulated flexible link in a kinematic chain

screw at point U of the end-plate of the first link is denoted by $\mathbf{u}$, the SAD screw at

point U' of the second end-plate is defined as

$$\mathbf{u}' = \mathbf{u} - [(\mathbf{s}^T \mathbf{u})\mathbf{s} - \beta\mathbf{s}] \quad (3.16)$$

in which $\mathbf{s}$ is the unit screw array of the revolute joint and β is a relative rotation due to the presence of the joint. This representation can be extended to all six LKPs. Figure 3.5 shows a flexible link in a kinematic chain which is connected to other links with a LKP of dof equals $6 - m$, imposing m constraints on the coupling; if the SAD screw at the second end point U_i of the $(l - 1)$ st link is denoted by $\mathbf{u}_i$, the SAD screw at the first end point of the l th link U'_i can be computed as

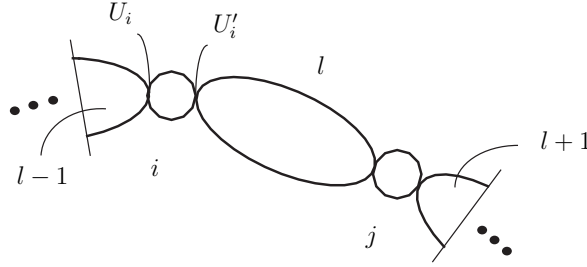


FIGURE 3.5. An articulated flexible link in a kinematic chain

$$\mathbf{u}'_i = \mathbf{u}_i - \sum_{k=1}^{6-m} [(\mathbf{s}_k^T \mathbf{u}_i)\mathbf{s}_k - \beta_k \mathbf{s}_k] \quad (3.17)$$

in which $\mathbf{s}_k$ is the unit screw array of the k th dof of the kinematic pair and β_k is an elastic dof due to the presence of this dof in the kinematic pair. Here, a unit screw is defined for the line of action of the k th kinematic pair. For the six different LKPs, shown in Fig. 3.6, $\mathbf{s}_k$ and β_k are listed in Table 3.1. In the foregoing elastostatic modelling, the multi-dof kinematic pairs such as C, F, and S joints are modelled as concatenation of single-dof joints; hence, instead of screw matrices $\mathbf{S}_C$, $\mathbf{S}_F$, and $\mathbf{S}_S$ for C, F and S joints, respectively, defined in Ch. 2, the screw arrays are used. By resorting to eq. (3.15), the potential energy produced by the small-amplitude motion

TABLE 3.1. Screw arrays of the six lower kinematic pairs

Kinematic Pair	m	Screw array(s)	Kinematic Pair	m	Screw array(s)
R	5	$\mathbf{s} = [\mathbf{e}^T \quad \mathbf{0}^T]^T$	C	4	$\mathbf{s}_1 = [\mathbf{e}_1^T \quad \mathbf{0}^T]^T$ $\mathbf{s}_2 = [\mathbf{0}^T \quad \mathbf{e}_2^T]^T$
P	5	$\mathbf{s} = [\mathbf{0}^T \quad \mathbf{e}^T]^T$	S	3	$\mathbf{s}_1 = [\mathbf{e}_1^T \quad \mathbf{0}^T]^T$ $\mathbf{s}_2 = [\mathbf{e}_2^T \quad \mathbf{0}^T]^T$ $\mathbf{s}_3 = [\mathbf{e}_3^T \quad \mathbf{0}^T]^T$
H	5	$\mathbf{s} = [\mathbf{e}^T \quad p\mathbf{e}^T]^T$	F	3	$\mathbf{s}_1 = [\mathbf{0}^T \quad \mathbf{e}_1^T]^T$ $\mathbf{s}_2 = [\mathbf{0}^T \quad \mathbf{e}_2^T]^T$ $\mathbf{s}_3 = [\mathbf{e}_3^T \quad \mathbf{0}^T]^T$

of the l th link is

$$V = \frac{1}{2} \left[\mathbf{u}_i^T - \sum_{k=1}^{6-m} ((\mathbf{s}_k^T \mathbf{u}_i) \mathbf{s}_k^T - \beta_k \mathbf{s}_k^T) \quad \mathbf{u}_j^T \right] \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ \mathbf{K}_{ij}^T & \mathbf{K}_{jj} \end{bmatrix} \begin{bmatrix} \mathbf{u}_i - \sum_{k=1}^{6-m} ((\mathbf{s}_k^T \mathbf{u}_i) \mathbf{s}_k - \beta_k \mathbf{s}_k) \\ \mathbf{u}_j \end{bmatrix} \quad (3.18)$$

Expansion of V in eq. (3.18) results in

$$\begin{aligned} V = & \frac{1}{2} (\mathbf{u}_i^T \mathbf{K}_{ii} \mathbf{u}_i + 2\mathbf{u}_i^T \mathbf{K}_{ij} \mathbf{u}_j + \mathbf{u}_j^T \mathbf{K}_{jj} \mathbf{u}_j) + \sum_{k=1}^{6-m} (\mathbf{u}_i^T \mathbf{K}_{ii} \mathbf{s}_k (-\mathbf{s}_k^T \mathbf{u}_i + \beta_k) \\ & + \mathbf{s}_k^T \mathbf{K}_{ij} \mathbf{u}_j (-\mathbf{s}_k^T \mathbf{u}_i + \beta_k)) + \frac{1}{2} \sum_{k=1}^{6-m} ((-\mathbf{s}_k^T \mathbf{u}_i + \beta_k) \mathbf{s}_k^T) \mathbf{K}_{ii} \sum_{k=1}^{6-m} (\mathbf{s}_k (-\mathbf{s}_k^T \mathbf{u}_i + \beta_k)) \end{aligned} \quad (3.19)$$

The gradient of the potential energy of a mechanical system with respect to its generalized coordinates, which equals the generalized forces applied on the system, is

obtained as

$$\begin{aligned}
 \frac{\partial V}{\partial \mathbf{u}_i} &= \mathbf{w}_i = \mathbf{K}_{ii} \mathbf{u}_i + \mathbf{K}_{ij} \mathbf{u}_j + \sum_{k=1}^{6-m} (\mathbf{s}_k \mathbf{s}_k^T) \mathbf{K}_{ii} \sum_{k=1}^{6-m} (\mathbf{s}_k \mathbf{s}_k^T) \mathbf{u}_i - \sum_{k=1}^{6-m} (\mathbf{s}_k \mathbf{s}_k^T) \mathbf{K}_{ii} \sum_{k=1}^{6-m} \mathbf{s}_k \beta_k \\
 &\quad - \sum_{k=1}^{6-m} (\mathbf{K}_{ii} \mathbf{s}_k \mathbf{s}_k^T \mathbf{u}_i + \mathbf{K}_{ii} \mathbf{s}_k \beta_k - \mathbf{s}_k (\mathbf{s}_k^T \mathbf{K}_{ii}) \mathbf{u}_i - \mathbf{s}_k (\mathbf{s}_k^T \mathbf{K}_{ij}) \mathbf{u}_j) \\
 \frac{\partial V}{\partial \mathbf{u}_j} &= \mathbf{w}_j = \mathbf{K}_{ij}^T \mathbf{u}_i + \mathbf{K}_{jj} \mathbf{u}_j + \sum_{k=1}^{6-m} (\mathbf{K}_{ij}^T \mathbf{s}_k \beta_k - \mathbf{K}_{ij}^T \mathbf{s}_k \mathbf{s}_k^T \mathbf{u}_i) \\
 \frac{\partial V}{\partial \beta_n} &= w_{\beta_n} = \sum_{k=1}^{6-m} (\mathbf{s}_k^T \mathbf{K}_{ii} \mathbf{u}_i + \mathbf{s}_k^T \mathbf{K}_{ij} \mathbf{u}_j) + \mathbf{s}_n^T \mathbf{K}_{ii} \mathbf{s}_n \beta_n - \mathbf{s}_n^T \mathbf{K}_{ii} \mathbf{s}_n \mathbf{s}_n^T \mathbf{u}_i \\
 &\quad + \sum_{k=1}^{6-m} \sum_{q=1, q \neq k}^{6-m} \mathbf{s}_k^T \mathbf{K}_{ii} \mathbf{s}_q \beta_k \beta_q, \quad n = 1, \dots, (6-m)
 \end{aligned} \tag{3.20}$$

where the first two partial derivatives, being taken with respect to a SAD screw, result in wrenches. The relations in eq. (3.20) can be cast in the form

$$\mathbf{w}_a = \mathbf{K}_a \mathbf{u}_a \tag{3.21}$$

in which $\mathbf{w}_a$, $\mathbf{K}_a$, and $\mathbf{u}_a$ are the generalized force array, the generalized stiffness matrix, and the generalized displacement array of the articulated flexible link, respectively, whose components are shown below:

$$\mathbf{w}_a = \begin{bmatrix} \mathbf{w}_i \\ \mathbf{w}_j \\ \mathbf{w}_\beta \end{bmatrix}, \quad \mathbf{K}_a = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} & \mathbf{G}_{13} \\ \mathbf{G}_{12}^T & \mathbf{G}_{22} & \mathbf{G}_{23} \\ \mathbf{G}_{13}^T & \mathbf{G}_{23}^T & \mathbf{G}_{33} \end{bmatrix}, \quad \mathbf{u}_a = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \boldsymbol{\beta} \end{bmatrix} \tag{3.22}$$

in which

$$\mathbf{w}_\beta = \begin{bmatrix} w_{\beta_1} & \dots & w_{\beta_n} \end{bmatrix}^T, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 & \dots & \beta_n \end{bmatrix}^T, \quad n = 1, \dots, 6-m \tag{3.23}$$

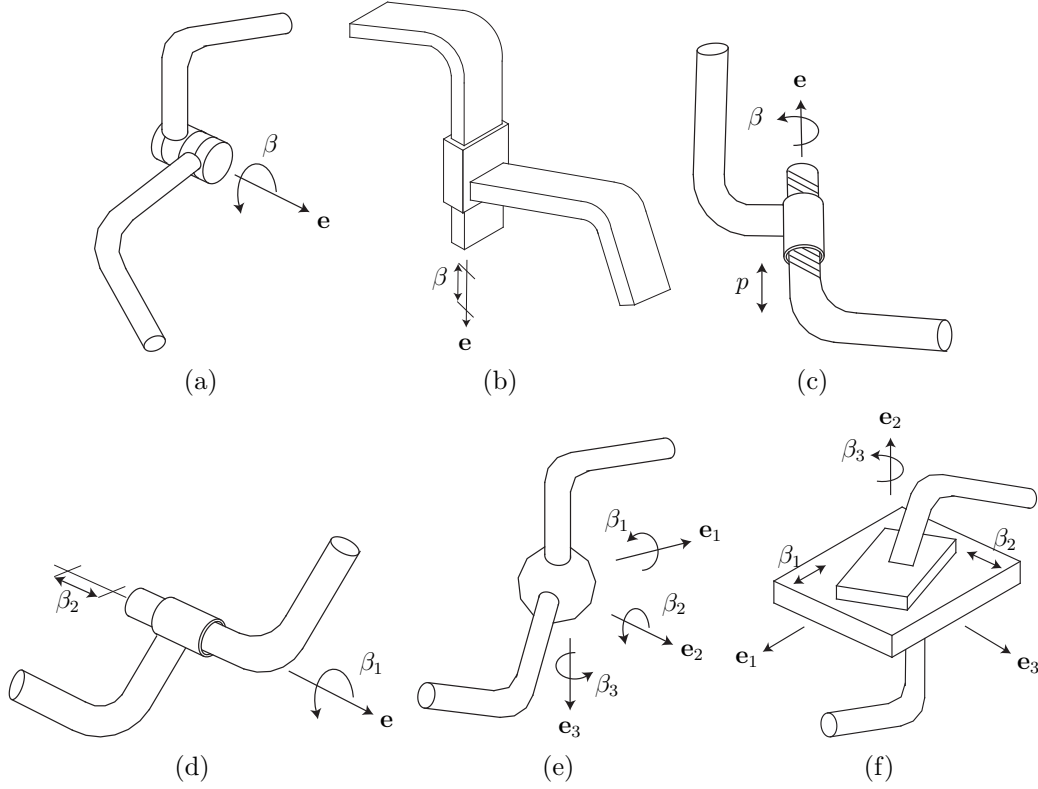


FIGURE 3.6. Six types of kinematic pairs: (a) Revolute (R); (f) Planar (F); (e) Spherical (S); (b) Prismatic (P); (d) Cylindrical (C); and (c) Helical (H)

Matrix $\mathbf{K}_a$ thus maps the $(18 - m)$ -dimensional generalized displacement array of the system into the generalized force array. The blocks of matrix $\mathbf{K}_a$ are displayed below:

$$\begin{aligned}
 \mathbf{G}_{11} &= \mathbf{K}_{ii} + \sum_{k=1}^{6-m} \mathbf{s}_k \mathbf{s}_k^T \mathbf{K}_{ii} \sum_{k=1}^{6-m} \mathbf{s}_k \mathbf{s}_k^T - \sum_{k=1}^{6-m} (\mathbf{K}_{ii} \mathbf{s}_k \mathbf{s}_k^T - \mathbf{s}_k \mathbf{s}_k^T \mathbf{K}_{ii}) \\
 \mathbf{G}_{12} &= \mathbf{K}_{ij} + \sum_{k=1}^{6-m} \mathbf{s}_k (\mathbf{s}_k^T \mathbf{K}_{ij}), \quad \mathbf{G}_{22} = \mathbf{K}_{jj} \\
 \mathbf{G}_{13} &= [\mathbf{g}_1 \quad \cdots \quad \mathbf{g}_n], \quad \mathbf{g}_n = \sum_{k=1}^{6-m} \mathbf{s}_k \mathbf{s}_k^T \mathbf{K}_{ii} \mathbf{s}_n, \quad n = 1, \dots, (6 - m) \\
 \mathbf{G}_{23} &= [\mathbf{g}_1 \quad \cdots \quad \mathbf{g}_n], \quad \mathbf{g}_n = \mathbf{K}_{ij}^T \mathbf{s}_n, \quad n = 1, \dots, (6 - m) \\
 \mathbf{G}_{33} &= \text{diag}(G_1, \dots, G_n), \quad G_n = \mathbf{s}_n^T \mathbf{K}_{ii} \mathbf{s}_n, \quad n = 1, \dots, (6 - m)
 \end{aligned} \tag{3.24}$$

Since the kinematic pair is assumed to be passive, the actuated generalized force exerted by the pair vanishes, i.e,

$$\mathbf{w}_{\beta_k} = \mathbf{0}_{\beta_k} \quad (3.25)$$

with $\mathbf{0}_{\beta_k}$ denoting the β_k -dimensional zero vector. As a result, the reduced 12×12 stiffness matrix $\mathbf{K}_r$, which maps the 12-dimensional SAD array $\bar{\mathbf{u}}$ into the 12-dimensional applied wrench array $\bar{\mathbf{w}}$, is obtained as

$$\mathbf{K}_r = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{12}^T & \mathbf{G}_{22} \end{bmatrix} - \begin{bmatrix} \mathbf{G}_{13} \\ \mathbf{G}_{23} \end{bmatrix} \mathbf{G}_{33}^{-1} \begin{bmatrix} \mathbf{G}_{13}^T & \mathbf{G}_{23}^T \end{bmatrix} \quad (3.26)$$

which accounts for the equivalent generalized spring of the articulated flexible link.

3.4 The Stiffness Matrix of the Π -joint

The Π -joint being an instance of the four-bar linkage, it is a one-dof joint, all four revolute joints rotating through the same angle. In reality, a Π -joint is a linkage with flexible links that can store potential energy. In this study, a Π joint is modeled as a four-bar linkage with two flexible links, the longer ones shown in Fig. 3.7. The flexible links are connected to the rigid bodies by means of four revolute joints whose axes are parallel. Here, in general, it is assumed that the revolute joints are passive. By

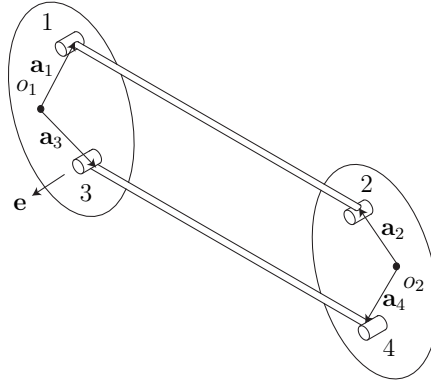


FIGURE 3.7. A Π -joint with four passive revolute joints

recalling eqs. (3.15) and (3.17) the potential energy of a Π joint, depicted in Fig. 3.7,

takes the form

$$V = \frac{1}{2} \begin{bmatrix} \mathbf{u}_1^T & \mathbf{u}_2^T \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{12}^T & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_1 \\ \mathbf{u}'_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{u}_3^T & \mathbf{u}_4^T \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{12}^T & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_3 \\ \mathbf{u}'_4 \end{bmatrix} \quad (3.27)$$

in which $\mathbf{K}_{ij}$, $i, j = 1, 2$, are the block matrices of the 12×12 stiffness matrix of the flexible links. By recalling eq. (3.16) for $\mathbf{u}'_i$, $i = 1, \dots, 4$, and upon expansion, the above expression results in

$$\begin{aligned} V = & \frac{1}{2} (\mathbf{u}_1^T \mathbf{K}_{11} \mathbf{u}_1 + 2\mathbf{u}_1^T \mathbf{K}_{12} \mathbf{u}_2 + \mathbf{u}_2^T \mathbf{K}_{22} \mathbf{u}_2) \\ & + \frac{1}{2} \left[\mathbf{s}^T \mathbf{K}_{11} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_1 + \beta_1)^2 + 2\mathbf{u}_1^T \mathbf{K}_{11} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_1 + \beta_1) + 2\mathbf{s}^T \mathbf{K}_{12} \mathbf{u}_2 (-\mathbf{s}^T \mathbf{u}_1 + \beta_1) \right] \\ & + \frac{1}{2} \left[\mathbf{s}^T \mathbf{K}_{22} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_2 + \beta_2)^2 + 2\mathbf{u}_1^T \mathbf{K}_{12} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_2 + \beta_2) + 2\mathbf{u}_2^T \mathbf{K}_{22} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_2 + \beta_2) \right] \\ & + \frac{1}{2} (\mathbf{u}_3^T \mathbf{K}_{11} \mathbf{u}_3 + 2\mathbf{u}_3^T \mathbf{K}_{12} \mathbf{u}_4 + \mathbf{u}_4^T \mathbf{K}_{22} \mathbf{u}_4) \\ & + \frac{1}{2} \left[\mathbf{s}^T \mathbf{K}_{22} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_4 + \beta_4)^2 + 2\mathbf{u}_3^T \mathbf{K}_{12} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_4 + \beta_4) + 2\mathbf{u}_4^T \mathbf{K}_{22} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_4 + \beta_4) \right] \\ & + \frac{1}{2} \left[\mathbf{s}^T \mathbf{K}_{11} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_3 + \beta_3)^2 + 2\mathbf{u}_3^T \mathbf{K}_{11} \mathbf{s} (-\mathbf{s}^T \mathbf{u}_3 + \beta_3) + 2\mathbf{s}^T \mathbf{K}_{12} \mathbf{u}_4 (-\mathbf{s}^T \mathbf{u}_3 + \beta_3) \right] \end{aligned} \quad (3.28)$$

The SAD screws of each rigid body are related by the formula introduced in eq. (3.10):

$$\begin{aligned} \mathbf{u}_1 &= \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{A}_1 & \mathbf{1} \end{bmatrix} \mathbf{u}_o \equiv \mathbf{W}_1 \mathbf{u}_{o1}, & \mathbf{u}_2 &= \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{A}_2 & \mathbf{1} \end{bmatrix} \mathbf{u}_o \equiv \mathbf{W}_2 \mathbf{u}_{o2} \\ \mathbf{u}_3 &= \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{A}_3 & \mathbf{1} \end{bmatrix} \mathbf{u}_o \equiv \mathbf{W}_3 \mathbf{u}_{o1}, & \mathbf{u}_4 &= \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{A}_4 & \mathbf{1} \end{bmatrix} \mathbf{u}_o \equiv \mathbf{W}_4 \mathbf{u}_{o2} \end{aligned}$$

The generalized coordinates are $\mathbf{u}_{O1}$ and $\mathbf{u}_{O2}$, which are SAD screws at points O_1 and O_2 , besides the “small”-amplitude rotations of the passive revolute joints β_i , for $i = 1, \dots, 4$. It is noteworthy that O_1 and O_2 can be any arbitrary points of the rigid bodies; for instance, when conducting a modal analysis, the COM of each rigid body is the best candidate. Finally, by calculating the gradient of the potential energy with respect to the generalized coordinates the relation between generalized force

and generalized displacement arrays is obtained:

$$\begin{bmatrix} \mathbf{w}_{O_1} \\ \mathbf{w}_{O_2} \\ \mathbf{m}_\beta \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11}^\Pi & \mathbf{K}_{12}^\Pi & \mathbf{K}_{13}^\Pi \\ \mathbf{K}_{12}^{\Pi T} & \mathbf{K}_{22}^\Pi & \mathbf{K}_{23}^\Pi \\ \mathbf{K}_{13}^{\Pi T} & \mathbf{K}_{23}^{\Pi T} & \mathbf{K}_{33}^\Pi \end{bmatrix} \begin{bmatrix} \mathbf{u}_{O_1} \\ \mathbf{u}_{O_2} \\ \boldsymbol{\beta} \end{bmatrix} \quad (3.29)$$

where

$$\mathbf{m}_\beta = \begin{bmatrix} m_{\beta_1} & m_{\beta_2} & m_{\beta_3} & m_{\beta_4} \end{bmatrix}^T, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}^T$$

in which m_{β_i} denotes the moment applied on the i th revolute joint. The block matrices of the 16×16 stiffness matrix are defined below:

$$\begin{aligned} \mathbf{K}_{11}^\Pi &= \mathbf{W}_1^T \mathbf{K}_{11} \mathbf{W}_1 + \mathbf{W}_3^T \mathbf{K}_{11} \mathbf{W}_3 + \mathbf{s}^T \mathbf{K}_{11} \mathbf{s} (\mathbf{W}_1^T \mathbf{s}) (\mathbf{s}^T \mathbf{W}_1) - \mathbf{W}_1 \mathbf{K}_{11} \mathbf{s} (\mathbf{s}^T \mathbf{W}_1) \\ &\quad - (\mathbf{W}_1^T \mathbf{s})^T \mathbf{s}^T \mathbf{K}_{11} \mathbf{W}_1 - \mathbf{W}_3^T \mathbf{K}_{11} \mathbf{s} (\mathbf{s}^T \mathbf{W}_3) + \mathbf{s}^T \mathbf{K}_{11} \mathbf{s} (\mathbf{W}_3^T \mathbf{s}) (\mathbf{s}^T \mathbf{W}_3) \\ \mathbf{K}_{12}^\Pi &= \mathbf{W}_1^T \mathbf{K}_{12} \mathbf{W}_2 + \mathbf{W}_3^T \mathbf{K}_{12} \mathbf{W}_4 + \mathbf{s}^T \mathbf{K}_{12} \mathbf{s} (\mathbf{W}_1^T \mathbf{s}) (\mathbf{s}^T \mathbf{W}_2) - \mathbf{W}_1 \mathbf{K}_{12} \mathbf{s} (\mathbf{s}^T \mathbf{W}_2) \\ &\quad - (\mathbf{W}_1^T \mathbf{s})^T \mathbf{s}^T \mathbf{K}_{12} \mathbf{W}_2 - \mathbf{W}_3^T \mathbf{K}_{12} \mathbf{s} (\mathbf{s}^T \mathbf{W}_4) + \mathbf{s}^T \mathbf{K}_{12} \mathbf{s} (\mathbf{W}_3^T \mathbf{s}) (\mathbf{s}^T \mathbf{W}_4) \\ \mathbf{K}_{22}^\Pi &= \mathbf{W}_2^T \mathbf{K}_{22} \mathbf{W}_2 + \mathbf{W}_4^T \mathbf{K}_{22} \mathbf{W}_4 + \mathbf{s}^T \mathbf{K}_{22} \mathbf{s} (\mathbf{W}_2^T \mathbf{s}) (\mathbf{s}^T \mathbf{W}_2) - \mathbf{W}_2 \mathbf{K}_{22} \mathbf{s} (\mathbf{s}^T \mathbf{W}_2) \\ &\quad - (\mathbf{W}_2^T \mathbf{s})^T \mathbf{s}^T \mathbf{K}_{22} \mathbf{W}_2 - \mathbf{W}_4^T \mathbf{K}_{22} \mathbf{s} (\mathbf{s}^T \mathbf{W}_4) + \mathbf{s}^T \mathbf{K}_{22} \mathbf{s} (\mathbf{W}_4^T \mathbf{s}) (\mathbf{s}^T \mathbf{W}_4) \\ \mathbf{K}_{13}^\Pi &= \begin{bmatrix} \mathbf{g}_{13} & \mathbf{g}_{14} & \mathbf{g}_{15} & \mathbf{g}_{16} \end{bmatrix}, \quad \mathbf{K}_{23}^\Pi = \begin{bmatrix} \mathbf{g}_{23} & \mathbf{g}_{24} & \mathbf{g}_{25} & \mathbf{g}_{26} \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} \mathbf{g}_{13} &= -(\mathbf{s}^T \mathbf{K}_{11} \mathbf{s}) (\mathbf{W}_1^T \mathbf{s}) + \mathbf{W}_1^T \mathbf{K}_{11} \mathbf{s}, \quad \mathbf{g}_{14} = \mathbf{W}_1^T \mathbf{K}_{12} \mathbf{s} - (\mathbf{s}^T \mathbf{K}_{12} \mathbf{s}) (\mathbf{W}_1^T \mathbf{s}) \\ \mathbf{g}_{15} &= -(\mathbf{s}^T \mathbf{K}_{11} \mathbf{s}) (\mathbf{W}_3^T \mathbf{s}) + \mathbf{W}_3^T \mathbf{K}_{11} \mathbf{s}, \quad \mathbf{g}_{16} = \mathbf{W}_3^T \mathbf{K}_{12} \mathbf{s} - (\mathbf{s}^T \mathbf{K}_{12} \mathbf{s}) (\mathbf{W}_3^T \mathbf{s}) \\ \mathbf{g}_{23} &= \mathbf{W}_2^T \mathbf{K}_{12} \mathbf{s} - (\mathbf{s}^T \mathbf{K}_{12} \mathbf{s}) (\mathbf{W}_2^T \mathbf{s}), \quad \mathbf{g}_{24} = -(\mathbf{s}^T \mathbf{K}_{22} \mathbf{s}) (\mathbf{W}_2^T \mathbf{s}) + \mathbf{W}_2^T \mathbf{K}_{22} \mathbf{s} \\ \mathbf{g}_{25} &= \mathbf{W}_4^T \mathbf{K}_{12} \mathbf{s} - (\mathbf{s}^T \mathbf{K}_{12} \mathbf{s}) (\mathbf{W}_4^T \mathbf{s}), \quad \mathbf{g}_{26} = -(\mathbf{s}^T \mathbf{K}_{12} \mathbf{s}) (\mathbf{W}_4^T \mathbf{s}) + \mathbf{W}_4^T \mathbf{K}_{22} \mathbf{s} \end{aligned} \quad (3.30)$$

and

$$\mathbf{K}_{33}^\Pi = \begin{bmatrix} G_{33} & G_{34} & 0 & 0 \\ G_{34}^T & G_{44} & 0 & 0 \\ 0 & 0 & G_{55} & G_{56} \\ 0 & 0 & G_{56}^T & G_{66} \end{bmatrix}$$

whose components are

$$G_{33} = G_{55} = \mathbf{s}^T \mathbf{K}_{11} \mathbf{s}, \quad G_{34} = G_{56} = \mathbf{s}^T \mathbf{K}_{12} \mathbf{s}, \quad G_{44} = G_{66} = \mathbf{s}^T \mathbf{K}_{22} \mathbf{s} \quad (3.31)$$

If the revolute joints are passive, the moment acting on them vanishes, i.e., $\mathbf{m}_\beta = \mathbf{0}$. Hence, the 12×12 reduced stiffness matrix of the Π joint with respect to points O_1 and O_2 , is obtained as

$$\mathbf{K}_r = \begin{bmatrix} \mathbf{K}_{11}^\Pi & \mathbf{K}_{12}^\Pi \\ (\mathbf{K}_{12}^\Pi)^T & \mathbf{K}_{22}^\Pi \end{bmatrix} - \begin{bmatrix} \mathbf{K}_{13}^\Pi \\ \mathbf{K}_{23}^\Pi \end{bmatrix} (\mathbf{K}_{33}^\Pi)^{-1} \begin{bmatrix} (\mathbf{K}_{13}^\Pi)^T & (\mathbf{K}_{23}^\Pi)^T \end{bmatrix} \quad (3.32)$$

The flexible links of the Π -joint can be made of any linearly elastic material and with any shape; in some cases, for example, when the Π joints are made of composite materials with complex shapes, it is not possible to have the stiffness matrix of the flexible link $\mathbf{K}$ in parametric form. The stiffness matrix must then be calculated with FEA.

3.5 Case Study: The McGill SMG

The foregoing results are used to calculate the stiffness matrix of the McGill SMG, consisting of one BP and one MP, the two are coupled by two limbs, shown in Fig. 2.10, which is modelled as the elastostatic system depicted in Fig. 3.8. Kinetostatically, each limb consists of two rigid links: the elbow brackets and the wrist brackets. The parallelograms are thus regarded as joints, namely, Π -joints. In the elastostatic model each Π -joint is replaced with its corresponding equivalent generalized spring. The McGill SMG has two kinds of Π -joints, proximal and distal, as depicted in Fig. 2.10. Each distal Π -joint is made of four identical aluminum rods, which are modeled as flexible beams; the stiffness of the equivalent single spring can then be obtained by means of the beam stiffness matrix, eq. (3.14). In the case of the proximal Π -joints, these include two boxes made of a composite material, for which no closed-form parametric stiffness matrix is available. Because of the complex shape and the complexity associated with the modelling of composite materials, FEA was used to

calculate the stiffness matrix of the composite box. In the ensuing elastostatic analysis we assume that the rigid bodies are massless; hence, the potential energy only comes from elastic energy of the flexible parts. Therefore, the potential energy of the McGill SMG is

$$V = \frac{1}{2} \sum_{J=I,II} (\mathbf{q}_{J1}^T (\mathbf{K}_{22}^{PJ} + \mathbf{K}_{11}^{DJ}) \mathbf{q}_{J1} + 2\mathbf{q}_{J1}^T \mathbf{K}_{12}^{DJ} \mathbf{q}_{J2} + \mathbf{q}_{J2}^T \mathbf{K}_{22}^{DJ} \mathbf{q}_{J2}) \quad (3.33)$$

in which $\mathbf{K}_{ij}^{PJ}$ and $\mathbf{K}_{ij}^D$ are the 6×6 block matrices of the reduced stiffness matrices of the proximal and distal Π -joints, respectively, of limb J , which are calculated via eq. (3.32). Moreover, $\mathbf{q}_{J1}$ and $\mathbf{q}_{J2}$ are the SAD screws defined at the COM of the EB and WB J th limb.

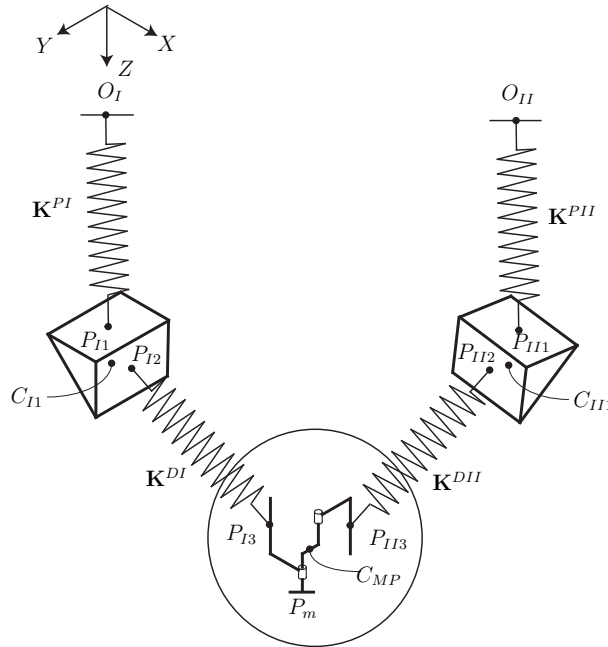


FIGURE 3.8. Elastostatic model of the McGill SMG

The wrist brackets, depicted in Fig. 3.8 and zoomed-in in Fig. 3.9, are connected to the MP by means of revolute joints; by invoking the rigidity assumption and because of the presence of the revolute joints, one has

$$\mathbf{q}_{J2} = \mathbf{G}_J (\mathbf{H}_J \mathbf{q}_m + \gamma_J \boldsymbol{\zeta}), \quad \boldsymbol{\zeta} = [\mathbf{e}_\gamma^T \quad \mathbf{0}^T]^T \quad (3.34)$$

in which $\mathbf{0}$ is the three-dimensional zero vector, γ_J is the “small”-amplitude angle of rotation of the revolute joint of the J th limb, and $\mathbf{e}_\gamma$ is the unit vector parallel to the revolute joint axis, which, in this case, is $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, and hence, constant. In fact, $\mathbf{e}_\gamma$ wobbles because of the flexible Π -joints; however, based on a first-order approximation, $\mathbf{e}_\gamma$ can be considered as a constant vector. Moreover,

$$\mathbf{G}_J = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{A}_J & \mathbf{1} \end{bmatrix}, \quad \mathbf{H}_J = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{D}_J & \mathbf{1} \end{bmatrix} \quad (3.35)$$

where $\mathbf{A}_J$ and $\mathbf{D}_J$ are the CPM of vectors $\mathbf{a}_J$ and $\mathbf{d}_J$, respectively, for $J = I, II$. Upon substituting eq. (3.34) into eq. (3.33) the potential energy of the robot becomes

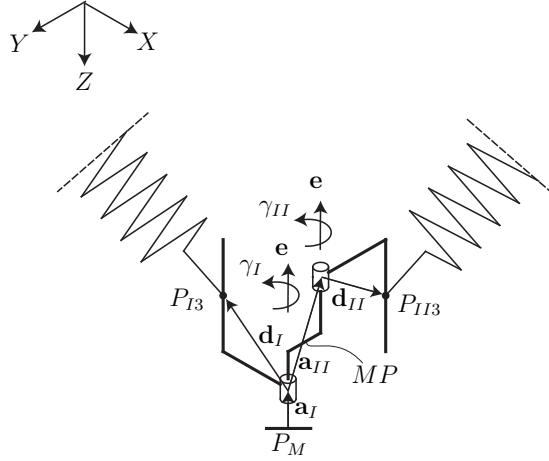


FIGURE 3.9. A zoom-in of the MP and the wrist brackets in the elastostatic model of the McGill SMG

$$\begin{aligned} V = \frac{1}{2} \sum_{J=I,II} \mathbf{q}_{J1}^T (\mathbf{K}_{22}^{PJ} + \mathbf{K}_{11}^{DJ}) \mathbf{q}_{J1} + 2 \mathbf{q}_{J1}^T \mathbf{K}_{12}^{DJ} \mathbf{G}_J (\mathbf{H}_J \mathbf{q}_m + \gamma_J \boldsymbol{\zeta}) \\ + (\mathbf{H}_J \mathbf{q}_m + \gamma_J \boldsymbol{\zeta})^T \mathbf{G}_J^T \mathbf{K}_{22}^{DJ} \mathbf{G}_J (\mathbf{H}_J \mathbf{q}_m + \gamma_J \boldsymbol{\zeta}) \end{aligned} \quad (3.36)$$

After taking the gradient of the potential energy with respect to the generalized coordinates, the matrix that maps the SMG generalized displacement array into the

generalized force array is obtained:

$$\begin{bmatrix} \mathbf{w}_m \\ \mathbf{w}_I \\ \mathbf{w}_{II} \\ m_{\gamma_I} \\ m_{\gamma_{II}} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} & \mathbf{K}_{14} & \mathbf{K}_{15} \\ \mathbf{K}_{12}^T & \mathbf{K}_{22} & \mathbf{0} & \mathbf{K}_{24} & 0 \\ \mathbf{K}_{13}^T & \mathbf{0} & \mathbf{K}_{33} & \mathbf{0} & \mathbf{K}_{35} \\ \mathbf{K}_{14}^T & \mathbf{K}_{24}^T & \mathbf{0}^T & \mathbf{K}_{44} & 0 \\ \mathbf{K}_{15}^T & \mathbf{0}^T & \mathbf{K}_{35}^T & 0 & \mathbf{K}_{55} \end{bmatrix} \begin{bmatrix} \mathbf{q}_m \\ \mathbf{q}_I \\ \mathbf{q}_{II} \\ \gamma_I \\ \gamma_{II} \end{bmatrix} \quad (3.37)$$

The block matrices of the stiffness matrix are shown below:

$$\begin{aligned} \mathbf{K}_{11} &= \mathbf{H}_I^T \mathbf{G}_I^T \mathbf{K}_{11}^{DI} \mathbf{G}_I \mathbf{H}_I + \mathbf{H}_{II}^T \mathbf{G}_{II}^T \mathbf{K}_{11}^{DII} \mathbf{G}_{II} \mathbf{H}_{II}, & \mathbf{K}_{12} &= \mathbf{H}_I^T \mathbf{G}_I^T \mathbf{K}_{12}^{DI} \\ \mathbf{K}_{13} &= \mathbf{H}_{II}^T \mathbf{G}_{II}^T \mathbf{K}_{12}^{DII}, & \mathbf{K}_{22} &= \mathbf{K}_{22}^{PI} + \mathbf{K}_{11}^{DI}, & \mathbf{K}_{24} &= \mathbf{K}_{12}^{DI} \mathbf{G}_I \zeta \\ \mathbf{K}_{33} &= \mathbf{K}_{22}^{PII} + \mathbf{K}_{11}^{PII}, & \mathbf{K}_{35} &= \mathbf{K}_{12}^{DII} \mathbf{G}_{II} \zeta \\ \mathbf{K}_{44} &= \zeta^T \mathbf{G}_I^T \mathbf{K}_{11}^{DI} \mathbf{G}_I \zeta, & \mathbf{K}_{55} &= \zeta^T \mathbf{G}_{II}^T \mathbf{K}_{11}^{DII} \mathbf{G}_{II} \zeta \end{aligned} \quad (3.38)$$

The Cartesian stiffness matrix is a mapping between the SAD of the MP and the actuator wrench applied onto it, namely,

$$\mathbf{w}_I = \mathbf{w}_{II} = \mathbf{0} \quad (3.39)$$

Moreover, the passive revolute joints transmit a zero moment about their axes, meaning that $m_{\gamma_I} = m_{\gamma_{II}} = 0$. Hence, the 6×6 Cartesian stiffness matrix of the overall robot becomes, finally,

$$\mathbf{K}_e = \mathbf{K}_{11} - \begin{bmatrix} \mathbf{K}_{12}^T & \mathbf{K}_{13}^T & \mathbf{K}_{14}^T & \mathbf{K}_{15}^T \end{bmatrix} \begin{bmatrix} \mathbf{K}_{22} & \mathbf{0} & \mathbf{K}_{24} & 0 \\ \mathbf{0} & \mathbf{K}_{33} & \mathbf{0} & \mathbf{K}_{35} \\ \mathbf{K}_{42} & \mathbf{0}^T & \mathbf{K}_{44} & 0 \\ \mathbf{0}^T & \mathbf{K}_{53} & 0 & \mathbf{K}_{55} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{12} \\ \mathbf{K}_{13} \\ \mathbf{K}_{14} \\ \mathbf{K}_{15} \end{bmatrix} \quad (3.40)$$

which relates the SAD of the MP, defined at point P_M , to the wrench applied onto the MP, with the force acting along a line that passes through P_M .

3.6 Stiffness Indices

In order to assess the elastostatic response of a robot to an external wrench, defining stiffness indices is essential. Here, the main issue is that a 6×6 Cartesian stiffness matrix is formed by four 3×3 block matrices whose components have different units, which is made apparent below:

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{f} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{12}^T & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\sigma} \end{bmatrix} \quad (3.41)$$

where $\mathbf{f}$ and $\mathbf{n}$ denote force and moment, respectively, the six dimensional array of the left-hand side being a wrench; $\boldsymbol{\sigma}$ and $\boldsymbol{\theta}$ define, in turn, a “small” point-translation—small with respect to the dimensions of the system bodies—and a “small”-amplitude rotation about an axis parallel to the unit vector $\boldsymbol{\theta}/\|\boldsymbol{\theta}\|$, respectively. The six-dimensional array of the right-hand side is thus a rigid-body SAD screw. Apparently, the block matrices $\mathbf{K}_{11}$, $\mathbf{K}_{12}$ and $\mathbf{K}_{22}$ have units of Nm, N and N/m, respectively. To define stiffness performance indices, first we should nondimensionalize the stiffness matrix. The method used here is based on the idea introduced by Kövecses and Ebrahimi (2009), to define dimensionless parameters for sensitivity analysis of mechanical systems. The two three-dimensional equations (3.41), one for moment and one for force, are displayed below:

$$\mathbf{n} = \mathbf{K}_{11}\boldsymbol{\theta} + \mathbf{K}_{12}\boldsymbol{\sigma} \quad (3.42a)$$

$$\mathbf{f} = \mathbf{K}_{12}^T\boldsymbol{\theta} + \mathbf{K}_{22}\boldsymbol{\sigma} \quad (3.42b)$$

Apparently, the force and moment vectors are made up of two independent components, namely,

$$\mathbf{n} = \mathbf{K}_{11}\boldsymbol{\theta} + \mathbf{K}_{12}\boldsymbol{\sigma} = \mathbf{n}_\theta + \mathbf{n}_\sigma \quad (3.43a)$$

$$\mathbf{f} = \mathbf{K}_{12}^T\boldsymbol{\theta} + \mathbf{K}_{22}\boldsymbol{\sigma} = \mathbf{f}_\theta + \mathbf{f}_\sigma \quad (3.43b)$$

We can associate each independent part of eqs. (3.43b) and (3.43a) with a physically meaningful quadratic form (Kövecses and Ebrahimi, 2009), such as $\|\mathbf{f}_\beta\|^2 =$

$\boldsymbol{\beta}^T \mathbf{K}_\beta^T \mathbf{K}_\beta \boldsymbol{\beta}$, which defines an ellipsoid in the space of $\boldsymbol{\beta}$, a dummy vector variable, defined as: $\boldsymbol{\beta} = \boldsymbol{\sigma}$ for a point-displacement, and $\boldsymbol{\beta} = \boldsymbol{\theta}$ for a “small”-angle rotation vector, similar to $\boldsymbol{\phi}$ of eq. (3.4). The eigenvalue problem of $\mathbf{K}_\beta^T \mathbf{K}_\beta$ are then used to define dimensionless parameters. A relation between two dimensionless parameter vectors $\mathbf{p}_\beta$ and $\boldsymbol{\beta}$ is now introduced:

$$\boldsymbol{\beta} = \mathbf{S}_\beta \mathbf{p}_\beta \quad (3.44)$$

where $\mathbf{S}_\beta$ is an orthogonal matrix whose columns are the eigenvectors of matrix $\mathbf{K}_\beta^T \mathbf{K}_\beta$. In our case, to define dimensionless parameters we will have two linear transformations regarding the force and moment equations; for the force equation the linear transformation leads to

$$\begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_\theta & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_\sigma \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_\theta \\ \boldsymbol{\psi}_\sigma \end{bmatrix} \quad (3.45)$$

while, for the moment equation,

$$\begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_\theta & \mathbf{O} \\ \mathbf{O} & \mathbf{H}_\sigma \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_\theta \\ \boldsymbol{\nu}_\sigma \end{bmatrix} \quad (3.46)$$

Thus, by substitution of eqs.(3.45) and (3.46) into eqs.(3.42b) and (3.42a), respectively, the force and moment vectors are transformed into dimensionless vectors by means of new linear transformations:

$$\mathbf{f} = \mathbf{G}_f \boldsymbol{\psi}, \quad \mathbf{n} = \mathbf{G}_n \boldsymbol{\nu} \quad (3.47)$$

in which $\mathbf{G}_f$ and $\mathbf{G}_n$ are dimensionally homogeneous coefficient matrices that have units corresponding to $\mathbf{f}$ and $\mathbf{n}$, respectively, i.e.,

$$\begin{aligned} \mathbf{G}_f &= \begin{bmatrix} \mathbf{K}_{12}^T \mathbf{S}_\theta & \mathbf{K}_{22} \mathbf{S}_\sigma \end{bmatrix}, \quad \boldsymbol{\psi} = \begin{bmatrix} \boldsymbol{\psi}_\theta^T & \boldsymbol{\psi}_\sigma^T \end{bmatrix}^T \\ \mathbf{G}_n &= \begin{bmatrix} \mathbf{K}_{11} \mathbf{H}_\theta & \mathbf{K}_{12} \mathbf{H}_\sigma \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} \boldsymbol{\nu}_\theta^T & \boldsymbol{\nu}_\sigma^T \end{bmatrix}^T \end{aligned} \quad (3.48)$$

Both matrices $\mathbf{G}_f^T \mathbf{G}_f$ and $\mathbf{G}_n^T \mathbf{G}_n$ have three mutually orthogonal eigenvectors corresponding to three non-zero eigenvalues that characterize the distortion of the unit

spheres $\|\mathbf{f}\| = 1$ (N) and $\|\mathbf{n}\| = 1$ (Nm), respectively. The minimum values of the eigenvalues of $\mathbf{G}_f^T \mathbf{G}_f$ and $\mathbf{G}_m^T \mathbf{G}_m$, λ_i and μ_i , $i = 1, \dots, 3$, respectively, are now defined as translational κ_t and rotational κ_r stiffness performance indices, respectively, i.e.,

$$\kappa_t = \min_i(\sqrt{\lambda_i}), \quad \kappa_r = \min_i(\sqrt{\mu_i}), \quad i = 1, \dots, 3 \quad (3.49)$$

where the square roots have been introduced because of the quadratic nature of $\mathbf{G}_f^T \mathbf{G}_f$ and $\mathbf{G}_m^T \mathbf{G}_m$.

3.7 Numerical Results

Now we compute the stiffness performance indices of the McGill SMG while going through a pick-and-place operation following the test trajectory. This trajectory was described in Sec. 2.5 and displayed in Fig. 2.14. The three translational and rotational eigenvalues λ_i and μ_i , $i = 1, \dots, 3$, respectively, are plotted in Fig. 3.10 through the test trajectory. The translational and rotational stiffness indices, κ_t and κ_r , respectively, of the McGill SMG through the test trajectory are also shown in Fig. 3.11. The behaviour of the two performance indices through the trajectory shows the points at which the robot has poor stiffness: the higher the index, the stiffer the structure. The indices reach their minimum and maximum values at different postures through the test trajectory. As shown in Fig. 3.11, in the foregoing case, the rotational and translational stiffness indices κ_r and κ_t , respectively, attain minimum values when the two limbs are coplanar; hence, this can be recognized as the weakest posture of the robot in terms of stiffness.

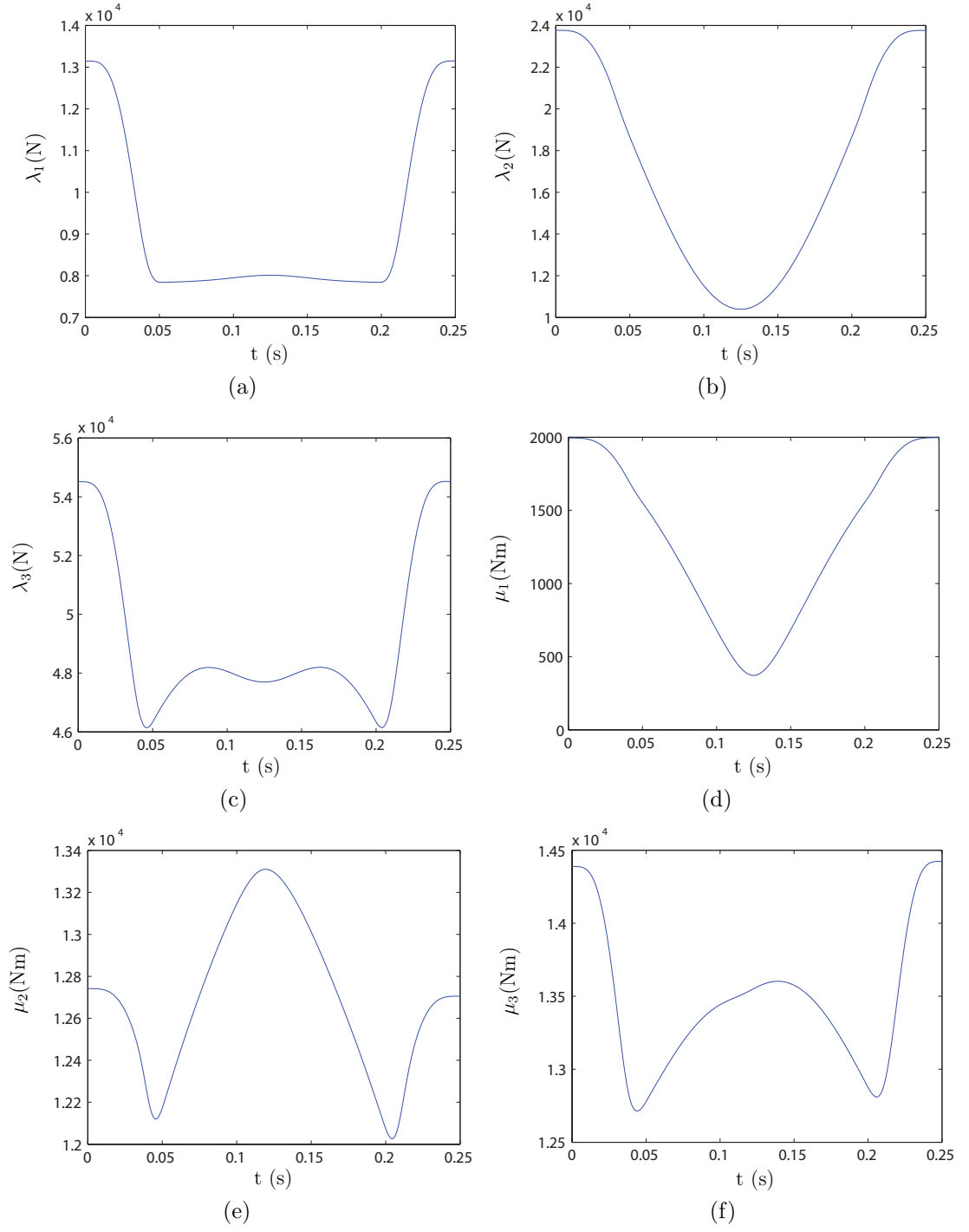


FIGURE 3.10. The variation of the translational and rotational eigenvalues, (a)–(c) λ_i and (d)–(f) μ_i , $i = 1, \dots, 3$, through the smoothed test trajectory

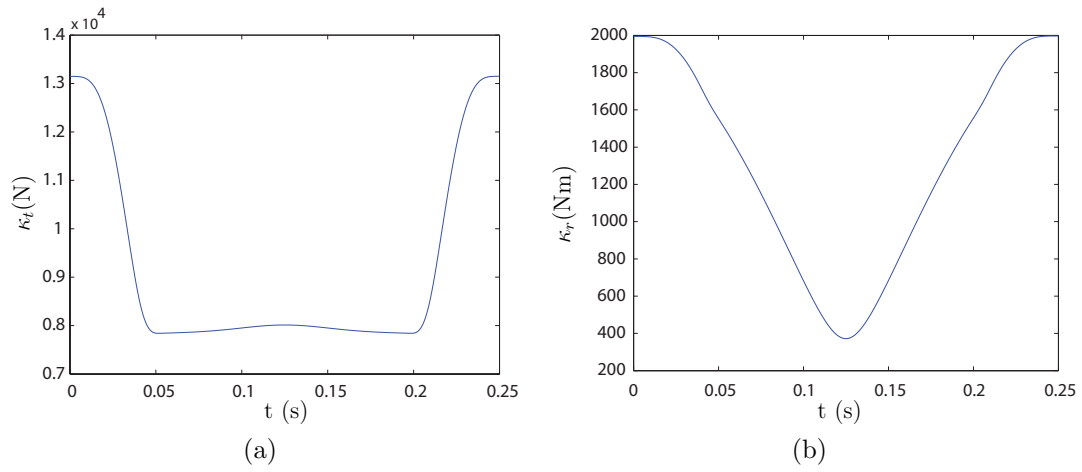


FIGURE 3.11. The variation of the stiffness indices, (a) translational and (b) rotational, through the smoothed test trajectory

CHAPTER 4

Elastodynamic Analysis of Parallel Manipulators

The elastodynamic analysis of parallel manipulators is the subject of this chapter. The methodology described here is exemplified with the analysis of the McGill SMG, a two-limb Schönflies motion generator. This analysis calls for the calculation of the stiffness and mass matrices. By resorting to the generalized spring concept, the posture-dependent stiffness matrix of the robot is computed. With the motors locked, the motion caused by the flexible components leads to the robot mass matrix. The generalized springs help to simplify the model. Although this simplification filters out the higher natural frequencies, it eases the computation of the posture-dependent stiffness and mass matrices. This provides a valuable tool to simplify the evaluation of the robot performance from an elastodynamic point of view, while the robot executes a given task. Finally, the modal analysis of the McGill SMG, while executing a pick-and-place operation, is conducted; under these conditions, the evolution of the first six natural frequencies is obtained. The elastodynamic performance of the robot for the given task is assessed using the results of the analysis.

4.1 Generalized Coordinates and Generalized Velocities

To formulate the problem, first the generalized coordinates and generalized velocities of the robot should be defined. Under the assumption that the flexible parts are massless, the McGill SMG is composed of five moving rigid bodies: two EBs; two WBs; and one MP, shown in Fig. 3.8 . The MP is connected to the WBs by means of revolute joints. In elastodynamics studies it is assumed that the rigid-body displacements are of “small” amplitude. The independent generalized-coordinate vector for the system is defined as

$$\mathbf{q} = \left[\mathbf{q}_{I1}^T \quad \gamma_I \quad \mathbf{q}_{II1}^T \quad \gamma_{II} \quad \mathbf{q}_M^T \right]^T \quad (4.1)$$

where $\mathbf{q}_{J1}$ is the six-dimensional array of generalized coordinates of the first rigid body of the J th limb, for $J = I, II$, due to the deformation of the first elastic body of this limb, i.e.,

$$\mathbf{q}_{J1} = \begin{bmatrix} \mathbf{r}_{J1} \\ \mathbf{p}_{J1} \end{bmatrix} \quad (4.2)$$

where $\mathbf{p}_{J1}$ and $\mathbf{r}_{J1}$ constitute the SAD screw from a reference pose of the body, defined at the COM of the $J1$ rigid link; these are the displacement of the COM and the “small”-rotation vectors, respectively, of the first body of the J th limb, respectively, due to the deformation of the first elastic body of the J th limb. Moreover, γ_I and γ_{II} are the small rotation angles of the revolute joints that connect the WB to the MP (Fig. 4.2). Finally, $\mathbf{q}_M$ is the six-dimensional array of generalized coordinates of the MP, due to the deformation of the second elastic body of the J th limb,

$$\mathbf{q}_M = \begin{bmatrix} \mathbf{r}_M \\ \mathbf{p}_M \end{bmatrix} \quad (4.3)$$

in which $\mathbf{p}_M$ and $\mathbf{r}_M$ compose the SAD screw, defined at the COM; these vectors are the counterparts of those displayed in eq. (4.2). All the SAD screws are measured from the equilibrium pose of each body when the motors are locked at a given robot posture. From eq. (4.1) the 20-dimensional generalized velocity vector $\dot{\mathbf{q}}$ of the robot

is obtained as

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\mathbf{q}}_{I1}^T & \dot{\gamma}_I & \dot{\mathbf{q}}_{II1}^T & \dot{\gamma}_{II} & \dot{\mathbf{q}}_M^T \end{bmatrix}^T \quad (4.4)$$

4.2 Calculation of the Stiffness Matrix

In Ch. 3 the concept of generalized spring was extensively studied; as a case study, the 6×6 Cartesian stiffness matrix of the McGill SMG was determined. In elastostatics, the mapping between the SAD screw of the MP and the wrench producing the SAD plays a central role. This mapping is represented by the 6×6 Cartesian stiffness matrix. In the elastodynamic analysis the dynamic response of the whole system, and not of a single body, under a dynamic load, is the main objective. In this case, the stiffness matrix maps the SAD screw of all rigid bodies into the wrenches producing these displacements. Hence, the potential energy of the robot is calculated in terms of the set of all independent generalized coordinates. The potential energy of the McGill SMG, according to its elastostatic model shown in Fig. 3.8, eq. (3.33) is recalled here:

$$V = \frac{1}{2} \sum_{J=I,II} [\mathbf{q}_{J1}^T (\mathbf{K}_{22}^{PJ} + \mathbf{K}_{11}^{DJ}) \mathbf{q}_{J1} + 2\mathbf{q}_{J1}^T \mathbf{K}_{12}^{DJ} \mathbf{q}_{J2} + \mathbf{q}_{J2}^T \mathbf{K}_{11}^{DJ} \mathbf{q}_{J2}] \quad (4.5)$$

in which $\mathbf{K}_{ij}^{PJ}$ and $\mathbf{K}_{ij}^{DJ}$, $i = 1, 2$, are the 6×6 block matrices of the equivalent stiffness matrices of the proximal and distal Π joints, respectively, of limb J . The two WBs are connected, in turn, to the MP by means of revolute joints; hence, by invoking the rigidity assumption and because of the presence of the revolute joints, eq. (3.34) holds. Upon substitution of eq. (3.34) into eq. (4.5), the potential energy of the system becomes

$$\begin{aligned} V = \frac{1}{2} \sum_{J=I,II} & \mathbf{q}_{J1}^T (\mathbf{K}_{22}^{PJ} + \mathbf{K}_{11}^{DJ}) \mathbf{q}_{J1} + 2\mathbf{q}_{J1}^T \mathbf{K}_{12}^{DJ} \mathbf{G}_J (\mathbf{H}_J \mathbf{q}_m + \gamma_J \boldsymbol{\zeta}) \\ & + (\mathbf{H}_J \mathbf{q}_m + \gamma_J \boldsymbol{\zeta})^T \mathbf{G}_J^T \mathbf{K}_{11}^{DJ} \mathbf{G}_J (\mathbf{H}_J \mathbf{q}_m + \gamma_J \boldsymbol{\zeta}) \end{aligned} \quad (4.6)$$

In eq. (4.6) the potential energy is a quadratic form in the generalized coordinates:

$$V = \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q} \quad (4.7)$$

in which $\mathbf{K}$ is the 20×20 stiffness matrix sought, namely,

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{k}_{12} & \mathbf{O}_{6 \times 6} & \mathbf{0}_6 & \mathbf{K}_{15} \\ \mathbf{k}_{12}^T & k_{22} & \mathbf{0}_6^T & 0 & \mathbf{k}_{25} \\ \mathbf{O}_{6 \times 6} & \mathbf{0}_6 & \mathbf{K}_{33} & \mathbf{k}_{34} & \mathbf{K}_{35} \\ \mathbf{0}_6^T & 0 & \mathbf{k}_{34}^T & k_{44} & \mathbf{k}_{45} \\ \mathbf{K}_{15}^T & \mathbf{k}_{25}^T & \mathbf{K}_{53}^T & \mathbf{k}_{45}^T & \mathbf{K}_{55} \end{bmatrix}$$

where $\mathbf{O}_{6 \times 6}$ is the 6×6 zero matrix and $\mathbf{0}_6$ is the six-dimensional zero vector. The various blocks of the stiffness matrix are described below:

$$\begin{aligned} \mathbf{K}_{11} &= \mathbf{K}_{22}^{PI} + \mathbf{K}_{11}^{DI}, \quad \mathbf{k}_{12} = \mathbf{K}_{12}^{DI} \mathbf{G}_I \boldsymbol{\zeta} \\ \mathbf{K}_{15} &= \mathbf{K}_{12}^{DI} \mathbf{G}_I \mathbf{H}_I, \quad k_{22} = \boldsymbol{\zeta}^T \mathbf{G}_I^T \mathbf{K}_{22}^{DI} \mathbf{G}_I \boldsymbol{\zeta} \\ \mathbf{k}_{25} &= \boldsymbol{\zeta}^T \mathbf{G}_I^T \mathbf{K}_{21}^{DI} \mathbf{G}_I \mathbf{H}_I, \quad \mathbf{K}_{33} = \mathbf{K}_{22}^{PII} + \mathbf{K}_{11}^{DII} \\ \mathbf{k}_{34} &= \mathbf{K}_{12}^{DII} \mathbf{G}_{II} \boldsymbol{\zeta}, \quad \mathbf{K}_{35} = \mathbf{K}_{12}^{DII} \mathbf{G}_{II} \mathbf{H}_{II} \\ k_{44} &= \boldsymbol{\zeta}^T \mathbf{G}_{II}^T \mathbf{K}_{22}^{DII} \mathbf{G}_{II} \boldsymbol{\zeta}, \quad \mathbf{k}_{45} = \boldsymbol{\zeta}^T \mathbf{G}_{II}^T \mathbf{K}_{21}^{DII} \mathbf{G}_{II} \mathbf{H}_{II} \\ \mathbf{K}_{55} &= \mathbf{H}_I^T \mathbf{G}_I^T \mathbf{K}_{22}^{DI} \mathbf{G}_I \mathbf{H}_I + \mathbf{H}_{II}^T \mathbf{G}_{II}^T \mathbf{K}_{22}^{DII} \mathbf{G}_{II} \mathbf{H}_{II} \end{aligned} \tag{4.8}$$

in which

$$\mathbf{K}_{21}^{DJ} = (\mathbf{K}_{12}^{DJ})^T, \quad J = I, II$$

In the next step, the previous expression is rearranged using the compact notation introduced below:

$$\mathbf{x}_I = \begin{bmatrix} \mathbf{q}_{I1} \\ \gamma_I \end{bmatrix}, \quad \mathbf{x}_{II} = \begin{bmatrix} \mathbf{q}_{II1} \\ \gamma_{II} \end{bmatrix}, \tag{4.9}$$

By partitioning the generalized coordinate vector $\mathbf{q}$ into three blocks, $\mathbf{x}_I$, $\mathbf{x}_{II}$, and $\mathbf{q}_M$, the potential energy can be cast in the form

$$V = \frac{1}{2} \begin{bmatrix} \mathbf{x}_I^T & \mathbf{x}_{II}^T & \mathbf{q}_M^T \end{bmatrix} \begin{bmatrix} \mathbf{K}_I & \mathbf{O}_{7 \times 7} & \mathbf{K}_{IM} \\ \mathbf{O}_{7 \times 7} & \mathbf{K}_{II} & \mathbf{K}_{IIM} \\ \mathbf{K}_{IM}^T & \mathbf{K}_{IIM}^T & \mathbf{K}_M \end{bmatrix} \begin{bmatrix} \mathbf{x}_I \\ \mathbf{x}_{II} \\ \mathbf{q}_M \end{bmatrix} \tag{4.10}$$

in which

$$\begin{aligned} \mathbf{K}_I &= \begin{bmatrix} \mathbf{K}_{11} & \mathbf{k}_{12} \\ \mathbf{k}_{12}^T & k_{22} \end{bmatrix}, \quad \mathbf{K}_{II} = \begin{bmatrix} \mathbf{K}_{33} & \mathbf{k}_{34} \\ \mathbf{k}_{34}^T & k_{44} \end{bmatrix}, \quad \mathbf{K}_M = \mathbf{K}_{55} \\ \mathbf{K}_{IM} &= \begin{bmatrix} \mathbf{K}_{15} \\ \mathbf{k}_{25} \end{bmatrix}, \quad \mathbf{K}_{IIM} = \begin{bmatrix} \mathbf{K}_{35} \\ \mathbf{k}_{45} \end{bmatrix} \end{aligned} \quad (4.11)$$

From eq. (4.10) the stiffness matrix $\mathbf{K}$ is

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_I & \mathbf{O}_{7 \times 7} & \mathbf{K}_{IM} \\ \mathbf{O}_{7 \times 7} & \mathbf{K}_{II} & \mathbf{K}_{IIM} \\ \mathbf{K}_{IM}^T & \mathbf{K}_{IIM}^T & \mathbf{K}_M \end{bmatrix} \quad (4.12)$$

where $\mathbf{O}_{7 \times 7}$ is the 7×7 zero matrix.

4.3 Calculation of the Mass Matrix

In an elastodynamic analysis, all the motors are locked at a particular robot posture, any possible motion being due only to the deformation of the elastic bodies. These bodies are, in our case, the proximal and the distal parallelograms of each limb, while the drive units, the EB and the WB of both limbs, and the MP, are modeled as rigid bodies, all bodies being displayed in Fig. 2.10. Because the stiffness of the drive units is much higher than the robot structure, their small-amplitude displacement are not taken into account when deriving the inertia matrix. In the first step the generalized velocities of each rigid body, along with the expressions of its twist array, are obtained. After this step, the expression for the kinetic energy of the robot is derived. The mass matrix of the SMG is calculated as the Hessian of the kinetic energy with respect to the generalized velocities. The fixed reference frame $\mathcal{F}_O = \{O, X_O, Y_O, Z_O\}$ is defined, with the Z -axis coincident with the axis of rotation of the first revolute joint of limb I ; the origin is located at the contact surface between the drive unit and the machine frame. To describe the effect of the deformation of the elastic bodies on the posture of the rigid bodies, two local reference frames, one before and one after the deformation, are defined, with origin at: C_{J1} , the COM of EB; C_{J2} ,

the COM of WB; and C_M the COM of the MP. $\mathcal{F}_{C_{J1}} = \{C_{J1}; X_{C_{J1}}, Y_{C_{J1}}, Z_{C_{J1}}\}$ is the local reference frame with origin at the COM of the EB before the deformation; $\mathcal{F}_{C_{J2}} = \{C_{J2}; X_{C_{J2}}, Y_{C_{J2}}, Z_{C_{J2}}\}$ is the local reference frame with origin at the COM of the WB before the deformation; $\mathcal{F}_{C_M} = \{C_M; X_{C_M}, Y_{C_M}, Z_{C_M}\}$ is the local reference frame with origin at the COM of the MP before the deformation. The counterpart frames $\mathcal{F}'_{C_{Ji}}$, for $i = 1, 2$, and $\mathcal{F}'_{C_{JM}}$ after deformation are defined likewise, their axes being labelled as those before deformation, but *primed*. The only variables of this model are θ_{J1} and θ_{J4} , for $J = I, II$, because the robot posture is known and the variables needed to describe the posture are θ_{J1} and ϕ ; θ_{J4} can be calculated via the relations

$$\theta_{I4} = \theta_{I1} + \phi \quad \theta_{II4} = \theta_{II1} + \phi - \pi \quad (4.13)$$

While eqs. (4.13) hold when the links are rigid, as it is assumed that all rigid-bodies

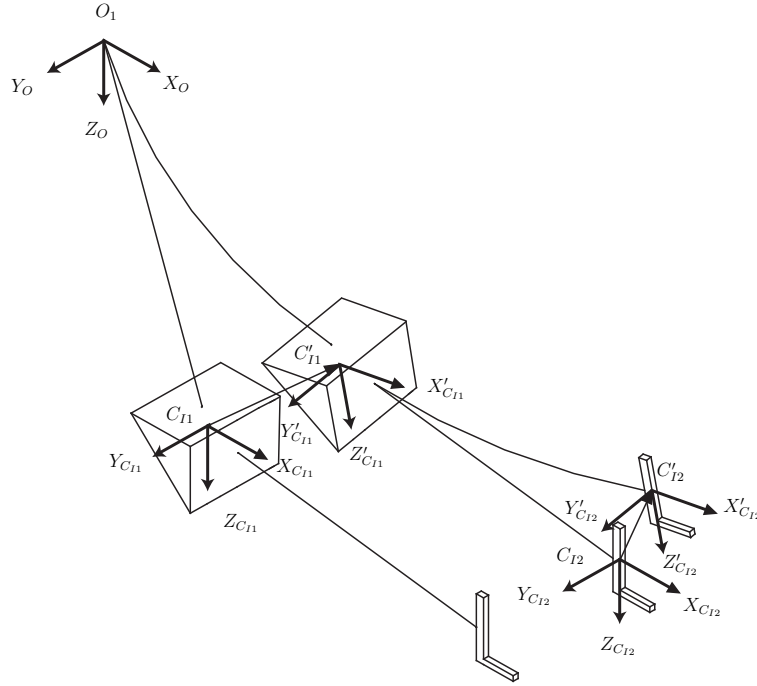


FIGURE 4.1. Coordinate frames for the first two rigid bodies of the first limb before and after deformation

undergo “small” amplitude motions, these relations can be approximately accepted. To describe the posture of the rigid bodies of the SMG, once the four motors are

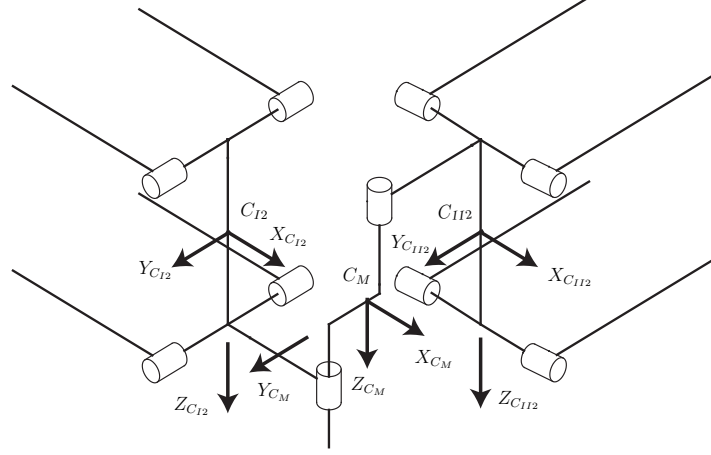


FIGURE 4.2. Coordinate frames for the WB and the MP

locked, the rotation matrices involved are now introduced: $\mathbf{R}_{J1,0}$ is the rotation matrix carrying $\mathcal{F}_{C_{J1}}$ into $\mathcal{F}_O$, namely,

$$\mathbf{R}_{J1,0} = \begin{bmatrix} \cos \theta_{J1} & -\sin \theta_{J1} & 0 \\ \sin \theta_{J1} & \cos \theta_{J1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.14)$$

$\mathbf{R}_{J2,1}$ is the the rotation matrix carrying $\mathcal{F}_{C_{J2}}$ into $\mathcal{F}_{C_{J1}}$, which have identical orientations, and hence,

$$\mathbf{R}_{J2,1} = \mathbf{1} \quad (4.15)$$

where $\mathbf{1}$ is the 3×3 identity matrix. At any posture prior to deformation, the corresponding axes of the two frames are parallel, their Y and Z axes lying in the plane of the J th limb, which explains eq. (4.15). $\mathbf{R}_{M1}$ and $\mathbf{R}_{M2}$ are the rotation matrices that carry $\mathcal{F}_{C_M}$ into $\mathcal{F}_{C_{I1}}$, and $\mathcal{F}_{C_M}$ into $\mathcal{F}_{C_{I2}}$, respectively; these two matrices are identical because the only difference between the frames $\mathcal{F}_{C_{I1}}$ and $\mathcal{F}_{C_{I2}}$ is the location of the origin. Therefore,

$$\mathbf{R}_{M1} = \mathbf{R}_{M2} = \begin{bmatrix} \cos \theta_{I4} & -\sin \theta_{I4} & 0 \\ \sin \theta_{I4} & \cos \theta_{I4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.16)$$

To describe the posture of the rigid bodies after deformation, the rotation matrix carrying $\mathcal{F}'_{C_{J1}}$ into $\mathcal{F}_{C_{J1}}$ is needed; this matrix is displayed below:

$$\mathbf{R}_J = \begin{bmatrix} 1 & -\phi_J e_{zJ1} & \phi_J e_{yJ1} \\ \phi_J e_{zJ1} & 1 & -\phi_J e_{xJ1} \\ -\phi_J e_{yJ1} & \phi_J e_{xJ1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\phi_{Jz} & \phi_{Jy} \\ \phi_{Jz} & 1 & -\phi_{Jx} \\ -\phi_{Jy} & \phi_{Jx} & 1 \end{bmatrix} \quad (4.17)$$

which is dependent on the small-rotation angles and is, hence, posture-dependent. According to eq. (4.17), the matrix $\mathbf{R}_J$ can be defined as

$$\mathbf{R}_J = \text{CPM}(\mathbf{r}_J) + \mathbf{1}, \quad \mathbf{r}_J = \begin{bmatrix} \phi_{Jx} & \phi_{Jy} & \phi_{Jz} \end{bmatrix}^T \quad (4.18)$$

Contrary to $\mathbf{R}_J$, the previous rotation matrices are constant under the small-amplitude assumption.

4.3.1 The Twist Arrays of the Rigid Bodies. Now we proceed to derive the twist arrays of the rigid bodies of the J th limb and of the MP, in base-frame coordinates. For the first rigid body, the elbow bracket, the angular velocity $\boldsymbol{\omega}_{J1}$ is

$$\boldsymbol{\omega}_{J1} = \mathbf{R}_{J1,0} \dot{\mathbf{r}}_{J1} \quad (4.19)$$

where $\dot{\mathbf{r}}_{J1}$ is the angular velocity of the elbow bracket of the J th limb, due to the deformation of the first elastic body of this limb. The displacement $\mathbf{c}_{J1}$ of the COM of the rigid body of the J th limb is, likewise,

$$\mathbf{c}_{J1} = \mathbf{R}_{J1,0} \mathbf{p}_{J1} \quad (4.20)$$

Differentiating $\mathbf{c}_{J1}$ with respect to time, the velocity $\dot{\mathbf{c}}_{J1}$ of the body COM in question is obtained as

$$\dot{\mathbf{c}}_{J1} = \mathbf{R}_{J1,0} \dot{\mathbf{p}}_{J1} \quad (4.21)$$

where $\dot{\mathbf{p}}_{J1}$ is the velocity of the COM of the same body, due to the deformation of the first elastic body of the J th limb. The angular velocity $\boldsymbol{\omega}_{J2}$ of the second rigid

body of the J th limb, the wrist bracket, is

$$\boldsymbol{\omega}_{J2} = \mathbf{R}_{J1,0} \dot{\mathbf{r}}_{J1} + \mathbf{R}_{J1,0} \mathbf{R}_J \dot{\mathbf{r}}_{J2} = \mathbf{R}_{J1,0} (\dot{\mathbf{r}}_{J1} + \mathbf{R}_J \dot{\mathbf{r}}_{J2}) \quad (4.22)$$

where $\dot{\mathbf{r}}_{J2}$ is the angular velocity of the body in question, due to the deformation of the second elastic body of the J th limb. The displacement $\mathbf{c}_{J2}$ of the COM of the second rigid body is, likewise,

$$\begin{aligned} \mathbf{c}_{J2} &= \mathbf{R}_{J1,0} (\mathbf{p}_{J1} + \mathbf{R}_J \mathbf{b}_{J2} + \mathbf{R}_J \mathbf{p}_{J2} - \mathbf{b}_{J2}) \\ &= \mathbf{R}_{J1,0} [\mathbf{p}_{J1} + (\mathbf{R}_J - \mathbf{1}) \mathbf{b}_{J2} + \mathbf{R}_J \mathbf{p}_{J2}] \end{aligned} \quad (4.23)$$

where $\mathbf{b}_{J2}$ is the vector directed from C_{J1} to C_{J2} ; differentiating $\mathbf{c}_{J2}$ with respect to time, the velocity $\dot{\mathbf{c}}_{J2}$ of C_{J2} is obtained as

$$\dot{\mathbf{c}}_{J2} = \mathbf{R}_{J1,0} (\dot{\mathbf{p}}_{J1} + \dot{\mathbf{R}}_J \mathbf{b}_{J2} + \dot{\mathbf{R}}_J \mathbf{p}_{J2} + \mathbf{R}_J \dot{\mathbf{p}}_{J2}) \quad (4.24)$$

where $\dot{\mathbf{p}}_{J2}$ is the velocity of the COM of the WB, due to the deformation of the second elastic body. The angular velocity $\boldsymbol{\omega}_M$ of the MP is

$$\boldsymbol{\omega}_M = \mathbf{R}_{I10} \dot{\mathbf{r}}_{I1} + \mathbf{R}_{I10} \mathbf{R}_I \mathbf{R}_{M1} \dot{\mathbf{r}}_M = \mathbf{R}_{I10} (\dot{\mathbf{r}}_{I1} + \mathbf{R}_I \mathbf{R}_{M1} \dot{\mathbf{r}}_M) \quad (4.25)$$

where $\dot{\mathbf{r}}_M$ is the angular velocity of the MP, due to the deformation of the second elastic body of limb I . The displacement $\mathbf{c}_M$ of the COM of the MP is

$$\begin{aligned} \mathbf{c}_M &= \mathbf{R}_{I10} (\mathbf{p}_{I1} + \mathbf{R}_I \mathbf{b}_M + \mathbf{R}_I \mathbf{R}_{M,1} \mathbf{p}_M - \mathbf{b}_M) \\ &= \mathbf{R}_{I10} [\mathbf{p}_{I1} + (\mathbf{R}_I - \mathbf{1}) \mathbf{b}_M + \mathbf{R}_I \mathbf{R}_{M,1} \mathbf{p}_M] \end{aligned} \quad (4.26)$$

where $\mathbf{p}_M$ is the displacement of the COM of the same body, due to the deformation of the second elastic body of limb I , and $\mathbf{b}_M$ is the vector directed from C_{I1} to C_M ; differentiating $\mathbf{c}_M$ with respect to time, the velocity $\dot{\mathbf{c}}_M$ of the COM of the MP is obtained as

$$\dot{\mathbf{c}}_M = \mathbf{R}_{I10} (\dot{\mathbf{p}}_{I1} + \dot{\mathbf{R}}_I \mathbf{b}_M + \dot{\mathbf{R}}_I \mathbf{R}_{M,1} \mathbf{p}_M + \mathbf{R}_I \mathbf{R}_{M,1} \dot{\mathbf{p}}_M) \quad (4.27)$$

where $\dot{\mathbf{p}}_M$ is the velocity of the COM of the second rigid body, due to the deformation of the second elastic body of limb I . Because of the presence of the term $\mathbf{R}_{J1,0}$ in eq. (4.27), the 6×6 matrix that transforms the components of the six-dimensional twist and wrench arrays, denoted by $\mathbf{T}_{J1,0}$, is

$$\mathbf{T}_{J1,0} = \begin{bmatrix} \mathbf{R}_{J1,0} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_{J1,0} \end{bmatrix} \quad (4.28)$$

By means of the above matrix, the twist array $\mathbf{t}_{J1}$ of the first rigid body of the J th limb is obtained as

$$\mathbf{t}_{J1} = \begin{bmatrix} \boldsymbol{\omega}_{J1} \\ \dot{\mathbf{c}}_{J1} \end{bmatrix} = \mathbf{T}_{J1,0} \dot{\mathbf{q}}_{J1} \quad (4.29)$$

For the second rigid body of the J th limb, the WB, the twist array $\mathbf{t}_{J2}$ is

$$\mathbf{t}_{J2} = \begin{bmatrix} \boldsymbol{\omega}_{J2} \\ \dot{\mathbf{c}}_{J2} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{J1,0} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_{J1,0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_{J1} + \mathbf{R}_J \dot{\mathbf{r}}_{J2} \\ \dot{\mathbf{p}}_{J1} + \dot{\mathbf{R}}_J \mathbf{b}_{J2} + \dot{\mathbf{R}}_J \mathbf{p}_{J2} + \mathbf{R}_J \dot{\mathbf{p}}_{J2} \end{bmatrix} \quad (4.30)$$

which can be rewritten as

$$\mathbf{t}_{J2} = \begin{bmatrix} \mathbf{R}_{J1,0} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_{J1,0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_{J1} + \mathbf{R}_J \dot{\mathbf{r}}_{J2} \\ \dot{\mathbf{p}}_{J1} - \mathbf{B}_{J2} \dot{\mathbf{r}}_{J1} + \mathbf{R}_J \dot{\mathbf{p}}_{J2} \end{bmatrix} \quad (4.31)$$

with $\mathbf{B}_{J2}$ denoting the CPM of $\mathbf{b}_{J2} + \mathbf{p}_{J2}$. The 6×6 matrices $\mathbf{T}_J$ and $\mathbf{Q}_{J2}$ are now introduced:

$$\mathbf{T}_J = \begin{bmatrix} \mathbf{R}_J & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_J \end{bmatrix}, \quad \mathbf{Q}_{J2} = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{B}_{J2} & \mathbf{O} \end{bmatrix} \quad (4.32)$$

The twist array $\mathbf{t}_{J2}$ of the WB of the J th limb can be written as a function of the generalized velocities, namely,

$$\mathbf{t}_{J2} = \mathbf{T}_{J1,0} (\mathbf{1} - \mathbf{Q}_{J2}) \dot{\mathbf{q}}_{J1} + \mathbf{T}_{J1,0} \mathbf{T}_J \dot{\mathbf{q}}_{J2} \quad (4.33)$$

For the MP, the twist array $\mathbf{t}_M$ is obtained as

$$\mathbf{t}_M = \begin{bmatrix} \boldsymbol{\omega}_M \\ \dot{\mathbf{c}}_M \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{I10} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_{I10} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_{I1} + \mathbf{R}_I \mathbf{R}_{M,1} \dot{\mathbf{r}}_M \\ \dot{\mathbf{p}}_{I1} + \dot{\mathbf{R}}_I \mathbf{b}_M + \dot{\mathbf{R}}_I \mathbf{R}_{M,1} \mathbf{p}_M + \mathbf{R}_I \mathbf{R}_{M,1} \dot{\mathbf{p}}_M \end{bmatrix} \quad (4.34)$$

which can be also rewritten as

$$\mathbf{t}_M = \begin{bmatrix} \mathbf{R}_{I10} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_{I10} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_{I1} + \mathbf{R}_I \mathbf{R}_{M,1} \dot{\mathbf{r}}_M \\ \dot{\mathbf{p}}_{I1} - \mathbf{B}_M \dot{\mathbf{r}}_{I1} + \mathbf{R}_I \mathbf{R}_{M,1} \dot{\mathbf{p}}_M \end{bmatrix} \quad (4.35)$$

with $\mathbf{B}_M$ denoting the CPM of $\mathbf{b}_M + \mathbf{R}_{M,1} \mathbf{p}_M$. Let the 6×6 matrices $\mathbf{T}_{M,1}$ and $\mathbf{Q}_M$ be

$$\mathbf{T}_{M,1} = \begin{bmatrix} \mathbf{R}_{M,1} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_{M,1} \end{bmatrix}, \quad \mathbf{Q}_M = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{B}_M & \mathbf{O} \end{bmatrix} \quad (4.36)$$

Hence, the twist array $\mathbf{t}_M$ of the MP can be expressed as a function of the generalized velocities, namely,

$$\mathbf{t}_M = \mathbf{T}_{I10} (\mathbf{1} - \mathbf{Q}_M) \dot{\mathbf{q}}_{I1} + \mathbf{T}_{I10} \mathbf{T}_I \mathbf{T}_{M,1} \dot{\mathbf{q}}_M \quad (4.37)$$

4.3.2 Kinetic Energy of the McGill SMG. Now it is possible to formulate the kinetic energy of each rigid body. For the whole robot, the kinetic energy T is the sum of the kinetic energies of: the EBs T_{J1} ; the WBs T_{J2} , of the J th limb, for $J = I, II$; and the MP T_M , i.e.,

$$T = \sum_{J=I,II} (T_{J1} + T_{J2}) + T_M \quad (4.38)$$

with

$$T_{J1} = \frac{1}{2} m_{J1} \|\dot{\mathbf{c}}_{J1}\|^2 + \frac{1}{2} \boldsymbol{\omega}_{J1}^T (\mathbf{R}_{J1,0}^T [\mathbf{I}_{C_{J1}}]_{C_O} \mathbf{R}_{J1,0}) \boldsymbol{\omega}_{J1} \quad (4.39)$$

where m_{J1} and $[\mathbf{I}_{C_{J1}}]_{C_O}$ denote, respectively, the mass and the moment of inertia matrix of the EB of the J th limb with respect to the fixed reference frame; moreover,

$$T_{J2} = \frac{1}{2} m_{J2} \|\dot{\mathbf{c}}_{J2}\|^2 + \frac{1}{2} \omega_{J2}^T (\mathbf{R}_{J1,0}^T \mathbf{R}_J^T [\mathbf{I}_{C_{J2}}]_{C_O} \mathbf{R}_J \mathbf{R}_{J1,0}) \omega_{J2} \quad (4.40)$$

where m_{J2} and $[\mathbf{I}_{C_{J2}}]_{C_O}$ denote, respectively, the mass and the moment of inertia matrix of the WB of the J th limb with respect to the fixed reference frame, the later in $\mathcal{F}_{C_{J2}}$, and

$$T_M = \frac{1}{2} m_M \|\dot{\mathbf{c}}_M\|^2 + \frac{1}{2} \boldsymbol{\omega}_M^T (\mathbf{R}_{I10}^T \mathbf{R}_I^T \mathbf{R}_{M,1}^T [\mathbf{I}_{C_M}]_{C_O} \mathbf{R}_{M,1} \mathbf{R}_I \mathbf{R}_{I10}) \boldsymbol{\omega}_M \quad (4.41)$$

where m_M and $\mathbf{I}_{C_M}$ denote, respectively, the mass and the moment of inertia matrix of the MP in the fixed reference frame. All the forgoing inertia matrices are evaluated at the local reference frame with origin at the COM of the corresponding body before deformation. It is thus necessary to define these matrices in the fixed reference frame, which is done below:

$$\begin{aligned} [\mathbf{I}_{C_{J1}}]_{C_O} &= \mathbf{R}_{J1,0} \mathbf{I}_{C_{J1}} \mathbf{R}_{J1,0}^T, \quad [\mathbf{I}_{C_{J2}}]_{C_O} = \mathbf{R}_{J1,0} \mathbf{R}_J \mathbf{I}_{C_{J2}} \mathbf{R}_J^T \mathbf{R}_{J1,0}^T \\ [\mathbf{I}_{C_M}]_{C_O} &= \mathbf{R}_{I10} \mathbf{R}_I \mathbf{R}_{M,1} \mathbf{I}_{C_M} \mathbf{R}_{M,1}^T \mathbf{R}_I^T \mathbf{R}_{I10}^T \end{aligned} \quad (4.42)$$

Using the relations of eq. (4.42), the expression for the kinetic energy of the whole robot is obtained as:

$$\begin{aligned} T &= \sum_{J=I,II} \left(\frac{1}{2} m_{J1} \|\dot{\mathbf{c}}_{J1}\|^2 + \frac{1}{2} \boldsymbol{\omega}_{J1}^T \mathbf{I}_{C_{J1}} \boldsymbol{\omega}_{J1} + \frac{1}{2} m_{J2} \|\dot{\mathbf{c}}_{J2}\|^2 + \frac{1}{2} \boldsymbol{\omega}_{J2}^T \mathbf{I}_{C_{J2}} \boldsymbol{\omega}_{J2} \right) \\ &\quad + \frac{1}{2} m_M \|\dot{\mathbf{c}}_M\|^2 + \frac{1}{2} \boldsymbol{\omega}_M^T [\mathbf{I}_{C_M}]_{C_O} \boldsymbol{\omega}_M \end{aligned} \quad (4.43)$$

Let us now introduce the 6×6 mass matrices $\mathbf{M}_{J1}$, $\mathbf{M}_{J2}$ of the EBs and WBs for the J th limb, respectively, and $\mathbf{M}_M$ for the MP:

$$\mathbf{M}_{J1} = \begin{bmatrix} \mathbf{I}_{C_{J1}} & \mathbf{O} \\ \mathbf{O} & m_{J1} \mathbf{1} \end{bmatrix}, \quad \mathbf{M}_{J2} = \begin{bmatrix} \mathbf{I}_{C_{J2}} & \mathbf{O} \\ \mathbf{O} & m_{J2} \mathbf{1} \end{bmatrix}, \quad \mathbf{M}_M = \begin{bmatrix} \mathbf{I}_{C_M} & \mathbf{O} \\ \mathbf{O} & m_M \mathbf{1} \end{bmatrix} \quad (4.44)$$

where $\mathbf{O}$ is the 3×3 zero matrix; using the above-defined twist arrays as functions of the generalized velocities, the expression of the kinetic energy is obtained:

$$\begin{aligned} T &= \sum_{J=I,II} \left(\frac{1}{2} \dot{\mathbf{q}}_{J1}^T \left[\mathbf{T}_{J1,0}^T \mathbf{M}_{J1} \mathbf{T}_{J1,0} + (\mathbf{1} - \mathbf{Q}_{J2})^T \mathbf{T}_{J1,0}^T \mathbf{M}_{J2} \mathbf{T}_{J1,0} (\mathbf{1} - \mathbf{Q}_{J2}) \right] \dot{\mathbf{q}}_{J1} \right. \\ &\quad \left. + \dot{\mathbf{q}}_{J1}^T (\mathbf{1} - \mathbf{Q}_{J2})^T \mathbf{T}_{J1,0}^T \mathbf{M}_{J2} \mathbf{T}_{J1,0} \mathbf{T}_J \dot{\mathbf{q}}_{J2} + \frac{1}{2} \dot{\mathbf{q}}_{J2}^T \mathbf{T}_J^T \mathbf{T}_{J1,0}^T \mathbf{M}_{J2} \mathbf{T}_{J1,0} \mathbf{T}_J \dot{\mathbf{q}}_{J2} \right) \\ &\quad + \frac{1}{2} \dot{\mathbf{q}}_{I1}^T (\mathbf{1} - \mathbf{Q}_M)^T \mathbf{T}_{I1,0}^T \mathbf{M}_M (\mathbf{1} - \mathbf{Q}_M) \dot{\mathbf{q}}_{I1} \\ &\quad + \frac{1}{2} 2 \dot{\mathbf{q}}_{I1}^T (\mathbf{1} - \mathbf{Q}_M)^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} \mathbf{T}_I \mathbf{T}_{M,1} \dot{\mathbf{q}}_M \\ &\quad + \frac{1}{2} \dot{\mathbf{q}}_M^T \mathbf{T}_{M,1}^T \mathbf{T}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} \mathbf{T}_I \mathbf{T}_{M,1} \dot{\mathbf{q}}_M \end{aligned} \quad (4.45)$$

Moreover, upon differentiation of the two sides of relation (3.34) with respect to time, the generalized-velocity vectors of the WBs are obtained as functions of the generalized velocity vectors of the MP and the revolute joint rates, namely,

$$\dot{\mathbf{q}}_{J2} = \mathbf{G}_J(\mathbf{H}_J\dot{\mathbf{q}}_M + \dot{\gamma}_J\boldsymbol{\zeta}) \quad (4.46)$$

Now, by substituting relation (4.46) into eq. (4.45) and multiplying the generalized velocities in the parentheses we obtain a new expression for the kinetic energy, namely,

$$\begin{aligned} T = & \frac{1}{2}\dot{\mathbf{q}}_{I1}^T \left[\mathbf{T}_{I1,0}^T \mathbf{M}_{I1} \mathbf{T}_{I10} + (\mathbf{1} - \mathbf{Q}_{I2})^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} (\mathbf{1} - \mathbf{Q}_{I2}) \right. \\ & \left. + (\mathbf{1} - \mathbf{Q}_M)^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} (\mathbf{1} - \mathbf{Q}_M) \right] \dot{\mathbf{q}}_{I1} \\ & + \frac{1}{2}2\dot{\mathbf{q}}_{I1}^T \left[(\mathbf{1} - \mathbf{A}_{I2})^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{G}_I \mathbf{H}_I + (\mathbf{1} - \mathbf{Q}_M)^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} \mathbf{T}_I \mathbf{T}_{M,1} \right] \dot{\mathbf{q}}_M \\ & + \frac{1}{2}2\dot{\mathbf{q}}_{I1}^T (\mathbf{1} - \mathbf{Q}_{I2})^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{G}_I \boldsymbol{\zeta} \dot{\gamma}_I \\ & + \frac{1}{2}\dot{\mathbf{q}}_M^T (\mathbf{H}_I^T \mathbf{G}_I^T \mathbf{T}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{G}_I \mathbf{H}_I + \mathbf{T}_{M,1}^T \mathbf{T}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} \mathbf{T}_I \mathbf{T}_{M,1} \\ & \quad + \mathbf{H}_{II}^T \mathbf{G}_{II}^T \mathbf{T}_{II}^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{G}_{II} \mathbf{H}_{II}) \dot{\mathbf{q}}_M \\ & + \frac{1}{2}2\dot{\mathbf{q}}_M^T \mathbf{H}_I^T \mathbf{G}_I^T \mathbf{T}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{G}_I \boldsymbol{\zeta} \dot{\gamma}_I + \frac{1}{2}\dot{\gamma}_I \boldsymbol{\zeta}^T \mathbf{G}_I^T \mathbf{T}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{G}_I \boldsymbol{\zeta} \dot{\gamma}_I \\ & + \frac{1}{2}\dot{\mathbf{q}}_{II1}^T \left[\mathbf{T}_{II1,0}^T \mathbf{M}_{II1} \mathbf{T}_{II10} + (\mathbf{1} - \mathbf{Q}_{II2})^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} (\mathbf{1} - \mathbf{Q}_{II2}) \right] \dot{\mathbf{q}}_{II1} \\ & + \frac{1}{2}2\dot{\mathbf{q}}_{II1}^T (\mathbf{1} - \mathbf{Q}_{II2})^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{G}_{II} \mathbf{H}_{II} \dot{\mathbf{q}}_M \\ & + \frac{1}{2}2\dot{\mathbf{q}}_{II1}^T (\mathbf{1} - \mathbf{Q}_{II2})^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{G}_{II} \boldsymbol{\zeta} \dot{\gamma}_{II} \\ & + \frac{1}{2}2\dot{\mathbf{q}}_M^T \mathbf{H}_{II}^T \mathbf{G}_{II}^T \mathbf{T}_{II}^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{G}_{II} \boldsymbol{\zeta} \dot{\gamma}_{II} \\ & + \frac{1}{2}\dot{\gamma}_{II}^2 \boldsymbol{\zeta}^T \mathbf{G}_{II}^T \mathbf{T}_{II}^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{G}_{II} \boldsymbol{\zeta} \end{aligned} \quad (4.47)$$

In eq. (4.47) the kinetic energy appears as a quadratic form in the generalized velocities:

$$T = \frac{1}{2}\mathbf{q}^T \mathbf{M} \mathbf{q} \quad (4.48)$$

the associated 20×20 matrix being the mass matrix sought, i.e.,

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{m}_{12} & \mathbf{O}_{6 \times 6} & \mathbf{0}_6 & \mathbf{M}_{15} \\ \mathbf{m}_{12}^T & m_{22} & \mathbf{0}_6^T & 0 & \mathbf{m}_{25} \\ \mathbf{O}_{6 \times 6} & \mathbf{0}_6 & \mathbf{M}_{33} & \mathbf{m}_{34} & \mathbf{M}_{35} \\ \mathbf{0}_6^T & 0 & \mathbf{m}_{34}^T & m_{44} & \mathbf{m}_{45} \\ \mathbf{M}_{15}^T & \mathbf{m}_{25}^T & \mathbf{M}_{53}^T & \mathbf{m}_{45}^T & \mathbf{M}_{55} \end{bmatrix}$$

The various blocks of the mass matrix are displayed below:

$$\begin{aligned} \mathbf{M}_{11} &= \mathbf{T}_{I1,0}^T \mathbf{M}_{I1} \mathbf{T}_{I10} + (\mathbf{1} - \mathbf{Q}_{I2})^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} (\mathbf{1} - \mathbf{Q}_{I2}) \\ &\quad + (\mathbf{1} - \mathbf{Q}_M)^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} (\mathbf{1} - \mathbf{Q}_M) \\ \mathbf{M}_{15} &= (\mathbf{1} - \mathbf{Q}_{I2})^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{G}_I \mathbf{H}_I + (\mathbf{1} - \mathbf{Q}_M)^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} \mathbf{T}_I \mathbf{T}_{M,1} \\ \mathbf{m}_{12} &= (\mathbf{1} - \mathbf{Q}_{I2})^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{G}_I \boldsymbol{\zeta} \\ \mathbf{M}_{55} &= \mathbf{H}_I^T \mathbf{G}_I^T \mathbf{T}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{G}_I \mathbf{H}_I + \mathbf{T}_{M,1}^T \mathbf{T}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} \mathbf{T}_I \mathbf{T}_{M,1} \\ &\quad + \mathbf{H}_{II}^T \mathbf{G}_{II}^T \mathbf{T}_{II}^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{G}_{II} \mathbf{H}_{II} \\ \mathbf{m}_{25} &= \boldsymbol{\zeta}^T \mathbf{G}_I^T \mathbf{T}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{H}_I \mathbf{G}_I \\ m_{22} &= \boldsymbol{\zeta}^T \mathbf{G}_I^T \mathbf{T}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{T}_I \mathbf{G}_I \boldsymbol{\zeta} \\ \mathbf{M}_{33} &= \mathbf{T}_{II1,0}^T \mathbf{M}_{II1} \mathbf{T}_{II10} + (\mathbf{1} - \mathbf{Q}_{II2})^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} (\mathbf{1} - \mathbf{Q}_{II2}) \\ \mathbf{M}_{35} &= (\mathbf{1} - \mathbf{Q}_{II2})^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{G}_{II} \mathbf{H}_{II} \\ \mathbf{m}_{34} &= (\mathbf{1} - \mathbf{Q}_{II2})^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{G}_{II} \boldsymbol{\zeta} \\ \mathbf{m}_{45} &= \boldsymbol{\zeta}^T \mathbf{G}_{II}^T \mathbf{T}_{II}^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{H}_{II}^T \mathbf{G}_{II}^T \\ m_{44} &= \boldsymbol{\zeta}^T \mathbf{G}_{II}^T \mathbf{T}_{II}^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{T}_{II} \mathbf{G}_{II} \boldsymbol{\zeta} \end{aligned} \tag{4.49}$$

Upon recalling the notation introduced in eq. (4.9) the kinetic energy expression can be cast in compact form:

$$T = \frac{1}{2} \dot{\mathbf{x}}^T \mathbf{M} \dot{\mathbf{x}}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{x}}_I \\ \dot{\mathbf{x}}_{II} \\ \dot{\mathbf{q}}_M \end{bmatrix} \quad (4.50)$$

where the mass matrix takes the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_I & \mathbf{O}_{7 \times 7} & \mathbf{M}_{IM} \\ \mathbf{O}_{7 \times 7} & \mathbf{M}_{II} & \mathbf{M}_{IIM} \\ \mathbf{M}_{IM}^T & \mathbf{M}_{IIM}^T & \mathbf{M}_M \end{bmatrix} \quad (4.51)$$

with

$$\begin{aligned} \mathbf{M}_I &= \begin{bmatrix} \mathbf{M}_{11} & \mathbf{m}_{12} \\ \mathbf{m}_{12}^T & m_{22} \end{bmatrix}, \quad \mathbf{M}_{II} = \begin{bmatrix} \mathbf{M}_{33} & \mathbf{m}_{34} \\ \mathbf{m}_{34}^T & m_{44} \end{bmatrix}, \quad \mathbf{M}_M = \mathbf{M}_{55} \\ \mathbf{M}_{IM} &= \begin{bmatrix} \mathbf{M}_{15} \\ \mathbf{m}_{25} \end{bmatrix}, \quad \mathbf{M}_{IIM} = \begin{bmatrix} \mathbf{M}_{35} \\ \mathbf{m}_{45} \end{bmatrix} \end{aligned} \quad (4.52)$$

4.3.3 Model Linearization. Before introducing the mass matrix into the linear elastodynamics model, some simplifications are in order. Observing the blocks of the mass matrix in eq. (4.49), some carry the coordinate-transformation matrix $\mathbf{T}_J$; in particular, the diagonal blocks are quadratic in $\mathbf{T}_J$, while some off-diagonal blocks are linear in the same transformation matrix. Taking eqs. (4.17) and (4.32) into account, $\mathbf{T}_J$ can be written as

$$\mathbf{T}_J = \mathbf{1} + \Delta \mathbf{T}_J = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{bmatrix} + \begin{bmatrix} \text{CPM}(\mathbf{r}_J) & \mathbf{O} \\ \mathbf{O} & \text{CPM}(\mathbf{r}_J) \end{bmatrix} \quad (4.53)$$

where $\mathbf{R}_J$ was defined in eq. (4.18). As the modal analysis is based on small-amplitude rotations, second-order terms can be dropped from the above expression. For the blocks linear in $\mathbf{T}_J$, the linear vibration model is recalled. If damping is neglected, this model reduces to

$$\mathbf{M}(\mathbf{x}) \delta \ddot{\mathbf{x}} + \mathbf{K}(\mathbf{x}) \delta \mathbf{x} = \mathbf{0} \quad (4.54)$$

with $\mathbf{K}(\mathbf{x})$ denoting the stiffness matrix, while $\delta\mathbf{x}$ and $\delta\ddot{\mathbf{x}}$, are the arrays of “small” generalized coordinates and “small” generalized accelerations, respectively. In eq. (4.54) both matrices $\mathbf{M}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ are posture-dependent, but the vibration analysis is studied for a specific posture given by $\mathbf{x} = \mathbf{x}_0$, so that these matrices become constant; the above equation can thus be rewritten as

$$\mathbf{M}_0\delta\ddot{\mathbf{x}} + \mathbf{K}_0\delta\mathbf{x} = \mathbf{0} \quad (4.55)$$

where $\mathbf{M}_0 = \mathbf{M}(\mathbf{x}_0)$ and $\mathbf{K}_0 = \mathbf{K}(\mathbf{x}_0)$. Now it is apparent that all the blocks of the mass matrix in eq. (4.55) are multiplied by a small generalized acceleration, the contribution of the full $\mathbf{T}_J$ matrix of eq. (4.53) in all the off-diagonal blocks of the left-hand side of eq. (4.55) thus giving rise to second-order terms, which are neglected in the elastodynamics model of interest, thereby obtaining the approximation of $\mathbf{T}_J$ given by

$$\mathbf{T}_J \approx \mathbf{1} \quad (4.56)$$

Moreover, the rigid body displacements are of “small” amplitude; this means that the rigid body displacements are much smaller than the robot dimensions, i.e.,

$$\mathbf{p}_{J2}, \mathbf{p}_M \ll \mathbf{b}_{J2} \quad (4.57)$$

Hence, for matters of linearization, in addition to the approximation of $\mathbf{T}_J$, $\mathbf{B}_{J2}$ and $\mathbf{B}_M$ are also approximated below:

$$\mathbf{B}_{J2} = \text{CPM}(\mathbf{b}_{J2} + \mathbf{p}_{J2}) \approx \text{CPM}(\mathbf{b}_{J2}) \quad \mathbf{B}_M = \text{CPM}(\mathbf{b}_{J2} + \mathbf{R}_{M,1}\mathbf{p}_M) \approx \text{CPM}(\mathbf{b}_M) \quad (4.58)$$

This consideration can be applied to matrices $\mathbf{Q}_M$ and $\mathbf{Q}_{J2}$, defined in eqs. (4.32) and (4.36), as well. Now the blocks of the mass matrix $\mathbf{M}$ can be rewritten as

$$\begin{aligned}
\mathbf{M}_{11} &= \mathbf{T}_{I1,0}^T \mathbf{M}_{I1} \mathbf{T}_{I10} + (\mathbf{1} - \mathbf{Q}_{I2})^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} (\mathbf{1} - \mathbf{Q}_{I2}) \\
&\quad + (\mathbf{1} - \mathbf{Q}_M)^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} (\mathbf{1} - \mathbf{Q}_M) \\
\mathbf{m}_{12} &= (\mathbf{1} - \mathbf{Q}_{I2})^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{G}_I \boldsymbol{\zeta} \\
\mathbf{M}_{15} &= (\mathbf{1} - \mathbf{Q}_{I2})^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{G}_I \mathbf{H}_I + (\mathbf{1} - \mathbf{Q}_M)^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} \mathbf{T}_{M,1} \\
m_{22} &= \boldsymbol{\zeta}^T \mathbf{G}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{G}_I \boldsymbol{\zeta} \\
\mathbf{m}_{25} &= \boldsymbol{\zeta}^T \mathbf{G}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{G}_I \mathbf{H}_I \\
\mathbf{M}_{33} &= \mathbf{T}_{II1,0}^T \mathbf{M}_{II1} \mathbf{T}_{II10} + (\mathbf{1} - \mathbf{Q}_{II2})^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} (\mathbf{1} - \mathbf{Q}_{II2}) \\
\mathbf{m}_{34} &= (\mathbf{1} - \mathbf{Q}_{II2})^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{G}_{II} \boldsymbol{\zeta} \\
\mathbf{M}_{35} &= (\mathbf{1} - \mathbf{Q}_{II2})^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{G}_{II} \mathbf{H}_{II} \\
m_{44} &= \boldsymbol{\zeta}^T \mathbf{G}_{II}^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{G}_{II} \boldsymbol{\zeta} \\
\mathbf{m}_{45} &= \mathbf{H}_{II}^T \mathbf{G}_{II}^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{G}_{II} \boldsymbol{\zeta} \\
\mathbf{M}_{55} &= \mathbf{H}_I^T \mathbf{G}_I^T \mathbf{T}_{I1,0}^T \mathbf{M}_{I2} \mathbf{T}_{I10} \mathbf{G}_I \mathbf{H}_I + \mathbf{T}_{M,1}^T \mathbf{T}_{I1,0}^T \mathbf{M}_M \mathbf{T}_{I10} \mathbf{T}_{M,1} \\
&\quad + \mathbf{H}_{II}^T \mathbf{G}_{II}^T \mathbf{T}_{II1,0}^T \mathbf{M}_{II2} \mathbf{T}_{II10} \mathbf{G}_{II} \mathbf{H}_{II}
\end{aligned} \tag{4.59}$$

4.4 Modal Analysis

After calculating the mass and stiffness matrices at an equilibrium state, the natural frequencies of the robot can be obtained. The mathematical model of the robot, at the equilibrium state, is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{4.60}$$

with $\mathbf{M}$ denoting the 20×20 mass matrix and $\mathbf{K}$ the 20×20 stiffness matrix evaluated at the equilibrium posture, both being constant, symmetric and positive-definite; moreover, for the sake of notation simplicity, the symbol δ has been dropped in the foregoing model. Since $\mathbf{M}$ is positive-definite, a factor $\mathbf{N}$ can be obtained as (Angeles

et al., 1999)

$$\mathbf{M} = \mathbf{N}^2, \quad \mathbf{N} \equiv \sqrt{\mathbf{M}} \quad (4.61)$$

To calculate the natural frequencies a change of coordinates is introduced in the foregoing reference:

$$\mathbf{y} = \mathbf{N}\mathbf{x} \quad (4.62)$$

Upon substitution of eq. (4.62) into eq. (4.60), the latter becomes

$$\ddot{\mathbf{y}} + \mathbf{\Omega}^2 \mathbf{y} = \mathbf{0} \quad (4.63)$$

where $\mathbf{\Omega}$ is the positive-definite *frequency matrix*, its square being defined as

$$\mathbf{\Omega}^2 = \mathbf{N}^{-1} \mathbf{K} \mathbf{N}^{-1} \quad (4.64)$$

The natural frequencies of the robot $\{\omega_i\}_1^{20}$ are the eigenvalues of $\mathbf{\Omega}$, the natural modes being defined by the set $\{\boldsymbol{\nu}_i\}_1^{20}$, with $\boldsymbol{\nu}_i$ given by linear transformations of the unit eigenvectors $\{\mathbf{v}_i\}_1^{20}$ — $\mathbf{v}$ is the Greek letter ypsilon—of $\mathbf{\Omega}$ i.e.,

$$\boldsymbol{\nu}_i = \mathbf{N}^{-1} \mathbf{v}_i \quad (4.65)$$

4.5 Fourier Analysis of the Test Cycle

For design purposes it is important to identify the excitation frequencies applied on the system. For a periodic motion, Fourier analysis provides the frequency spectrum of the motion. A periodic function $f(t)$ with a fundamental frequency ω can be represented in the standard form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \quad (4.66)$$

where

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt, \quad a_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega t) dt, \quad b_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega t) dt \quad (4.67)$$

in which n and T are the harmonic number and the period of function $f(t)$, respectively. As depicted in Fig. 2.15, the MP starts the motion along the trajectory from $(x, y, z, \phi) = (0, 0.15, 0.025, -\pi/2)$ at $t = 0$ and completes half a cycle at $(x, y, z, \phi) = (0, -0.15, 0.025, \pi/2)$. In the next half, it goes back over the same trajectory from the latter to the former position. This implies that the test cycle is composed of even periodic functions with a period equal to or a fraction of that of the test cycle. For instance, in the foregoing case, the test cycle is made of three even periodic functions, two translations along the Y - and Z -axes with the time period of $T_y = T$ and $T_z = T/2$, respectively, and a rotation about the Z -axis with the time period of $T_\phi = T$. As an even function is described only by cosine waves, the coefficients $b_{zn}, b_{yn}, b_{\phi n}$ of eq. (4.66) vanish. By choosing a finite number of harmonics N_h , a periodic function can be approximated, within an error e_{N_h} . By resorting to *Parseval's Theorem* (Strang, 1986), this error is given by

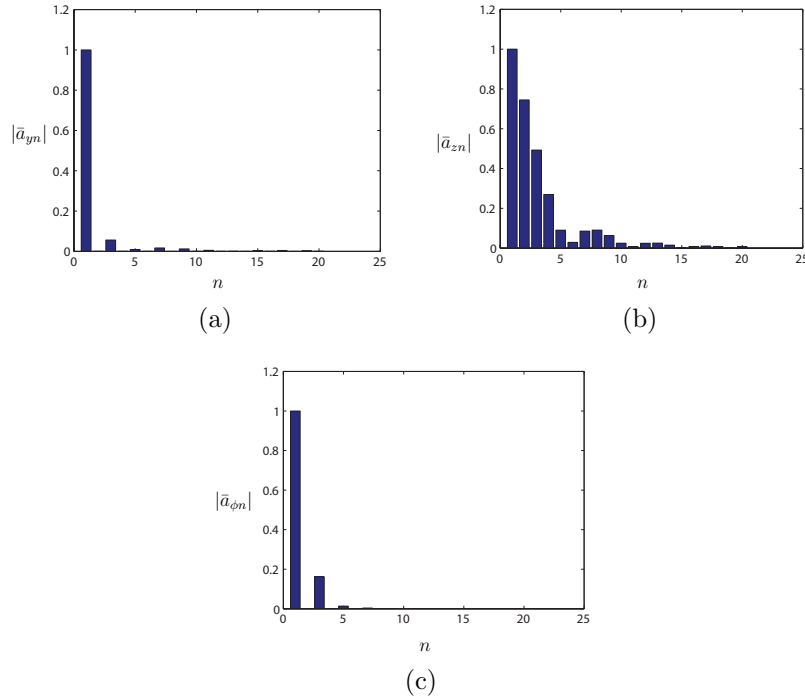


FIGURE 4.3. Amplitudes of the harmonics of the three independent motions vs. the frequency number n : (a) translation along the Y -axis; (b) translation along the Z -axis; (c) rotation about the Z -axis

$$e_{N_h} = [f(t)]_{\text{rms}} - \sqrt{a_0^2 + \frac{1}{2} \sum_{n=1}^{N_h} a_n^2 + \frac{1}{2} \sum_{n=1}^{N_h} b_n^2} > 0 \quad (4.68)$$

with $[f(t)]_{\text{rms}}$ defined as the *root-mean square* of $f(t)$, i.e.,

$$[f(t)]_{\text{rms}} = \sqrt{\frac{1}{T} \int_{-T/2}^{T/2} f^2(t) dt} \quad (4.69)$$

By resorting to eq. (4.68), for a maximum error of 0.001 for each function, $e_{N_h}^y$, $e_{N_h}^z$, and $e_{N_h}^\phi < 0.001$, the minimum number of harmonics $N_h = 20$ is obtained. In Figs. 4.3(a), 4.3(b) and 4.3(c) the absolute value distributions of normalized $\bar{a}_{zn}$, $\bar{a}_{yn}$, and $\bar{a}_{\phi n}$ over the harmonic number n are plotted. Apparently, the translation along the Z -axis is affected more than the two other motions by harmonics with high frequency. As depicted in the same plot, fundamentals with frequency number above three contribute below 1% of the fundamental harmonic frequency to the translation along the Y -axis and rotation about the Z -axis, while, the translational motion along the Z -axis is still affected around 6% by the fifth harmonic, as shown in Fig. 4.3(b). This result implies that a harmful vibration can be produced on a robot with a fundametal frequency below 5ω while working under the foregoing test cycle.

4.6 Numerical Results

Now we let the McGill SMG operate over the smoothed test trajectory displayed in Fig. 2.14. The evolution of the first six natural frequencies is shown in Figs. 4.4. As plotted in Fig. 4.4(a), when the MP is at the middle of the test trajectory, where the two limbs are coplanar, the first natural frequency is at its minimum value of around 13 Hz. The corresponding frequency numbers for $y(t)$, $z(t)$ and $\phi(t)$ are 6, 3, and 6, respectively. For translation along the Y -axis and rotation about the Z -axis the McGill SMG is on the safe side; however, for the translation along the Z -axis, the normalized amplitude $\bar{a}_{zn}$ of the corresponding harmonic contributes 50% to the motion; this means that an external excitation with a frequency of 13 Hz can stimulate this mode, which can lead to a harmful vibration.

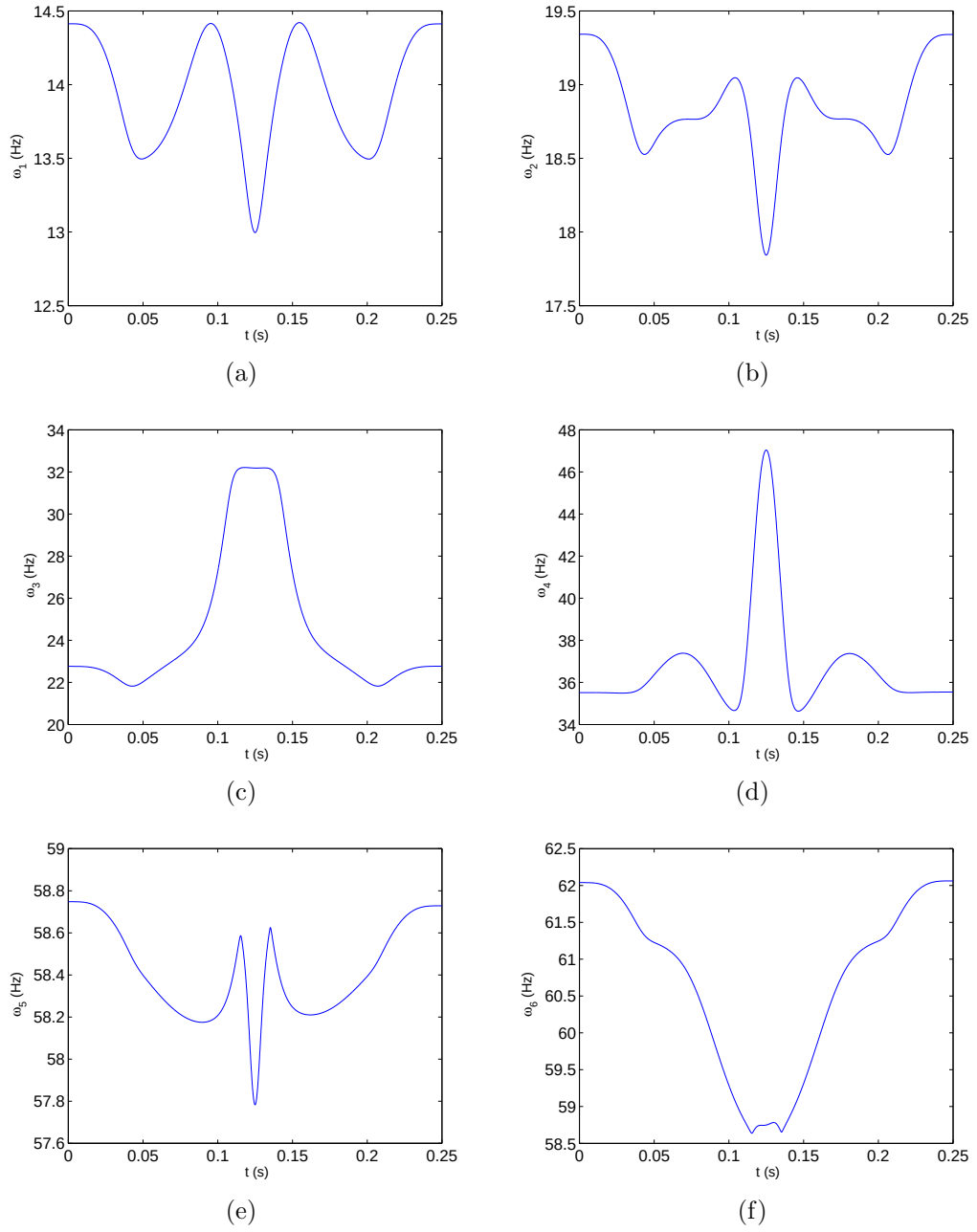


FIGURE 4.4. The evolution of the natural frequencies of the McGill SMG over the test trajectory: (a) first natural frequency, ω_1 (Hz); (b) second natural frequency, ω_2 (Hz); (c) third natural frequency, ω_3 (Hz); (d) fourth natural frequency, ω_4 (Hz); (e) fifth natural frequency, ω_5 (Hz); (f) sixth natural frequency, ω_6 (Hz)

CHAPTER 5

Optimum Structural Design of the McGill SMG

Obtaining the optimum dimensions of the McGill SMG to maximize the overall stiffness of the robot structure is the subject of this chapter. In six-dimensional Cartesian space, for a mechanical system, six independent stiffnesses can be defined: three translational along independent directions and three rotational about non-coplanar axes. In this study, the objective is to maximize the maximum translational and rotational stiffnesses. To this end, the stiffness matrix of the robot is obtained using the concept of the generalized spring, introduced in Ch. 3; the translational and rotational stiffness indices κ_t and κ_r , respectively, are considered as the two objective functions. In order to avoid obtaining a design with unacceptable workspace volume, a pertinent constraint should be introduced. Hence, it is assumed that, regardless of its dimensions, the robot should be able to pick up an object located at a specific pose within a prescribed workspace. For the optimization procedure, a genetic algorithm (GA) is used. In the end, three different designs are obtained, their stiffness performance over a test trajectory then being analyzed.

5.1 A Multi-objective Optimization Problem

A multi-objective optimization problem consists of a number of independent objective functions that are to be minimized or maximized (Deb, 2001). In a general form, the problem can be defined as,

$$\begin{aligned}
 \min_{\mathbf{x}} \quad & f_m(\mathbf{x}), \quad m = 1, \dots, M \\
 & g_j(\mathbf{x}) \geq 0, \quad j = 1, \dots, J \\
 & h_k(\mathbf{x}) = 0, \quad k = 1, \dots, K \\
 & x_i^L < x_i < x_i^U, \quad i = 1, \dots, N
 \end{aligned} \tag{5.1}$$

in which $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_N]^T$ is the *design variable vector*; the objective functions can also be arrayed in a vector, $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \ f_2(\mathbf{x}) \ \dots \ f_M(\mathbf{x})]^T$, which is called the *vector of objectives*. Accordingly, g_j and h_k are inequality and equality constraint functions, respectively. Moreover, x_i^L and x_i^U are design-variable lower and upper bounds, respectively. These bounds define a hyperparallelepiped $\mathcal{R}$ in the design-variable space. The objective functions are most frequently conflicting, and hence, it is rarely possible to find a unique solution that simultaneously optimizes all objectives. Therefore, in these cases, the concept of *domination* is used (Deb, 2001). According to this concept, a solution $\mathbf{x}_1$ dominates $\mathbf{x}_2$ ($\mathbf{x}_1 \succ \mathbf{x}_2$) if $\mathbf{x}_1$ is better or equal to $\mathbf{x}_2$ in all objectives, and strictly better in at least one objective. That is, for M objectives to be minimized, one has

$$\begin{aligned}
 f_m(\mathbf{x}_1) &\leq f_m(\mathbf{x}_2), \quad \forall m = 1, \dots, M \\
 \exists m \mid f_m(\mathbf{x}_1) &< f_m(\mathbf{x}_2), \quad m = 1, \dots, M
 \end{aligned} \tag{5.2}$$

A set of dominant solutions and the corresponding set of objective vectors are known as the *Pareto optimal solutions* and the *Pareto optimal frontier*, respectively. In a design optimization problem, probably, the final design should be a member of the Pareto set.

5.2 Genetic Algorithms (GA)

GA and evolutionary algorithms were first introduced by Rechenberg (1973); Holland (1975). In the late 80's, Goldberg (1989) developed GA for solving an engineering optimization problem. GA are based on the process of natural selection. Each optimization parameter is encoded into a gene, i.e., a real number or string of bits. Then, a chromosome is formed by the corresponding genes for all variables $x_1, x_2, \dots, x_N$; each chromosome describes an individual. Depending on the nature of a problem, a chromosome could be defined as an array of real numbers or a binary string. Finally, the individuals, which represent possible solutions, form a population. In the next step, the fittest individuals of a population are chosen for mating; each couple of selected individuals produces a child. The children are generated by combining genes from different parents, which is called *crossover*. At the end, a few children are chosen to mutate randomly, a procedure called *mutation*. Finally, the new individuals form a new population and the algorithm starts all over again. The optimization procedure stops when either a maximum number of generations has been reached, or a fitness criterion has been met.

GA provide a useful tool for all kinds of optimization problems, especially for those with a number of objectives, constraint functions, and design variables that cannot be described by simple formulas. Since the early 80's, researchers developed many GA methods in multi-objective optimization problems that can be separated into two main categories: 1) non-Pareto-based approaches such as VEGA (Schaffer, 1985) and lexicographic ordering (Fourman, 1985) and 2) Pareto-based approaches, such as NSGA (Srinivas and Deb, 1994), SPEA (Zitzler, 1999) and NSGA-II (Deb et al., 2002). As Pareto-based approaches provide diverse solutions, they have been found to be more acceptable than their non-Pareto-based counterparts. One of the latest Pareto-based methods is NSGA-II which has been widely used since it was introduced by Deb et al. (2002). This method is computationally faster than others; it is based on the non-dominated sorting approach. The complete procedure can be found in

(Deb et al., 2002). Because of the foregoing merits of the NSGA-II, it is chosen as the optimization technique in this study.

5.3 The Multi-objective Optimization of the McGill SMG

5.3.1 Problem Definition. This work aims to modify the link lengths of the McGill SMG in order to improve its translational and rotational stiffnesses when its MP is at a specific pose, called the *reference pose*, denoted by $\mathcal{P}$. The pose of the MP of the current prototype when it is at the home posture is considered as $\mathcal{P}$. The home posture of the robot is defined as the posture farthest from parallel singularities, as reported by (Alizadeh et al., 2009). The joint angles of the current McGill SMG at the home posture are:

$$\theta_{I1} = \theta_{II1} = 0, \quad \theta_{I2} = \theta_{II2} = 5.28^\circ, \quad \theta_{I3} = \theta_{II3} = 45^\circ, \quad \theta_{I4} = 0, \quad \theta_{II4} = 180^\circ \quad (5.3)$$

Hence, with the current dimensions, shown in Table 5.1, the reference pose $\mathcal{P}$ is obtained as

$$\mathcal{P} : (X_{\mathcal{P}}, Y_{\mathcal{P}}, Z_{\mathcal{P}}, \phi_{\mathcal{P}}) = \left(\frac{l_0}{2}, 0, 0.991, 0 \right) \quad (5.4)$$

where l_0 is the distance between the two limbs, as depicted in Fig. 5.1. At this reference pose, the robot is symmetric with respect to a plane parallel to the ZY -plane passing through the point $\mathcal{P} : (X_{\mathcal{P}}, Y_{\mathcal{P}}, Z_{\mathcal{P}})$; hence, the joint angles that define $\mathcal{P}$ are

$$\theta_{I1} = \theta_{II1} = 0, \quad \theta_{I2} = \theta_{II2} = \theta_2, \quad \theta_{I3} = \theta_{II3} = \theta_3, \quad \theta_{I4} = 0, \quad \theta_{II4} = 180^\circ \quad (5.5)$$

whence,

$$a_{I4} = a_{II4} + a_{II5} \quad (5.6)$$

In this problem, six dimensions affect significantly the stiffness of the robot: l_0 , l_2 , l_3 , a_{I4} , a_{II4} , and a_{I5} , all shown in Fig. 5.1; by considering eq. (5.6), there are five independent dimensions, which are the design variables. In Ch. 3, two indices, κ_t and κ_r , the translational and rotational stiffnesses of a robot, respectively, were defined.

TABLE 5.1. The dimensions of the current prototype of the McGill SMG

Dimensions	l_0	l_1	l_2	l_3	a_{II4}	a_{II5}	a_1	h
Values (m)	0.6	0.06	0.6329	0.3	0.0621	0.0254	0.12	0.03

Here, these two indices are chosen as the objective functions. Accordingly, the design variable vector $\mathbf{x}$ and the objective vector $\mathbf{f}$ are:

$$\mathbf{x} = \begin{bmatrix} l_0 & l_2 & l_3 & a_{II4} & a_{II5} \end{bmatrix}^T, \quad \mathbf{f} = \begin{bmatrix} \kappa_t & \kappa_r \end{bmatrix}^T \quad (5.7)$$

By resorting to the constraints in eqs. (5.5), at the reference pose, lengths l_2 and l_3

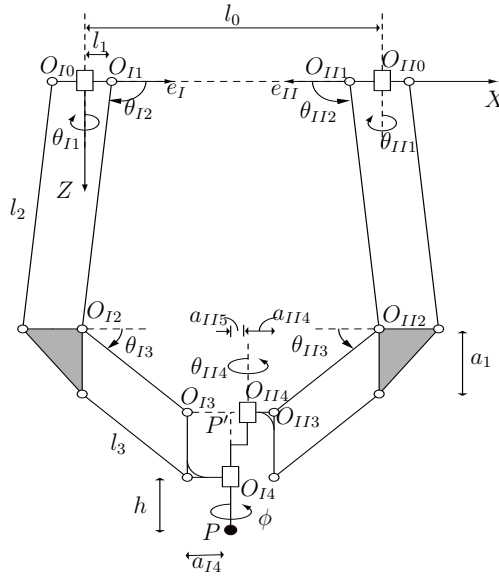


FIGURE 5.1. Front view of the kinematic chain of the McGill SMG at the reference pose

can be obtained in terms of θ_2 and θ_3 , namely,

$$l_2 = \frac{(a_1 + h - Z_P) \cos \theta_3 - (l_1 + a_{I4} - l_0/2) \sin \theta_3}{\sin(\theta_3 - \theta_2)} \quad (5.8)$$

$$l_3 = \frac{(a_1 + h - Z_P) \cos \theta_2 - (l_1 + a_{I4} - l_0/2) \sin \theta_2}{\sin(\theta_2 - \theta_3)} \quad (5.9)$$

Parameters a_1 , h and l_1 are constant, and equal to the values used in the current prototype (Table 5.1). In fact, according to the eq. (5.5), at the reference pose $\mathcal{P}$, θ_{J2} and θ_{J3} , $J = I, II$, are called θ_2 and θ_3 , respectively. For each set of design

variables, in order to calculate the objectives, θ_2 and θ_3 should be obtained from the trigonometric eqs. (5.8) and (5.9). In the optimization procedure, instead of l_2 and l_3 , it will prove convenient to use θ_2 and θ_3 as design variables. In other words, the variables l_2 and l_3 change according to θ_2 and θ_3 . The significant merit of this alternative is that there is no need to solve the trigonometric equations at each step, which makes the optimization procedure faster and more accurate. Hence, the design-variable vector becomes

$$\mathbf{x} = \begin{bmatrix} l_0 & \theta_2 & \theta_3 & a_{II4} & a_{II5} \end{bmatrix}^T \quad (5.10)$$

The design-variable lower and upper bounds are then cast in the form¹

$$0.5\text{m} \leq l_0 \leq 0.7\text{m}$$

$$90^\circ \leq \theta_{I2} \leq 130^\circ$$

$$30^\circ \leq \theta_{I3} \leq 60^\circ$$

$$0.02\text{m} \leq a_{II4} \leq 0.05\text{m}$$

$$0.01\text{m} \leq a_{II5} \leq 0.1\text{m}$$

5.4 Numerical Results

When adopting the NSGA-II procedure, the maximum number of generations is prescribed as 1000, each made of 100 individuals. The terms generation and individual in the GA stand for iteration and design-variable vectors. Hence, in the foregoing optimization procedure, the maximum number of 1000 iterations are run, each iteration containing 100 different designs. The crossover, where two parents mate to produce children, is performed via the simulated binary crossover (SBX) operator (Deb and Agrawal, 1995). To evaluate the objectives, the simplified model of the McGill SMG is used. In this model, depicted in Fig. 5.2, the proximal Π joints are modeled by aluminum frames instead of complex composite boxes. As GA multi-objective opti-

¹The current prototype of the McGill SMG was initially designed to have the same reach as Adept Cobra 600s. These bounds are chosen in a sense to explore a better architecture with the specifications close to the current prototype of the McGill SMG.

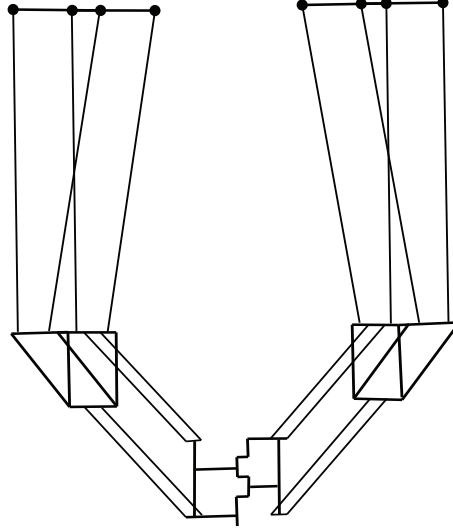


FIGURE 5.2. The simplified model of the McGill SMG robot

TABLE 5.2. The objectives and volume of the current McGill SMG

κ_t^0 (N)	κ_r^0 (Nm)	Volume ⁰ m ³
291.63	10.09	0.68

mization is a stochastic, time-demanding procedure, the objective evaluations inside the algorithm can tremendously slow it down. Using a simplified model makes the evaluation process much faster, so as to be able to conduct a less time-consuming, yet complete search procedure. This also helps to sharply perceive the effect of the dimensions on the robot stiffness. For the simplified model with the current dimensions, the values of κ_t and κ_r and the workspace volume are shown in Table 5.2. The optimization procedure is run for a maximum of 1000 generations with the crossover and mutation probabilities of 80% and 5%, respectively. This means that, at each generation, 80% of the individuals mate, while the remaining 20% are transferred directly to the next generation; and 5% of the children undergo random mutations in order to conduct a complete search on the design-variable space. As the number of generations grows, the final individuals approach the Pareto optimal set and the corresponding objective to the Pareto frontier. In Fig. 5.3(a), the evolution of the objectives trade-off after 20, 200, 800 and 1000 generations is plotted. By examining

Fig. 5.3(a), it seems that the objective trade-off variation from the generation number 800 to 1000 is approaching stable values. Moreover, Fig. 5.3(b) shows the number

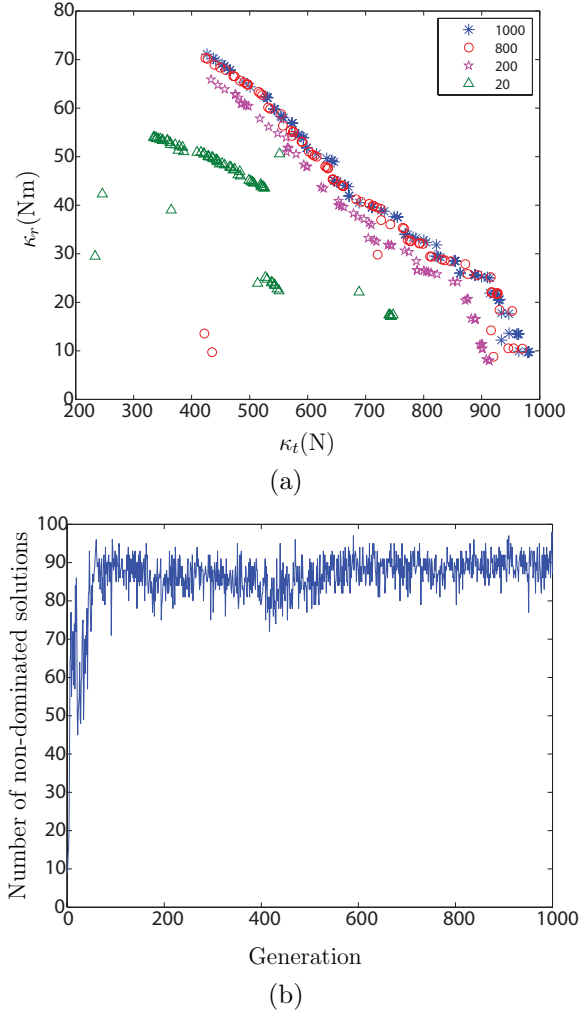


FIGURE 5.3. (a) The convergence of the Pareto frontier after 20, 200, 800 and 1000 generations; and (b) The number of non-dominated solutions with respect to the generation number

of non-dominant solutions with respect to the number of generations. The percentage of non-dominant solutions to the total number of individuals at each generation can be regarded as the closeness of the individuals to the actual Pareto optimal set. Figure 5.3(b) indicates that after about 600 generations, at each generation, in the average, 90% of individuals are non-dominant; this implies a satisfactory convergence

of the individuals to the actual Pareto optimal set. For the maximum number of 1000 generations, the Pareto frontier obtained is shown in Fig. 5.4. Here, a new index κ which is called the *stiffness improvement index* (SII) is defined:

$$\kappa = \frac{1}{2} \left(\frac{\kappa_t}{\kappa_t^0} + \frac{\kappa_r}{\kappa_r^0} \right) \quad (5.11)$$

in which κ_t^0 and κ_r^0 are the translational and rotational stiffness indices of the reference

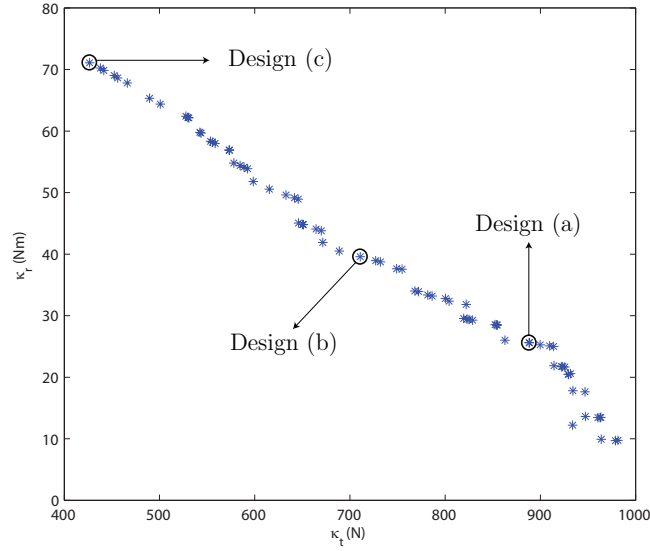


FIGURE 5.4. The Pareto frontier obtained after 1000 generations

design, namely, the current one. The workspace volume and the SII of the Pareto frontier members are depicted in Figs 5.5(a) and 5.5(b). The workspace volume has a minimum and a maximum of 0.61 m^3 and 0.66 m^3 , respectively. Compared to the workspace volume of the current prototype, which is 0.68 m^3 , in terms of the size of the workspace, all the members of the Pareto frontier are acceptable. However, the SII changes in a broad range from a minimum value of 2 to a maximum of 4.5. Although the SII value provides a good measurement of the overall stiffness improvement, a design with a large SII is not necessarily the best. To ensure that it is, in addition to the design with a maximum SII, design (c), two members of the Pareto optimal set, designs (a) and (b), with different values of SII are arbitrarily chosen for comparison;

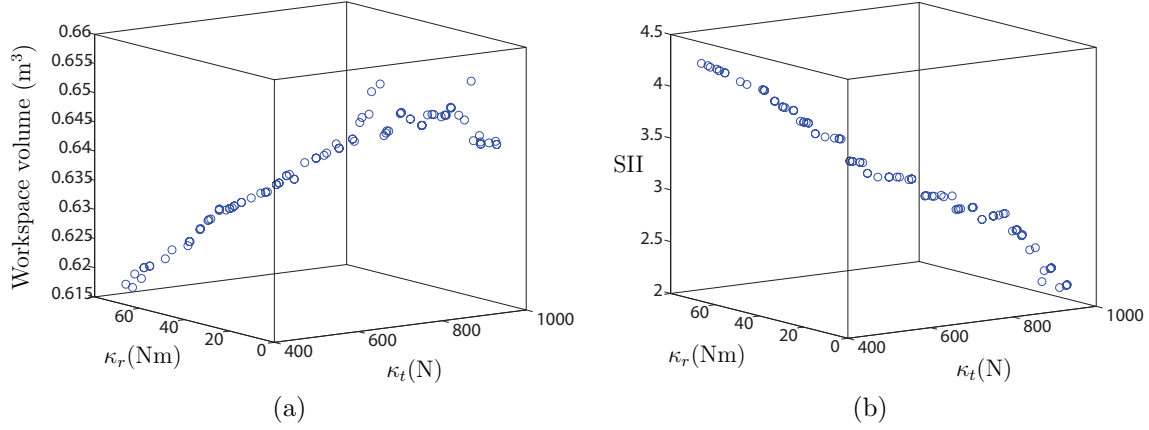


FIGURE 5.5. (a) The workspace volume and (b) the stiffness improvement index of the Pareto frontier members

the features of these designs are shown in Table 5.3. Now, to compare the designs

TABLE 5.3. The specifications of the three different designs of the McGill SMG

Specifications	Design (a)	Design (b)	Design (c)
l_0 (m)	0.6988	0.6988	0.6988
l_2 (m)	0.4533	0.4875	0.5804
l_3 (m)	0.4487	0.4092	0.3017
a_{II4} (m)	0.0241	0.0254	0.0497
a_{II5} (m)	0.0471	0.0673	0.0986
κ_t (N)	888.3	710.9	426.5
κ_r (Nm)	25.62	39.59	71.10
V (m ³)	0.6452	0.6398	0.6178
SII	2.793	3.181	4.255

(a), (b) and (c), the stiffness performance indices are calculated while the robot is operating through the standard test trajectory introduced in Sec. 2.4. The time-history of the two objective functions κ_t and κ_r for the design with current dimensions and three optimum designs are plotted in Figs. 5.6(a) and 5.6(b). Apparently, design (c) shows a significant improvement of κ_r over the test trajectory, while designs (a) and (b) are even weaker than the design with the current dimensions. Although designs (a) and (b) show a greater translational stiffness with respect to the others, because of the weak rotational stiffnesses in some part of the trajectory, they are discarded. Hence, the dimensions of design (c), which is the best, in terms of the

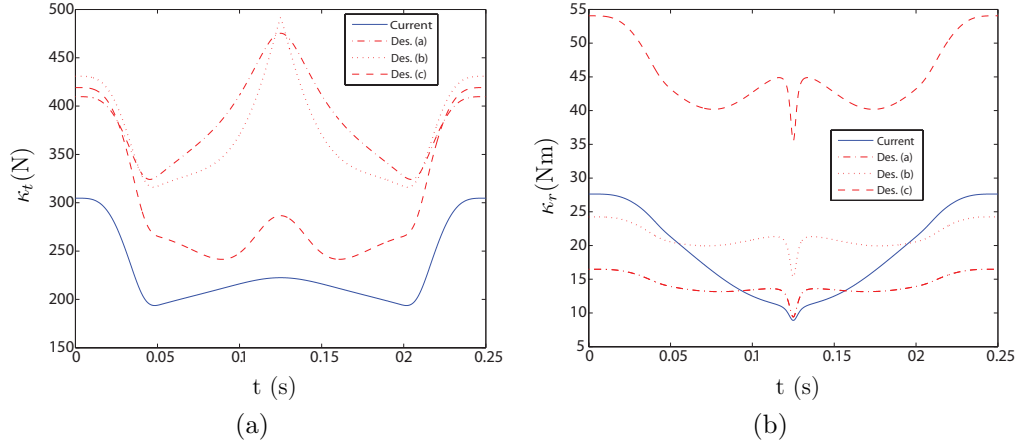


FIGURE 5.6. The evolution of (a) translational κ_t and (b) rotational κ_r stiffness indices for three different designs

rotational stiffness index, and better than the design with the current dimensions in terms of the translational stiffness index, is the chosen optimum solution. The 2D sketch of the McGill SMG with optimum dimensions is depicted in Fig. 5.7. The

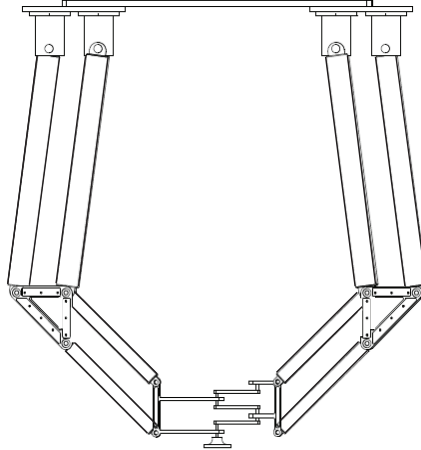


FIGURE 5.7. The McGill SMG with optimum dimensions

workspace of the robot with the optimum dimensions is plotted in Fig. 5.8. With the workspace volume of 0.62 m^3 , the McGill SMG with optimum dimensions still has a larger workspace than its competitor the Adept's Quattro with its workspace volume of 0.52 m^3 .

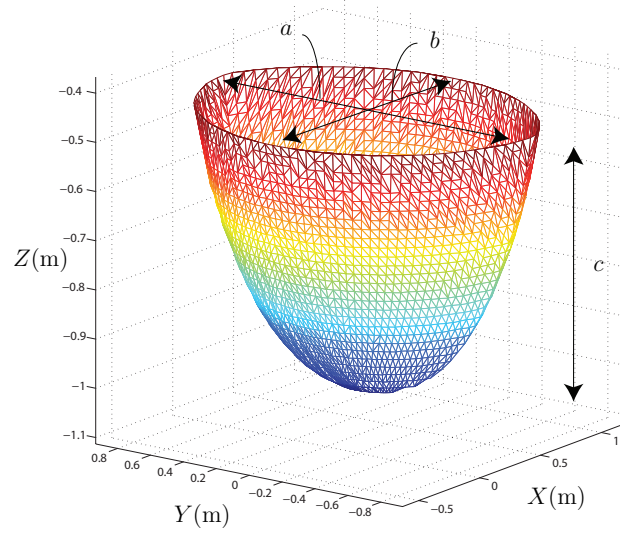


FIGURE 5.8. The workspace of the McGill SMG with optimum dimensions, $a = 1.84$ m, $b = 1.48$ m and $c = 0.64$ m

CHAPTER 6

Design Recommendations for the McGill SMG

After the calculation of the McGill SMG optimum dimensions, in the next level, the geometry and the material optimization of the flexible components, two proximal and two distal II-joints, are the main issues. Hence, conducting a multi-level optimization on the robot under study is an essential task. With the essential performance indices at hand, the stiffness sensitivity analysis can be an interesting topic as well. Since the latter provides an engineering insight, in the sequel, a novel stiffness sensitivity analysis over the McGill SMG is initiated. The proposed method is not yet complete, and hence, its completion is strongly recommended.

6.1 Stiffness Sensitivity Analysis of the McGill SMG

The stiffness matrix of a flexible part is posture-dependent, meaning that its entries are not constant. However, in the case of material or shape optimization of a flexible component, entries of the stiffness matrix computed in the reference frame attached to the body can be considered as indicators.

For the proximal II-joint of the current McGill SMG prototype, depicted in Fig. 6.1, which is fixed on one end to the machine frame (MF), the 6×6 stiffness matrix $\mathbf{K}_P$ is defined as the mapping between the SAD screw and wrench arrays at point P

with respect to the reference frame attached to the link. In this case, for instance, the

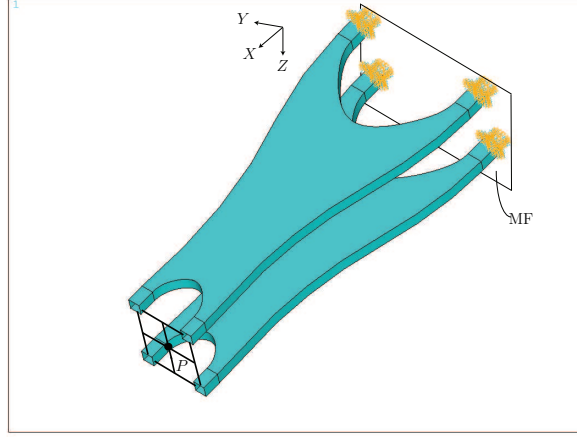


FIGURE 6.1. The proximal II-joint of the current McGill SMG prototype

entry K_{44} can be accounted for the translational stiffness along X -axis of the attached reference frame. Hence, any modification in the shape or the material of the II-joint which increases/decreases the translational stiffness along the X -axis will lead to a higher/lower absolute value of K_{44} . The 6×6 stiffness matrix of the proximal II-joint, $\mathbf{K}_P$, contains 36 entries, of which only 21 are independent, because of the inherent symmetry of stiffness matrices. It is expected that the six translational and rotational stiffnesses of the proximal II-joint do not contribute equally on the robot stiffness in a specific direction. This calls for a stiffness sensitivity analysis of the McGill SMG. This study aims to obtain the sensitivity of the two stiffness indices, translational κ_t and rotational κ_r , to the six diagonal entries of the Cartesian stiffness matrix of the proximal and distal II-joints computed with respect to the reference frame attached to the body, namely,

$$(\kappa_{j,ii})_P = \frac{\partial \kappa_j}{\partial k_{P,ii}}, \quad (\kappa_{j,ii})_D = \frac{\partial \kappa_j}{\partial k_{D,ii}}, \quad j = t, r \quad i = 1 \dots 6, \quad (6.1)$$

where $k_{P,ii}$ and $k_{D,ii}$ $i = 1 \dots 6$ are the diagonal entries of the proximal and distal II-joint stiffness matrices, respectively. Although all 21 entries of each flexible part stiffness matrix affect the stiffness indices, the effect of the six diagonal entries are more significant. It is claimed that the six diagonal entries of the stiffness matrix

of a flexible body with respect to the reference frame attached to the body can be considered as the indicators for the three translational and the three rotational stiffnesses of that flexible body.

Here, by resorting to finite differences, the stiffness sensitivity analysis of the McGill SMG with optimum dimensions, as obtained in Ch. 5, is conducted. As the number of objectives is two and the number of design variables is 12, the analysis results in 24 sensitivity values.

Now, while the McGill SMG is operating through the smoothed test trajectory, $(\kappa_{j,ii})_P$ and $(\kappa_{j,ii})_D$ where $j = t, r$ and $i = 1, \dots, 6$, are calculated. The units of κ_t and κ_r are N and Nm, respectively; as the first three and the last three diagonal entries of the stiffness matrix, corresponding to the translational and rotational motions, have units of N/m and Nm, respectively, the 24 sensitivity values are divided into four sets with different units. Figures. 6.2(a)–6.2(d) show the 12 sensitivity values regarding the proximal Π -joints; apparently, $(\kappa_{t,66})_P$, $(\kappa_{t,22})_P$, $(\kappa_{r,55})_P$, and $(\kappa_{r,11})_P$ are significant. It is important to note that, at each figure, some sensitivities are very close to zero; hence, in the black-and-white figures, they become hidden behind the time axis. The results corresponding to the distal Π -joints are shown in Figs. 6.3(a)–6.3(d). As the results show, in the McGill SMG with the new dimensions, the four diagonal entries of the proximal Π -joint stiffness matrix $k_{P,55}$, $k_{P,66}$, $k_{P,11}$, $k_{P,22}$ and the two of the distal Π -joints $k_{D,66}$ and $k_{D,33}$ play critical roles. In other words, the translational stiffness of both types of Π -joints along the X -axis, $k_{P,44}$ and $k_{D,44}$, do not affect the robot stiffness significantly, while the bending stiffnesses in the ZY -plane of both Π -joints, $k_{P,66}$ and $k_{D,66}$, are important variables. Moreover, by comparing the results, apparently, the stiffness of the proximal Π -joints affects the overall stiffness of the robot more than the distal Π -joint. Hence, the six components, $k_{P,55}$, $k_{P,66}$, $k_{P,11}$, $k_{P,22}$, $k_{D,66}$ and $k_{D,33}$, can be considered as performance indices by which different designs of the proximal and distal Π -joints can be compared. In the sequel, two designs for the proximal and two for the distal Π -joints are evaluated; by comparing the proposed objectives, the best design can be recognized.

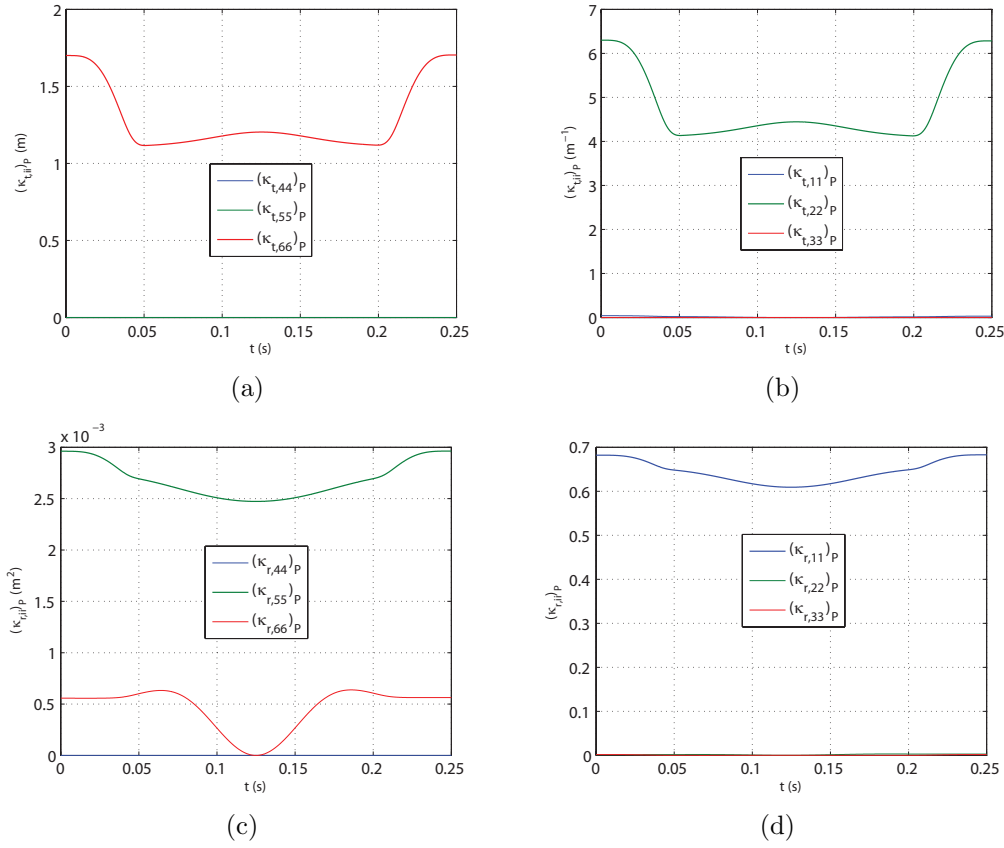


FIGURE 6.2. The sensitivity of κ_t and κ_r to the diagonal entries of the proximal Π -joint over the smoothed test trajectory: (a) $(\kappa_{t,ii})_P$; (b) $(\kappa_{t,ii})_P$; (c) $(\kappa_{r,ii})_P$; (d) $(\kappa_{r,ii})_P$

TABLE 6.1. Properties of LTM25 unidirectional prepreg (Daniel and Ishai, 1994)

Fiber Direction Modulus (E_x)	126.2 GPa
Matrix Direction Modulus (E_y)	10 GPa
Shear Modulus (E_s)	2.14 GPa
Poisson Ratio (ν_x)	0.3
Density (ρ)	1547 kg/m ³
Ply Thickness (h_0)	0.17 mm

6.2 Two Design Variants for the Proximal Π -joints

In the current prototype, the proximal Π -joints are made of two parallel identical trapezoid boxes fabricated of carbon-fibre composite laminate, as depicted in Fig. 6.4(a). Each box has walls made of a three-layer laminate with the lay-up

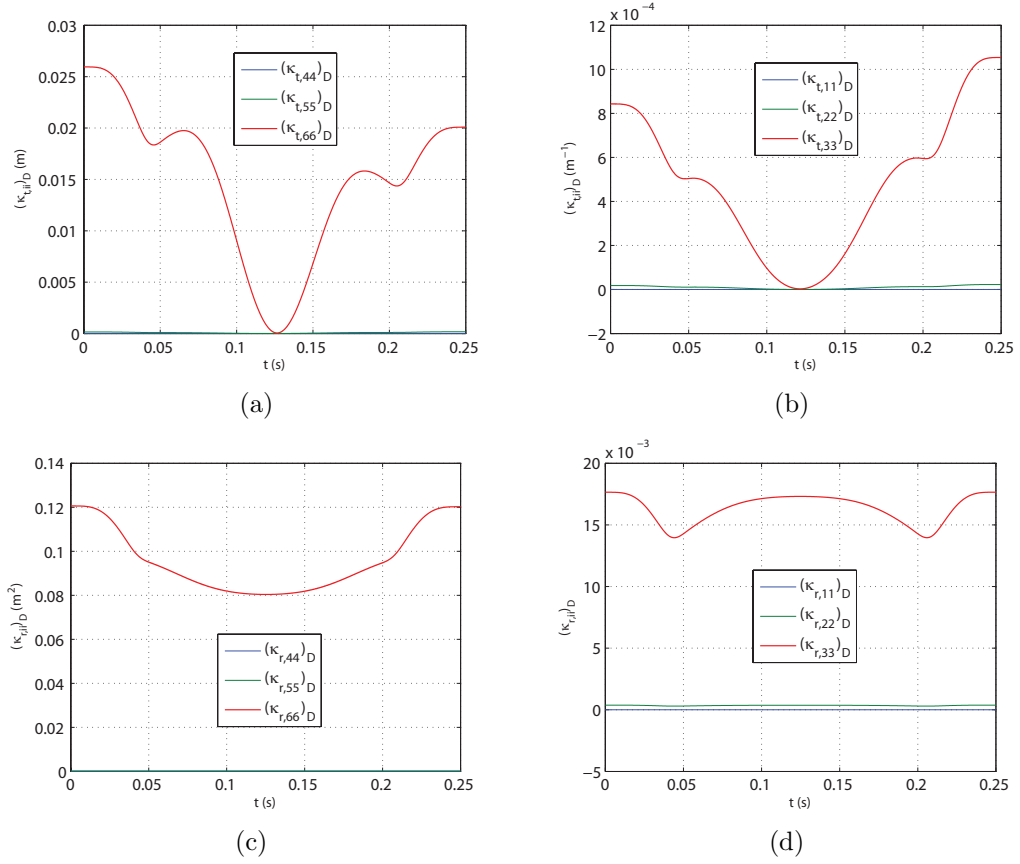


FIGURE 6.3. The sensitivity of κ_t and κ_r to the diagonal entries of the distal II-joint over the smoothed test trajectory: (a) $(\kappa_{t,ii})_D$; (b) $(\kappa_{t,ii})_D$; (c) $(\kappa_{r,ii})_D$; (d) $(\kappa_{r,ii})_D$

configuration $[0/90/0]$. The layers are made of LTM25 unidirectional prepreg carbon/epoxy composite material, with their corresponding mechanical properties shown in Table. 6.1. The second design, shown in Fig. 6.4(b), is composed of three parallel identical tubes made of carbon-fiber composite materials. The tube diameter is 0.05 m, and made of four layers of LTM25 unidirectional prepreg carbon/epoxy with the lay-up configuration $[0/90/90/0]$. The four effective stiffness components, $k_{P,55}$, $k_{P,66}$, $k_{P,11}$ and $k_{P,22}$, of both designs are compared in Table. 6.2. Apparently, except for $k_{P,55}$, the results from the two composite boxes dominate over the values from the three tubes. However, the latter shows significantly higher $k_{P,55}$. In addition to the foregoing issues, the manufacturing of a trapezoid box is more complex than that

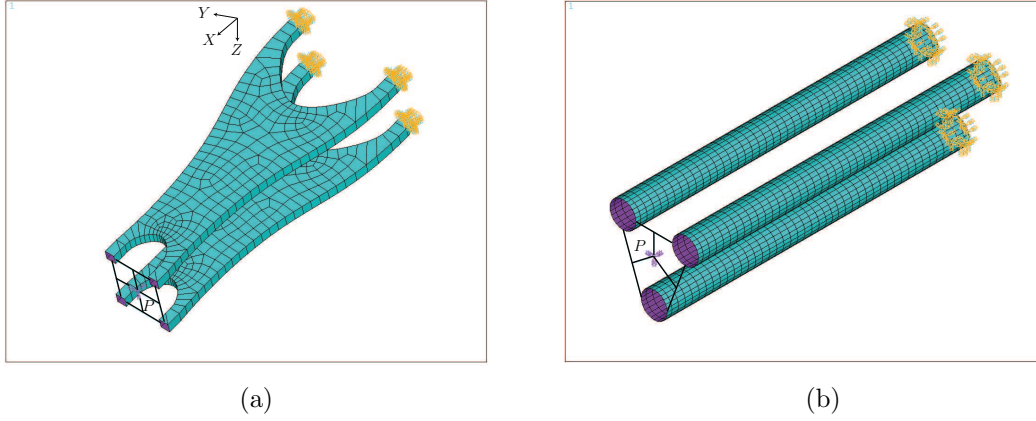


FIGURE 6.4. Two design variants of the proximal II-joint: (a) Two composite boxes ; (b) Three composite tubes

TABLE 6.2. The four effective stiffness components of the Proximal II-joints

Stiffness Component	Two Boxes	Three Tubes
$k_{P,55}$ (N/m)	0.8×10^4	6.9×10^4
$k_{P,66}$ (N/m)	19.6×10^4	7.5×10^4
$k_{P,11}$ (Nm)	348.4	318.3
$k_{P,22}$ (Nm)	362.2	353.6
Mass (kg)	0.275	0.284

of a tube. Hence, the proximal II-joint made of three composite tubes has some advantages, even though it may not be the stiffest choice.

6.3 Two Design Variants for the Distal II-joints

According to the sensitivity results, although the proximal II-joints have more influence on the overall stiffness of the robot, the stiffness improvement of the distal II-joints is also a crucial matter. In the current prototype, each distal II-joint is made of four parallel aluminum bars, shown in Fig. 6.5(a), each bar is 0.3 m long. As an alternative, the four aluminum bars can be replaced with three tubes made of composite materials. In this case, each tube is made of four LTM25 unidirectional prepreg carbon/epoxy with the lay-up configuration $[0/90/90/0]$ and a diameter of 0.03 m, as depicted in Fig. 6.5(b). Based on the results of a sensitivity analysis, the overall stiffness of the robot is more sensitive to the two of six diagonal entries of the

TABLE 6.3. The two effective stiffness components of the Distal Π -joints designs

Stiffness Component	Four Aluminum Bars	Three Composite Tubes
$k_{D,66}$ (N/m)	50.6×10^4	11×10^4
$k_{D,33}$ (N/m)	7179.6	127.8
Mass (kg)	0.3	0.09

distal Π -joint stiffness matrix, $k_{D,66}$ and $k_{D,33}$. These two components of both distal Π -joint designs are compared in Table. 6.3. Apparently, the design composed of

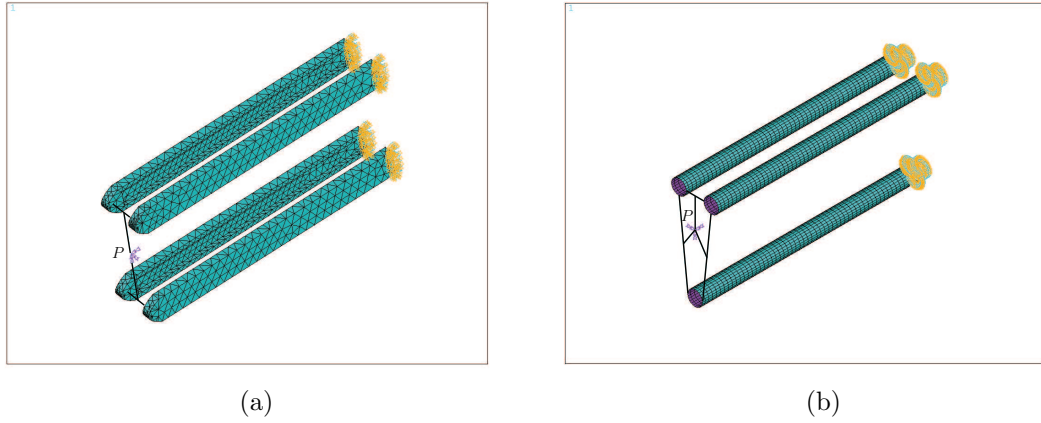


FIGURE 6.5. Two design variants of the distal Π -joint: (a) four aluminum bars ; (b) three composite tubes

three tubes cannot compete with their aluminum counterpart; however, the former is much lighter than the latter. Even if the number of layers is increased to eight, the stiffness components are still much lower than that of the aluminum bars. Therefore, the distal Π -joint with four aluminum bars is recommended to be retained in future designs.

6.4 The CAD model of the New Designs

Figures 6.6 and 6.7 depict the CAD model of the two recommended designs of the McGill SMG with optimum dimensions. In the former design, the proximal Π -joints are made of carbon-fibre composite tubes and the distal Π -joints are made of aluminum bars, while in the latter, both versions of the Π -joints are composed of carbon-fibre tubes.



FIGURE 6.6. The new design of the McGill SMG robot with the proximal II-joints made of carbon-fibre composite tubes

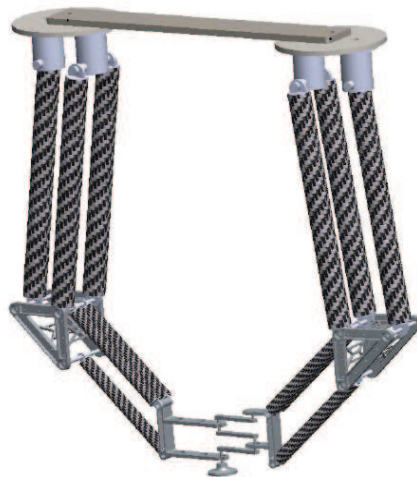


FIGURE 6.7. The new design of the McGill SMG robot with the proximal and distal II-joints made of carbon-fibre composite tubes

The design depicted in Fig. 6.6 is expected to be stiffer than its counterpart shown in Fig. 6.7, while the latter is lighter than the former. As the robot mass and stiffness are conflicting, obtaining the lightest yet stiffest design is a challenging problem which calls for further research.

CHAPTER 7

Closing Remarks and Recommendations for Future Research

7.1 Conclusions

This thesis reported novel methods in constraint-wrench, elastostatic and elastodynamic analyses of parallel manipulators, and the design optimization of the McGill SMG, a novel two-limb parallel manipulator. To this end, in Ch. 2 a methodology for the constraint-wrench analysis of single-loop parallel robots was developed. In this vein, first, the kinematic-constraint equations of eight kinematic pairs, six LKPs, i.e., revolute (R), prismatic (P), helical (H), cylindrical (C), spherical (S), and planar (F) joint, and two HKPs, the universal (U) and the Π -joint, were derived. Hence, the constraint matrix of a mechanical system is readily obtained upon assembly of the constraint matrices of the kinematic pairs. Then, by introducing the actuator wrench-shaping matrix, the constraint wrench analysis led to a system of linear equations in which the actuator and the constraint wrenches are the unknown variables. As a case study, the constraint-wrench analysis of the McGill SMG, when its MP moves over an industry-adopted test trajectory, was included. However, because the McGill SMG is an overconstrained mechanism, the number of equations derived is lower than the number of unknowns. To cope with this issue, isoconstrained versions of the robot were introduced, with which the constraint wrench analysis is possible.

Although the isoconstrained SMG differs from its overconstrained counterpart, its motion compatibilities are identical to those of the original overconstrained structure. The isoconstrained version allows not only for constraint-wrench analysis, but also for assembly.

In Ch. 3, by resorting to the concept of generalized spring, the calculation of the stiffness matrix of complex multibody systems with passive kinematic pairs was discussed. A new terminology, based on the concept of element assembly, as used in FEA, was proposed, by which the stiffness matrix of an articulated flexible body can be simply modelled. This terminology helps derive a general formulation for the stiffness matrix of two flexible bodies connected by any of the six LKPs. Especially in the case of a complex mechanism, the proposed method provides significant insight of the calculation of the stiffness matrix. In addition, when the stiffness of the robot is to be monitored along a continuous trajectory, substantial savings in time and calculation cost are obtained. As a case study, the posture-dependent Cartesian stiffness matrix of the McGill SMG was computed by the proposed method. In the second part, by means of a nondimensionalization of the stiffness matrix, two stiffness performance indices, translational and rotational, were introduced. The stiffness indices were computed along a standard test trajectory for the above-mentioned robot. The indices can be used either for assessment of the behaviour of the robot at different points of the trajectory or for design optimization purposes, the two stiffness indices help the designer analyze the stiffness of the structure accurately and comprehensively.

In Ch. 4, the elastodynamic analysis of parallel Schönflies motion generators, using as a case study the McGill SMG, was reported. By using the generalized-spring concept, introduced in Ch. 3, the stiffness matrix of the SMG was calculated. A systematic step-by-step procedure was introduced that eases the derivation of the linearized elastodynamics model of the system under “small”-amplitude vibration, around a given robot posture, with all motors locked. The mass matrix of a parallel

SMG in which a rigid MP is connected to the motors by flexible parts can be a cumbersome task. Moreover, as the mass matrix is posture-dependent, when it comes to many postures, modal analysis can be prohibitively expensive. The procedure introduced here appears significantly useful for complex parallel SMG whose kinematic chains contain flexible components. The method can effectively provide a symbolic posture-dependent formula for the mass matrix. Having the essential ingredients, the stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$, the modal analysis of a parallel SMG when operating over a trajectory is possible. The first six natural frequencies of the McGill SMG over a test trajectory were calculated. As the mass and stiffness matrices are both posture-dependent, the natural frequencies of the robot attain different values for different robot postures over the test trajectory. The foregoing analysis can be used as an efficient tool to assess the performance of a robot from the elastodynamic point of view at the design stage. Moreover, for structural optimization, finding the critical postures of a mechanism, i.e., postures at which a mechanism attains its lowest fundamental frequencies, is a crucial design step.

The stiffness optimization of the McGill SMG was the subject of Ch. 5. By resorting to the translational and rotational stiffness indices, a multi-objective optimization problem was defined. To avoid obtaining solutions with the desired workspace volume, a pertinent condition was imposed. By the use of GAs, the foregoing multi-objective optimization led to a set of Pareto optimal solutions. Each member of the Pareto optimal set can fit an additional criterion, and thus, can be considered as an optimum solution. The stiffness improvement index (SII) was defined in order to pick up a unique solution. Finally, the stiffness performance of the design with maximum SII and two other members of the Pareto optimal set over a standard test trajectory were examined, and the best design was introduced.

Finally, in Ch. 6, the stiffness sensitivity analysis of the McGill SMG was conducted. Based on this analysis, recommendations for the next design of the McGill SMG were also proposed.

7.2 Recommendations for Future Research

There is a lot of room in this thesis for future research; in the sequel, some recommendations are discussed.

The kinematic constraint matrices of eight kinematic pairs, six LKPs and two HKPs, were introduced in Ch. 2. This method can be generalized to model any other type of kinematic pair such as gears and cams. Moreover, in Ch. 2, the constraint-wrench analysis was conducted by calculation of the constraint matrix of the robot under the rigidity assumption of the robot links. However, in reality, the links are flexible. A possible recommendation for future research is to define new kineto-elastic constraint equations for each type of kinematic pairs; hence, upon assembly of these new matrices, the kineto-elastic constraint matrix of a robot can be obtained. This modifies the method proposed in Ch. 2, which leads to more accurate results. In Ch. 5, the optimum dimensions of the McGill SMG were calculated. In the next level, optimization of the geometry and the material of the robot flexible components, the two proximal and the two distal II-joints, is recommended. This calls for a multi-level optimization. In this case, the parametric stiffness matrix for the flexible components is needed; however, as the flexible components are made of composite materials, a more detailed modeling effort is required.

With the essential performance indices at hand, the stiffness sensitivity analysis also can be undertaken in future research. In Ch. 6, a novel conceptual stiffness sensitivity analysis over the McGill SMG was already initiated. However, the method and its implementation into the design procedure are incomplete; its completion is strongly recommended for future research.

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APPENDIX A

Derivation of The Twist Shaping Matrices

As shown in eq. (2.53), the joint-rate vector is mapped onto the twist vector of a robot by means of the twist shaping matrix. This matrix is composed of the twist-shaping matrices $\mathbf{T}_i$ of all rigid bodies, namely,

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \vdots \\ \mathbf{T}_n \end{bmatrix} \quad (\text{A.1})$$

which is a $6(n-1) \times (d+p)$ matrix. The time derivative of the twist-shaping matrix is simply obtained by the differentiation of its components with respect to time:

$$\dot{\mathbf{T}} = \begin{bmatrix} \dot{\mathbf{T}}_1 \\ \dot{\mathbf{T}}_2 \\ \vdots \\ \dot{\mathbf{T}}_n \end{bmatrix} \quad (\text{A.2})$$

In the sequel, the twist-shaping matrices of the seven rigid bodies of the McGill SMG, along with their time derivatives, are derived. The parameters regarding the geometry of the McGill SMG that are used in the formulas below are depicted in Figs. A.1 and

A.2.

$$\mathbf{T}_1 = \begin{bmatrix} \mathbf{0}_2 & \mathbf{O}_{2 \times 7} \\ 1 & \mathbf{0}_7^T \\ \mathbf{0}_3 & \mathbf{O}_{3 \times 7} \end{bmatrix}, \quad \dot{\mathbf{T}}_1 = \mathbf{O}_{6 \times 8}$$

where $\mathbf{O}_{n \times m}$ denotes the $n \times m$ zero matrix, and $\mathbf{0}_m$ the m -dimensional zero vector.

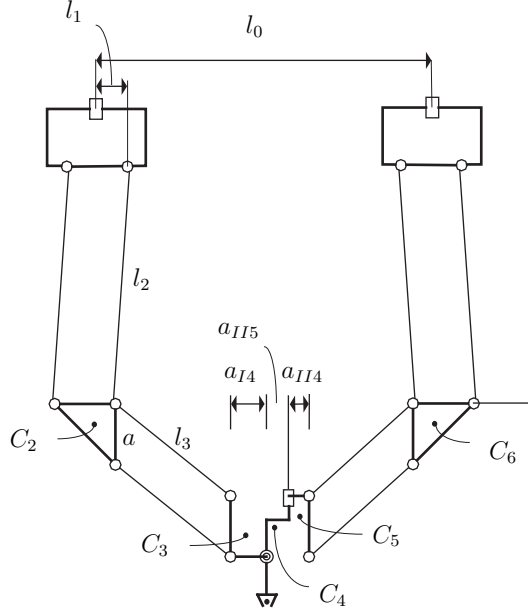


FIGURE A.1. The McGill SMG dimensions

$$\mathbf{T}_2 = \begin{bmatrix} 0 & 0 & \mathbf{0}_6^T \\ 0 & 0 & \mathbf{0}_6^T \\ 1 & 0 & \mathbf{0}_6^T \\ t_{41} & t_{42} & \mathbf{0}_6^T \\ t_{51} & t_{52} & \mathbf{0}_6^T \\ 0 & t_{62} & \mathbf{0}_6^T \end{bmatrix}, \quad \dot{\mathbf{T}}_2 = \begin{bmatrix} 0 & 0 & \mathbf{0}_6^T \\ 0 & 0 & \mathbf{0}_6^T \\ 0 & 0 & \mathbf{0}_6^T \\ \dot{t}_{41} & \dot{t}_{42} & \mathbf{0}_6^T \\ \dot{t}_{51} & \dot{t}_{52} & \mathbf{0}_6^T \\ 0 & \dot{t}_{62} & \mathbf{0}_6^T \end{bmatrix}$$

with

with

$$t_{41} = -l_1 s\theta_{I1} - l_2 s\theta_{I1} s\theta_{I2} - l_3 s\theta_{I1} c\theta_{I3} - a_{I5} s\theta_{I1}$$

$$t_{42} = l_2 c\theta_{I1} c\theta_{I2}$$

$$t_{43} = -l_3 c\theta_{I1} s\theta_{I3}$$

$$t_{51} = l_1 c\theta_{I1} + l_2 c\theta_{I1} s\theta_{I2} + l_3 c\theta_{I1} c\theta_{I3} + a_{I5} c\theta_{I1}$$

$$t_{52} = l_2 s\theta_{I1} c\theta_{I2}$$

$$t_{53} = -l_3 s\theta_{I1} s\theta_{I3}$$

$$t_{62} = -l_2 s\theta_{I2}$$

$$t_{63} = -l_3 c\theta_{I3}$$

$$\dot{t}_{41} = -l_1 \dot{\theta}_{I1} c\theta_{I1} - l_2 \dot{\theta}_{I1} c\theta_{I1} s\theta_{I2} - l_2 \dot{\theta}_{I2} s\theta_{I1} c\theta_{I2} - l_3 \dot{\theta}_{I1} c\theta_{I1} c\theta_{I3} + l_3 \dot{\theta}_{I3} s\theta_{I1} s\theta_{I3} - a_{I5} \dot{\theta}_{I1} c\theta_{I1}$$

$$\dot{t}_{42} = -l_2 \dot{\theta}_{I1} s\theta_{I1} c\theta_{I2} - l_2 \dot{\theta}_{I2} c\theta_{I1} s\theta_{I2}$$

$$\dot{t}_{43} = l_3 \dot{\theta}_{I1} s\theta_{I1} s\theta_{I3} - l_3 \dot{\theta}_{I3} c\theta_{I1} c\theta_{I3}$$

$$\dot{t}_{51} = -l_1 \dot{\theta}_{I1} s\theta_{I1} - l_2 \dot{\theta}_{I1} s\theta_{I1} s\theta_{I2} + l_2 \dot{\theta}_{I2} c\theta_{I1} c\theta_{I2} - l_3 \dot{\theta}_{I1} s\theta_{I1} c\theta_{I3} - l_3 \dot{\theta}_{I3} c\theta_{I1} s\theta_{I3} - a_{I5} \dot{\theta}_{I1} s\theta_{I1}$$

$$\dot{t}_{52} = l_2 \dot{\theta}_{I1} c\theta_{I1} c\theta_{I2} - l_2 \dot{\theta}_{I2} s\theta_{I1} s\theta_{I2}$$

$$\dot{t}_{53} = -l_3 \dot{\theta}_{I1} c\theta_{I1} s\theta_{I3} - l_3 \dot{\theta}_{I3} s\theta_{I1} c\theta_{I3}$$

$$\dot{t}_{62} = -l_2 \dot{\theta}_{I2} c\theta_{I2}$$

$$\dot{t}_{63} = l_3 \dot{\theta}_{I3} s\theta_{I3}$$

$$\mathbf{T}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{0}_4^T \\ 0 & 0 & 0 & 0 & \mathbf{0}_4^T \\ 1 & 0 & 0 & 0 & \mathbf{0}_4^T \\ t_{41} & t_{42} & t_{43} & t_{44} & \mathbf{0}_4^T \\ t_{51} & t_{52} & t_{53} & t_{54} & \mathbf{0}_4^T \\ 0 & t_{62} & t_{63} & 0 & \mathbf{0}_4^T \end{bmatrix}, \quad \dot{\mathbf{T}}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{0}_4^T \\ 0 & 0 & 0 & 0 & \mathbf{0}_4^T \\ 0 & 0 & 0 & 0 & \mathbf{0}_4^T \\ \dot{t}_{41} & \dot{t}_{42} & \dot{t}_{43} & \dot{t}_{44} & \mathbf{0}_4^T \\ \dot{t}_{51} & \dot{t}_{52} & \dot{t}_{53} & \dot{t}_{54} & \mathbf{0}_4^T \\ 0 & \dot{t}_{62} & \dot{t}_{63} & 0 & \mathbf{0}_4^T \end{bmatrix}$$

with

$$t_{41} = -l_1 s\theta_{I1} - l_2 s\theta_{I1} s\theta_{I2} - l_3 s\theta_{I1} c\theta_{I3} - a_{I4} c\theta_{I1} + a_{II7} \sin(\theta_{II1} + \theta_{II4})$$

$$t_{42} = l_2 c\theta_{I1} c\theta_{I2}$$

$$t_{43} = -l_3 c\theta_{I1} s\theta_{I3}$$

$$t_{44} = a_{II7} \sin(\theta_{II1} + \theta_{II4})$$

$$t_{51} = l_1 c\theta_{I1} + l_2 c\theta_{I1} s\theta_{I2} + l_3 c\theta_{I1} c\theta_{I3} + a_{I4} c\theta_{I1} - a_{II7} \cos(\theta_{II1} + \theta_{II4})$$

$$t_{52} = l_2 s\theta_{I1} c\theta_{I2}$$

$$t_{53} = -l_3 s\theta_{I1} s\theta_{I3}$$

$$t_{54} = -a_{II7} \cos(\theta_{II1} + \theta_{II4})$$

$$t_{62} = -l_2 s\theta_{I2}$$

$$t_{63} = -l_3 c\theta_{I3}$$

$$\begin{aligned} \dot{t}_{41} = & -l_1 \dot{\theta}_{I1} c\theta_{I1} - l_2 \dot{\theta}_{I1} c\theta_{I1} s\theta_{I2} - l_2 \dot{\theta}_{I2} s\theta_{I1} c\theta_{I2} - l_3 \dot{\theta}_{I1} c\theta_{I1} c\theta_{I3} + l_3 \dot{\theta}_{I3} s\theta_{I1} s\theta_{I3} + a_{I4} \dot{\theta}_{I1} \\ & s\theta_{I1} + a_{II7} (\dot{\theta}_{I1} + \dot{\theta}_{I4}) \cos(\theta_{II1} + \theta_{II4}) \end{aligned}$$

$$\dot{t}_{42} = -l_2 \dot{\theta}_{I1} s\theta_{I1} c\theta_{I2} - l_2 \dot{\theta}_{I2} c\theta_{I1} s\theta_{I2}$$

$$\dot{t}_{43} = l_3 \dot{\theta}_{I1} s\theta_{I1} s\theta_{I3} - l_3 \dot{\theta}_{I3} c\theta_{I3} c\theta_{I1}$$

$$\dot{t}_{44} = a_{II7} (\dot{\theta}_{I1} + \dot{\theta}_{I4}) \cos(\theta_{II1} + \theta_{II4})$$

$$\begin{aligned} \dot{t}_{51} = & -l_1 \dot{\theta}_{I1} s\theta_{I1} + l_2 \dot{\theta}_{I2} c\theta_{I2} c\theta_{I1} - l_2 \dot{\theta}_{I1} s\theta_{I1} s\theta_{I2} - l_3 \dot{\theta}_{I1} s\theta_{I1} c\theta_{I3} - l_3 \dot{\theta}_{I3} c\theta_{I1} s\theta_{I3} - a_{I4} \dot{\theta}_{I1} \\ & s\theta_{I1} + a_{II7} (\dot{\theta}_{I1} + \dot{\theta}_{I4}) \sin(\theta_{II1} + \theta_{II4}) \end{aligned}$$

$$\dot{t}_{52} = l_2 \dot{\theta}_{I1} c\theta_{I1} c\theta_{I2} - l_2 \dot{\theta}_{I2} s\theta_{I1} s\theta_{I2}$$

$$\dot{t}_{53} = -l_3 \dot{\theta}_{I1} c\theta_{I1} s\theta_{I3} - l_3 \dot{\theta}_{I3} s\theta_{I1} c\theta_{I3}$$

$$\dot{t}_{54} = a_{II7} (\dot{\theta}_{I1} + \dot{\theta}_{I4}) \sin(\theta_{II1} + \theta_{II4})$$

$$\dot{t}_{62} = -l_2 \dot{\theta}_{I2} c\theta_{I2}$$

$$\dot{t}_{63} = l_3 \dot{\theta}_{I3} s\theta_{I3}$$

$$\mathbf{T}_5 = \begin{bmatrix} \mathbf{0}_5^T 0 & 0 & 0 & \\ \mathbf{0}_5^T 0 & 0 & 0 & \\ \mathbf{0}_5^T & 1 & 0 & 0 \\ \mathbf{0}_5^T & t_{41} & t_{42} & t_{43} \\ \mathbf{0}_5^T & t_{51} & t_{52} & t_{53} \\ \mathbf{0}_5^T & 0 & t_{62} & t_{63} \end{bmatrix}, \quad \dot{\mathbf{T}}_5 = \begin{bmatrix} \mathbf{0}_5^T & 0 & 0 & 0 \\ \mathbf{0}_5^T & 0 & 0 & 0 \\ \mathbf{0}_5^T & 0 & 0 & 0 \\ \mathbf{0}_5^T & \dot{t}_{41} & \dot{t}_{42} & \dot{t}_{43} \\ \mathbf{0}_5^T & \dot{t}_{51} & \dot{t}_{52} & \dot{t}_{53} \\ \mathbf{0}_5^T & 0 & \dot{t}_{62} & \dot{t}_{63} \end{bmatrix}$$

with

$$\begin{aligned}
t_{41} &= -l_1 s\theta_{II1} - l_2 s\theta_{II1} s\theta_{II2} - l_3 s\theta_{II1} c\theta_{II3} - a_{II6} s\theta_{II1} \\
t_{42} &= l_2 c\theta_{II1} c\theta_{II2} \\
t_{43} &= -l_3 c\theta_{II1} s\theta_{II3} \\
t_{51} &= l_1 c\theta_{II1} + l_2 c\theta_{II1} s\theta_{II2} + l_3 c\theta_{II1} c\theta_{II3} + a_{II6} c\theta_{II1} \\
t_{52} &= l_2 s\theta_{II1} c\theta_{II2} \\
t_{53} &= -l_3 s\theta_{II1} s\theta_{II3} \\
t_{62} &= -l_2 s\theta_{II2} \\
t_{63} &= -l_3 c\theta_{II3} \\
\dot{t}_{41} &= -l_1 \dot{\theta}_{II1} c\theta_{II1} - l_2 \dot{\theta}_{II1} c\theta_{II1} s\theta_{II2} - l_2 \dot{\theta}_{II2} s\theta_{II1} c\theta_{II2} - l_3 \dot{\theta}_{II1} c\theta_{II1} c\theta_{II3} \\
&\quad + l_3 \dot{\theta}_{II3} s\theta_{II1} s\theta_{II3} - a_{II6} \dot{\theta}_{II1} c\theta_{II1} \\
\dot{t}_{42} &= -l_2 \dot{\theta}_{II1} s\theta_{II1} c\theta_{II2} - l_2 \dot{\theta}_{II2} c\theta_{II1} s\theta_{II2} \\
\dot{t}_{43} &= l_3 \dot{\theta}_{II1} s\theta_{II1} s\theta_{II3} - l_3 \dot{\theta}_{II3} c\theta_{II1} c\theta_{II3} \\
\dot{t}_{51} &= -l_1 \dot{\theta}_{II1} s\theta_{II1} - l_2 \dot{\theta}_{II1} s\theta_{II1} s\theta_{II2} + l_2 \dot{\theta}_{II2} c\theta_{II1} c\theta_{II2} - l_3 \dot{\theta}_{II1} s\theta_{II1} c\theta_{II3} \\
&\quad - l_3 \dot{\theta}_{II3} c\theta_{II1} s\theta_{II3} + a_{II6} \dot{\theta}_{II1} s\theta_{II1} \\
\dot{t}_{52} &= l_2 \dot{\theta}_{II1} c\theta_{II1} c\theta_{II2} - l_2 \dot{\theta}_{II2} s\theta_{II1} s\theta_{II2} \\
\dot{t}_{53} &= -l_3 \dot{\theta}_{II1} c\theta_{II1} s\theta_{II3} - l_3 \dot{\theta}_{II3} s\theta_{II1} c\theta_{II3} \\
\dot{t}_{62} &= -l_2 \dot{\theta}_{II2} c\theta_{II2} \\
\dot{t}_{63} &= l_3 \dot{\theta}_{II3} s\theta_{II3}
\end{aligned}$$

$$\mathbf{T}_6 = \begin{bmatrix} \mathbf{0}_6^T & 0 & 0 \\ \mathbf{0}_6^T & 0 & 0 \\ \mathbf{0}_6^T & 1 & 0 \\ \mathbf{0}_6^T & t_{41} & t_{42} \\ \mathbf{0}_6^T & t_{51} & t_{52} \\ \mathbf{0}_6^T & 0 & t_{62} \end{bmatrix}, \quad \dot{\mathbf{T}}_6 = \begin{bmatrix} \mathbf{0}_6^T & 0 & 0 \\ \mathbf{0}_6^T & 0 & 0 \\ \mathbf{0}_6^T & 0 & 0 \\ \mathbf{0}_6^T & \dot{t}_{41} & \dot{t}_{42} \\ \mathbf{0}_6^T & \dot{t}_{51} & \dot{t}_{52} \\ \mathbf{0}_6^T & 0 & \dot{t}_{62} \end{bmatrix}$$

with

$$t_{41} = -l_1 s\theta_{II1} - l_2 s\theta_{II1} s\theta_{II2} + 0.47as\theta_{II1}$$

$$t_{42} = l_2 c\theta_{II1} c\theta_{II2}$$

$$t_{51} = l_1 c\theta_{II1} + l_2 c\theta_{II1} s\theta_{II2} - 0.47ac\theta_{II1}$$

$$t_{52} = l_2 s\theta_{II1} c\theta_{II2}$$

$$t_{62} = -l_2 s\theta_{II2}$$

$$\dot{t}_{41} = -l_1 \dot{\theta}_{II1} c\theta_{II1} - l_2 \dot{\theta}_{II1} c\theta_{II1} s\theta_{II2} - l_2 \dot{\theta}_{II2} s\theta_{II1} c\theta_{II2} + 0.47a\dot{\theta}_{II1} c\theta_{II1}$$

$$\dot{t}_{42} = -l_2 \dot{\theta}_{II1} s\theta_{II1} c\theta_{II2} - l_2 \dot{\theta}_{II2} c\theta_{II1} s\theta_{II2}$$

$$\dot{t}_{51} = -l_1 \dot{\theta}_{II1} s\theta_{II1} - l_2 \dot{\theta}_{II1} s\theta_{II1} s\theta_{II2} + l_2 \dot{\theta}_{II2} c\theta_{II1} c\theta_{II2} + 0.4a\dot{\theta}_{II1} s\theta_{II1}$$

$$\dot{t}_{52} = l_2 \dot{\theta}_{II1} c\theta_{II1} c\theta_{II2} - l_2 \dot{\theta}_{II2} s\theta_{II1} s\theta_{II2}$$

$$\dot{t}_{62} = -l_2 \dot{\theta}_{II2} c\theta_{II2}$$

Finally,

$$\mathbf{T}_7 = \begin{bmatrix} \mathbf{O}_{2 \times 7} & 0 \\ \mathbf{0}_7^T & 0 \\ \mathbf{O}_{3 \times 7} & 1 \end{bmatrix}, \quad \dot{\mathbf{T}}_7 = \mathbf{O}_{6 \times 8}$$

APPENDIX B

Derivation of The Angular Velocity Dyads

All the rigid bodies in the McGill SMG move under the same Schönflies displacement subgroup, which contains three translations and one rotation about the Z -axis of Fig. 2.12. Therefore, the angular velocity of each rigid body of the J th limb is the same, i.e., for the first three rigid bodies, which belong to limb I ,

$$\boldsymbol{\omega}_i = \begin{bmatrix} 0 & 0 & \dot{\theta}_{I1} \end{bmatrix}^T, \quad i = 1, 2, 3 \quad (\text{B.1})$$

The last three rigid bodies also rotate with the same angular velocity $\dot{\theta}_{II1}$, whence,

$$\boldsymbol{\omega}_i = \begin{bmatrix} 0 & 0 & \dot{\theta}_{II1} \end{bmatrix}^T, \quad i = 5, 6, 7 \quad (\text{B.2})$$

The fourth rigid body, the MP, rotates through angle $\theta_{I1} - \theta_{I4}$, and hence,

$$\boldsymbol{\omega}_4 = \begin{bmatrix} 0 & 0 & \dot{\theta}_{I1} - \dot{\theta}_{I4} \end{bmatrix}^T, \quad (\text{B.3})$$

Angeles (2007) defined the angular-velocity dyad of an angular velocity vector as

$$\mathbf{W} = \begin{bmatrix} \boldsymbol{\Omega} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \quad (\text{B.4})$$

in which $\boldsymbol{\Omega}$ is the CPM of the angular velocity vector $\boldsymbol{\omega}$ and $\mathbf{O}$ is the 3×3 zero matrix. It is important to note that since the Newton-Euler equations for a rigid body are cast in terms of the twist, which is a 6-dimensional array, $\boldsymbol{\Omega}$ is defined as a

matrix of 6×6 . Thus, the seven angular velocity dyads corresponding to the robot rigid bodies are

$$\mathbf{W}_i = \begin{bmatrix} \boldsymbol{\Omega}_i & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}, \quad i = 1, \dots, 7$$

with $\boldsymbol{\Omega}_i = \text{CPM}(\boldsymbol{\omega})$, and hence,

$$\boldsymbol{\Omega}_i = \begin{bmatrix} 0 & -\dot{\theta}_{I1} & 0 \\ \dot{\theta}_{I1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i = 1, 2, 3 \quad (\text{B.5a})$$

$$\boldsymbol{\Omega}_4 = \begin{bmatrix} 0 & -(\dot{\theta}_{I1} - \dot{\theta}_{I4}) & 0 \\ (\dot{\theta}_{I1} - \dot{\theta}_{I4}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{B.5b})$$

$$\boldsymbol{\Omega}_i = \begin{bmatrix} 0 & -\dot{\theta}_{II1} & 0 \\ \dot{\theta}_{I1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i = 5, 6, 7 \quad (\text{B.5c})$$

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